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Cohomological classification of Ann-categories

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Abstract. An Ann-category is a categorification of rings. Regular Ann-categories were classified by Shukla cohomology of algebras. In this paper, we state and prove the precise theorem on classification of Ann-categories in the general case based on Mac Lane cohomology of rings.

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1. Introduction

Categories with monoidal structures $\oplus, \otimes$ (or *categories with distributivity constraints*) were originally considered by Laplaza in [4]. Kapranov and Voevodsky [3] omitted conditions related to the commutativity constraint with respect to $\otimes$ in the axioms of Laplaza and called these categories *ring categories*.

In an alternative approach, monoidal categories can be "refined" to become *categories with group structure* if the objects are all invertible (see [5, 12]). When the underlying category is a *groupoid* (that is, every morphism is an isomorphism), we obtain the notion of *monoidal category group-like* [1], or *Gr-category* [14]. These categories can be classified by the cohomology group $H^3(\Pi, A)$ of groups.

In 1987, Quang [8] introduced the notion of *Ann-category* which is a categorification of rings. Ann-categories are symmetric Gr-categories (or Picard categories) equipped with a monoidal structure $\otimes$. Since all objects are invertible and all morphisms are isomorphisms, the axioms of an Ann-category are much fewer than those of a ring category (see [7]). The first two invariants of an Ann-category $\mathcal{A}$ are the ring $R = \pi_0\mathcal{A}$ of isomorphism classes of the objects in $\mathcal{A}$ and the R-bimodule $M = \pi_1\mathcal{A} = \mathrm{Aut}_{\mathcal{A}}(0)$. Via the structure transport, we can construct an Ann-category of type (R, M) which is Ann-equivalent to $\mathcal{A}$. A family of constraints of $\mathcal{A}$ induces a 5-tuple of functions $(\xi, \alpha, \lambda, \rho : R^3 \to M, \eta : R^2 \to M)$ satisfying certain relations. This 5-tuple is called a *structure* of an Ann-category of type (R, M). Our purpose is to classify these categories by an appropriate cohomology group.

First we deal with the category of *regular* Ann-categories (they satisfy $c_{A,A} = id$ for all objects A) which arises from the ring extension problem. In [9], these categories were classified by the cohomology group H^3_{Sh} of the ring R (regarded as an

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$\mathbb{Z}$-algebra) in the sense of Shukla [13] (that we misleadingly call Mac Lane-Shukla). This result shows the relation between the notion of a regular Ann-category and the theory of Shukla cohomology. Note that the structure $(\xi, \eta, \alpha, \lambda, \rho)$ of a regular Ann-category has an extra condition $\eta(x, x) = 0$ for the symmetry constraint. This condition is similar to the requirement $f\begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix} = 0$ so that a 3-cocycle f of Mac Lane cohomology has a realization [6].

In 2007, Jibladze and Pirashvili [2] introduced the notion of *categorical ring* as a slightly modified version of the notion of Ann-category and classified categorical rings by the cohomology group $H^3_{MaL}(R, M)$. The condition $(Ann - 1)$ and the compatibility of $\otimes$ with associativity and commutativity constraints with respect to $\oplus$ are replaced by the compatibility of $\otimes$ with the "associativity - commutativity" constraint. We prove in [10] that the category of all Ann-categories is a subcategory of the category of all categorical rings. We also show that there exists a serious gap in the proof of Proposition 2.3 [2]. The authors of [2] did not prove the existence of the isomorphisms

$$A \otimes 0 \to 0, \quad 0 \otimes A \to 0,$$

so that the distributivity constraints induce the $\otimes$-functors which are compatible with the unit constraints. Thus the $\pi_0\mathcal{A}$-bimodule structure of the abelian group $\pi_1\mathcal{A}$ cannot be deduced from axioms of a categorical ring, and therefore results on cohomological classification of categorical rings can not be stated precisely. In the appendix, we give an example of a categorical ring which is not an Ann-category, and prove that the classification theorem in [2] is wrong.

The main result of this paper is the cohomological classification theorem for Ann-categories (Theorem 12) in the general case. It is not only a continuation of the results in [9] and in [11], but it also gives a new interpretation of low-dimensional Mac Lane cohomology groups.

After this introductory Section 1, Section 2 is devoted to recalling some well-known results: i) the construction of an Ann-category of type (R, M) which is the reduced Ann-category of an arbitrary one and the determination of a structure on such an Ann-category of type (R, M); ii) the Mac Lane cohomology and the obstruction theory of Ann-functors. In Section 3 we prove that there is a bijection

$$\mathrm{Struct}[R, M] \leftrightarrow H^3_{MacL}(R, M)$$

between the set of cohomology classes of structures on (R, M) and the Mac Lane cohomology group of the ring R with coefficients in the R-bimodule M, and therefore we obtain the precise theorem on classification of Ann-categories and Ann-functors.

In short, sometimes we write AB or $A.B$ instead of $A \otimes B$.

2. Ann-categories of type (R, M)

Let us recall some necessary concepts and facts in this section from [8, 9].

A monoidal category is called a *Gr-category* (or a *categorical group*) if every object is invertible and the background category is a groupoid. A *Picard* category (or a *symmetric* categorical group) is a Gr-category equipped with a symmetry constraint which is compatible with associativity constraint.

2.1. Ann-categories and Ann-functors

Definition 1. *An Ann-category consists of*

i) *a category* $\mathcal{A}$ *together with two bifunctors* $\oplus, \otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$*;*

ii) *a fixed object* $0 \in \mathcal{A}$ *together with natural isomorphisms* $\mathbf{a}_+, \mathbf{c}, \mathbf{g}, \mathbf{d}$ *such that* $(\mathcal{A}, \oplus, \mathbf{a}_+, \mathbf{c}, (0, \mathbf{g}, \mathbf{d}))$ *is a Picard category;*

iii) *a fixed object* $1 \in \mathcal{A}$ *together with natural isomorphisms* $\mathbf{a}, \mathbf{l}, \mathbf{r}$ *such that* $(\mathcal{A}, \otimes, \mathbf{a}, (1, \mathbf{l}, \mathbf{r}))$ *is a monoidal category;*

iv) *natural isomorphisms* $\mathfrak{L}, \mathfrak{R}$ *given by*

$$\begin{aligned} \mathfrak{L}_{A,X,Y} &: A \otimes (X \oplus Y) \longrightarrow (A \otimes X) \oplus (A \otimes Y), \\ \mathfrak{R}_{X,Y,A} &: (X \oplus Y) \otimes A \longrightarrow (X \otimes A) \oplus (Y \otimes A) \end{aligned}$$

such that the following conditions hold:

$(Ann-1)$ *for* $A \in \mathcal{A}$*, the pairs* $(L^A, \breve{L}^A), (R^A, \breve{R}^A)$ *defined by*

$$\begin{array}{llll} L^A &= A \otimes - \qquad\qquad & R^A &= - \otimes A \\ \breve{L}^A_{X,Y} &= \mathfrak{L}_{A,X,Y} & \breve{R}^A_{X,Y} &= \mathfrak{R}_{X,Y,A} \end{array}$$

are $\oplus$*-functors which are compatible with* $\mathbf{a}_+$ *and* $\mathbf{c}$*;*
$(Ann-2)$ *for all* $A, B, X, Y \in \mathcal{A}$*, the following diagrams commute*

$$\begin{tikzcd}
(AB)(X \oplus Y) \arrow[d, "\breve{L}^{AB}"'] & A(B(X \oplus Y)) \arrow[l, "\mathbf{a}_{A,B,X\oplus Y}"'] \arrow[r, "id_A \otimes \breve{L}^B"] & A(BX \oplus BY) \arrow[d, "\breve{L}^A"] \\
(AB)X \oplus (AB)Y & & A(BX) \oplus A(BY) \arrow[ll, "\mathbf{a}_{A,B,X} \oplus \mathbf{a}_{A,B,Y}"']
\end{tikzcd}$$

$$\begin{tikzcd}
(X \oplus Y)(BA) \arrow[r, "\mathbf{a}_{X\oplus Y,B,A}"] \arrow[d, "\breve{R}^{BA}"'] & ((X \oplus Y)B)A \arrow[r, "\breve{R}^B \otimes id_A"] & (XB \oplus YB)A \arrow[d, "\breve{R}^A"] \\
X(BA) \oplus Y(BA) \arrow[rr, "\mathbf{a}_{X,B,A} \oplus \mathbf{a}_{Y,B,A}"] & & (XB)A \oplus (YB)A
\end{tikzcd}$$

$$\begin{tikzcd}
(A(X \oplus Y)B \arrow[d, "\breve{L}^A \otimes id_B"'] & A((X \oplus Y)B) \arrow[l, "\mathbf{a}_{A,X\oplus Y,B}"'] \arrow[r, "id_A \otimes \breve{R}^B"] & A(XB \oplus YB) \arrow[d, "\breve{L}^A"] \\
(AX \oplus AY)B \arrow[r, "\breve{R}^B"] & (AX)B \oplus (AY)B & A(XB) \oplus A(YB) \arrow[l, "\mathbf{a} \oplus \mathbf{a}"']
\end{tikzcd}$$

$$\begin{tikzcd}
(A\oplus B)X\oplus(A\oplus B)Y \arrow[d, "\breve{R}^X\oplus\breve{R}^Y"'] & (A\oplus B)(X\oplus Y) \arrow[l, "\breve{L}^{A\oplus B}"'] \arrow[r, "\breve{R}^{X\oplus Y}"] & A(X\oplus Y)\oplus B(X\oplus Y) \arrow[d, "\breve{L}^A\oplus\breve{L}^B"] \\
(AX\oplus BX)\oplus(AY\oplus BY) \arrow[rr, "\mathbf{v}"] & & (AX\oplus AY)\oplus(BX\oplus BY)
\end{tikzcd}$$

where $\mathbf{v}=\mathbf{v}_{U,V,Z,T}:(U\oplus V)\oplus(Z\oplus T)\longrightarrow(U\oplus Z)\oplus(V\oplus T)$ *is a unique morphism constructed from* $\oplus,\mathbf{a}_+,\mathbf{c},id$ *of the symmetric monoidal category* $(\mathcal{A},\oplus)$*;*
$(Ann-3)$ *for the unit* $1\in\mathcal{A}$ *of the operation* $\otimes$*, the following diagrams commute*

$$\begin{tikzcd}
1(X\oplus Y) \arrow[rr, "\breve{L}^1"] \arrow[dr, "\mathbf{l}_{X\oplus Y}"'] & & 1X\oplus 1Y \arrow[dl, "\mathbf{l}_X\oplus\mathbf{l}_Y"] \\
& X\oplus Y &
\end{tikzcd}
\qquad
\begin{tikzcd}
(X\oplus Y)1 \arrow[rr, "\breve{R}^1"] \arrow[dr, "\mathbf{r}_{X\oplus Y}"'] & & X1\oplus Y1 \arrow[dl, "\mathbf{r}_X\oplus\mathbf{r}_Y"] \\
& X\oplus Y. &
\end{tikzcd}$$

Since each of pairs $(L^A,\hat{L}^A)$, $(R^A,\hat{R}^A)$ is an $\oplus$-functor which is compatible with the associativity constraint in the Picard category $\mathcal{A}$, it is also compatible with the unit constraint $(0,\mathbf{g},\mathbf{d})$, so we obtain the following result.

Lemma 1. *In an Ann-category* $\mathcal{A}$ *there exist unique isomorphisms*

$$\widehat{L}^A\ :\ A\otimes 0\longrightarrow 0\ ,\widehat{R}^A\ :\ 0\otimes A\longrightarrow 0$$

such that the following diagrams commute

$$\begin{tikzcd}
AX & A(0\oplus X) \arrow[l, "L^A(\mathbf{g})"'] \arrow[d, "\breve{L}^A"] \\
0\oplus AX \arrow[u, "\mathbf{g}"] & A0\oplus AX \arrow[l, "\hat{L}^A\oplus id"']
\end{tikzcd}
\qquad
\begin{tikzcd}
AX & A(X\oplus 0) \arrow[l, "L^A(\mathbf{d})"'] \arrow[d, "\breve{L}^A"] \\
AX\oplus 0 \arrow[u, "\mathbf{d}"] & AX\oplus A0 \arrow[l, "id\oplus\hat{L}^A"']
\end{tikzcd}$$

$$\begin{tikzcd}
XA & (0\oplus X)A \arrow[l, "R^A(\mathbf{g})"'] \arrow[d, "\breve{R}^A"] \\
0\oplus XA \arrow[u, "\mathbf{g}"] & 0A\oplus XA \arrow[l, "\hat{R}^A\oplus id"']
\end{tikzcd}
\qquad
\begin{tikzcd}
XA & (X\oplus 0)A \arrow[l, "R^A(\mathbf{d})"'] \arrow[d, "\breve{R}^A"] \\
XA\oplus 0 \arrow[u, "\mathbf{d}"] & XA\oplus 0A. \arrow[l, "id\oplus\hat{R}^A"']
\end{tikzcd}$$

It is easy to see that if $(F,\breve{F},\widehat{F}):(\mathcal{A},\oplus)\to(\mathcal{A}',\oplus)$ is a monoidal functor between two Gr-categories, then the canonical isomorphism $\widehat{F}:F0\to 0'$ can be deduced from others. Thus, we state the following definition.

Definition 2. *Let* $\mathcal{A}$ *and* $\mathcal{A}'$ *be Ann-categories. An Ann-functor* $(F,\breve{F},\widetilde{F},F_*):\mathcal{A}\to\mathcal{A}'$ *consists of a functor* $F:\mathcal{A}\to\mathcal{A}'$*, natural isomorphisms*

$$\breve{F}_{X,Y}:F(X\oplus Y)\to F(X)\oplus F(Y),\ \widetilde{F}_{X,Y}:F(X\otimes Y)\to F(X)\otimes F(Y),$$

and an isomorphism $F_*\ :\ F(1)\to 1'$ *such that* $(F,\breve{F})$ *is a symmetric monoidal functor with respect to the operation* $\oplus$*,* $(F,\widetilde{F},F_*)$ *is a monoidal functor with respect* *to the operation* $\otimes$*, and* $(F, \breve{F}, \widetilde{F})$ *satisfies two following commutative diagrams*

$$\begin{tikzcd}
F(X(Y\oplus Z)) \arrow[r, "\widetilde{F}"] \arrow[d, "F(\mathfrak{L})"'] & FX.F(Y\oplus Z) \arrow[r, "id\otimes\breve{F}"] & FX(FY\oplus FZ) \arrow[d, "\mathfrak{L}'"] \\
F(XY\oplus XZ) \arrow[r, "\breve{F}"] & F(XY)\oplus F(XZ) \arrow[r, "\widetilde{F}\oplus\widetilde{F}"] & FX.FY\oplus FX.FZ
\end{tikzcd}$$

$$\begin{tikzcd}
F((X\oplus Y)Z) \arrow[r, "\widetilde{F}"] \arrow[d, "F(\mathfrak{R})"'] & F(X\oplus Y).FZ \arrow[r, "\breve{F}\otimes id"] & (FX\oplus FY)FZ \arrow[d, "\mathfrak{R}'"] \\
F(XZ\oplus YZ) \arrow[r, "\breve{F}"] & F(XZ)\oplus F(YZ) \arrow[r, "\widetilde{F}\oplus\widetilde{F}"] & FX.FZ\oplus FY.FZ.
\end{tikzcd}$$

These diagrams are called the compatibility of the functor F *with the distributivity constraints.*

An Ann-morphism (or a homotopy)

$$\theta : (F, \breve{F}, \widetilde{F}, F_*) \to (F', \breve{F}', \widetilde{F}', F'_*)$$

between Ann-functors is an $\oplus$*-morphism, as well as an* $\otimes$*-morphism.*

If there exists an Ann-functor $(F', \breve{F}', \widetilde{F}', F'_*) : \mathcal{A}' \to \mathcal{A}$ *and Ann-morphisms* $F'F \xrightarrow{\sim} id_{\mathcal{A}}$, $FF' \xrightarrow{\sim} id_{\mathcal{A}'}$*, we say that* $(F, \breve{F}, \widetilde{F}, F_*)$ *is an Ann-equivalence, and* $\mathcal{A}$*,* $\mathcal{A}'$ *are Ann-equivalent.*

It can be proved that each Ann-functor is an Ann-equivalence if and only if F is a categorical equivalence.

Lemma 2. *Any Ann-functor* $F = (F, \breve{F}, \widetilde{F}, F_*) : \mathcal{A} \to \mathcal{A}'$ *is homotopic to an Ann-functor* $F' = (F', \breve{F}', \widetilde{F}', F'_*)$*, where* $F'0 = 0', \widehat{F}' = id_{0'}$*, and* $F'1 = 1', F'_* = id_{1'}$*.*

Proof. Consider a family of isomorphisms in $\mathcal{A}'$:

$$\theta_X = \begin{cases} id_{FX} & \text{if } X \neq 0,\ X \neq 1, \\ \widehat{F} & \text{if } X = 0, \\ F_* & \text{if } X = 1, \end{cases}$$

for $X \in \mathcal{A}$. Then, the Ann-functor F' can be constructed in a unique way such that $\theta : F \to F'$ becomes a homotopy. Namely,

$$F'X = \begin{cases} FX \text{if } X \neq 0, X \neq 1, \\ 0' \quad \text{if } X = 0, \\ 1' \quad \text{if } X = 1, \end{cases}$$
$$F'(f : X \to Y) = \theta_Y F(f)(\theta_X)^{-1} : F'X \to F'Y,$$
$$\breve{F}'_{X,Y} = (\theta_X \oplus \theta_Y)\breve{F}_{X,Y}\theta^{-1}_{X\oplus Y},$$
$$\widetilde{F'}_{X,Y} = (\theta_X \otimes \theta_Y)\widetilde{F}_{X,Y}\theta^{-1}_{XY},$$
$$\widehat{F'} = \widehat{F}\theta_0^{-1} = id_{0'},\ F'_* = F_*\theta_1^{-1} = id_{1'}.$$

□

Based on Lemma 2, we refer to $(F, \breve{F}, \widetilde{F})$ as an Ann-functor.

2.2. Reduced Ann-categories

For an Ann-category $\mathcal{A}$, the set $R = \pi_0\mathcal{A}$ of isomorphism classes of the objects in $\mathcal{A}$ is a ring where the operations $+, \times$ are induced by $\oplus, \otimes$ on $\mathcal{A}$, and $M = \pi_1\mathcal{A} = \mathrm{Aut}(0)$ is an abelian group where the operation, denoted by $+$, is just the composition. Moreover, $M = \pi_1\mathcal{A}$ is an R-bimodule with the actions

$$sa = \lambda_X(a), \quad as = \rho_X(a),$$

where $X \in s, s \in \pi_0\mathcal{A}, a \in \pi_1\mathcal{A}$ and λ_X, ρ_X satisfy the commutative diagrams

$$\begin{tikzcd} X.0 \arrow[r, "\hat{L}^X"] \arrow[d, "id\otimes a"'] & 0 \arrow[d, "\lambda_X(a)"] \\ X.0 \arrow[r, "\hat{L}^X"] & 0, \end{tikzcd} \qquad \begin{tikzcd} 0.X \arrow[r, "\hat{R}^X"] \arrow[d, "a\otimes id"'] & 0 \arrow[d, "\rho_X(a)"] \\ 0.X \arrow[r, "\hat{R}^X"] & 0. \end{tikzcd}$$

We recall briefly some main facts of the construction of the reduced Ann-category $S_\mathcal{A}$ of $\mathcal{A}$ via the structure transport (for details, see [9]). The objects of $S_\mathcal{A}$ are the elements of the ring $\pi_0\mathcal{A}$. A morphism is an automorphism $(s, a) : s \to s,\ s \in \pi_0\mathcal{A}, a \in \pi_1\mathcal{A}$. The composition of morphisms is given by

$$(s, a) \circ (s, b) = (s, a + b).$$

For each $s \in \pi_0\mathcal{A}$, choose an object $X_s \in \mathcal{A}$ such that $X_0 = 0, X_1 = 1$, and choose an isomorphism $i_X : X \to X_s$ such that $i_{X_s} = id_{X_s}$. We obtain two functors

$$\begin{cases} G : \mathcal{A} \to S_\mathcal{A} \\ G(X) = [X] = s \\ G(X \xrightarrow{f} Y) = (s, \gamma^{-1}_{X_s}(i_Y f i_X^{-1})), \end{cases} \qquad \begin{cases} H : S_\mathcal{A} \to \mathcal{A} \\ H(s) = X_s \\ H(s, a) = \gamma_{X_s}(a), \end{cases} \tag{1}$$

where $X, Y \in s$ and $f : X \to Y$, and γ_X is a map defined by the following commutative diagram

$$\begin{tikzcd} X \arrow[r, "\gamma_X(a)"] & X \\ 0\oplus X \arrow[u, "\mathbf{g}_X"] \arrow[r, "a\oplus id"] & 0\oplus X \arrow[u, "\mathbf{g}_X"'] \end{tikzcd}$$

Diagram 1

The operations on $S_{\mathcal{A}}$ are defined by

$$\begin{aligned} s\oplus t &= G(H(s)\oplus H(t)) = s+t,\\ (s,a)\oplus(t,b) &= G(H(s,a)\oplus H(t,b)) = (s+t, a+b),\\ s\otimes t &= G(H(s)\otimes H(t)) = st,\\ (s,a)\otimes(t,b) &= G(H(s,a)\otimes H(t,b)) = (st, sb+at), \end{aligned}$$

where $s,t\in\pi_0\mathcal{A}$, $a,b\in\pi_1\mathcal{A}$. Obviously, these operations do not depend on the choice of the set of representatives (X_s, i_X).

The constraints in $S_{\mathcal{A}}$ are defined by those in $\mathcal{A}$ by means of the notion of *stick*. A *stick* in $\mathcal{A}$ is a set of representatives (X_s, i_X) such that

$$\begin{aligned} &i_{0\oplus X_t} = \mathbf{g}_{X_t}, \quad i_{X_s\oplus 0} = \mathbf{d}_{X_s},\\ &i_{1\otimes X_t} = \mathbf{l}_{X_t}, \quad i_{X_s\otimes 1} = \mathbf{r}_{X_s}, \quad i_{0\otimes X_t} = \widehat{R}^{X_t}, \quad i_{X_s\otimes 0} = \widehat{L}^{X_s}. \end{aligned}$$

The unit constraints for two operations $\oplus,\otimes$ in $S_{\mathcal{A}}$ are $(0, id, id)$ and $(1, id, id)$, respectively. The functor H and isomorphisms

$$\breve{H} = i^{-1}_{X_s\oplus X_t},\ \widetilde{H} = i^{-1}_{X_s\otimes X_t} \tag{2}$$

transport the constraints $\mathbf{a}_+, \mathbf{c}, \mathbf{a}, \mathfrak{L}, \mathfrak{R}$ of $\mathcal{A}$ to those $\xi, \eta, \alpha, \lambda, \rho$ of $S_{\mathcal{A}}$. Then, the category

$$(S_{\mathcal{A}}, \xi, \eta, (0, id, id), \alpha, (1, id, id), \lambda, \rho)$$

is an Ann-category which is equivalent to $\mathcal{A}$ by the Ann-equivalence $(H, \breve{H}, \widetilde{H}) : S_{\mathcal{A}} \to \mathcal{A}$. Besides, the functor $G : \mathcal{A} \to S_{\mathcal{A}}$ together with isomorphisms

$$\breve{G}_{X,Y} = G(i_X \oplus i_Y),\ \widetilde{G}_{X,Y} = G(i_X \otimes i_Y) \tag{3}$$

is also an Ann-equivalence. We refer to $S_{\mathcal{A}}$ as an Ann-category of *type* (R, M), called a *reduction* of $\mathcal{A}$. We also call $(H, \breve{H}, \widetilde{H})$ and $(G, \breve{G}, \widetilde{G})$ *canonical* Ann-equivalences, the family of constraints $h = (\xi, \eta, \alpha, \lambda, \rho)$ of $S_{\mathcal{A}}$ a *structure* of the Ann-category of type (R, M), or simply a *structure on* (R, M).

The following result follows from the axioms of an Ann-category.

Theorem 1 ([9, Theorem 3.1])**.** *In the reduced Ann-category $S_{\mathcal{A}}$ of an Ann-category $\mathcal{A}$, the structure $(\xi, \eta, \alpha, \lambda, \rho)$ consists of functions with values in $\pi_1\mathcal{A}$ such that for any $x, y, z, t \in \pi_0\mathcal{A}$, the following conditions hold:*

A_1. $\xi(y,z,t) - \xi(x+y,z,t) + \xi(x,y+z,t) - \xi(x,y,z+t) + \xi(x,y,z) = 0,$

A_2. $\xi(x,y,z) - \xi(x,z,y) + \xi(z,x,y) + \eta(x+y,z) - \eta(x,z) - \eta(y,z) = 0,$

A_3. $\eta(x,y)+\eta(y,x)=0$,

A_4. $x\eta(y,z)-\eta(xy,xz)=\lambda(x,y,z)-\lambda(x,z,y)$,

A_5. $\eta(x,y)z-\eta(xz,yz)=\rho(x,y,z)-\rho(y,x,z)$,

A_6. $x\xi(y,z,t)-\xi(xy,xz,xt)=\lambda(x,z,t)-\lambda(x,y+z,t)+\lambda(x,y,z+t)-\lambda(x,y,z)$,

A_7. $\xi(x,y,z)t-\xi(xt,yt,zt)=\rho(y,z,t)-\rho(x+y,z,t)+\rho(x,y+z,t)-\rho(x,y,z)$,

A_8. $\rho(x,y,z+t)-\rho(x,y,z)-\rho(x,y,t)+\lambda(x,z,t)+\lambda(y,z,t)-\lambda(x+y,z,t)$
$= \xi(xz+xt,yz,yt)+\xi(xz,xt,yz)-\eta(xt,yz)+\xi(xz+yz,xt,yt)-\xi(xz,yz,xt)$,

A_9. $\alpha(x,y,z+t)-\alpha(x,y,z)-\alpha(x,y,t)=x\lambda(y,z,t)+\lambda(x,yz,yt)-\lambda(xy,z,t)$,

A_{10}. $\alpha(x,y+z,t)-\alpha(x,y,t)-\alpha(x,z,t)=x\rho(y,z,t)-\rho(xy,xz,t)+\lambda(x,yt,zt)$
$-\lambda(x,y,z)t$,

A_{11}. $\alpha(x+y,z,t)-\alpha(x,y,t)-\alpha(y,z,t)=-\rho(x,y,z)t-\rho(xz,yz,t)+\rho(x,y,zt)$,

A_{12}. $x\alpha(y,z,t)-\alpha(xy,z,t)+\alpha(x,yz,t)-\alpha(x,y,zt)+\alpha(x,y,z)t=0$.

Further, these functions satisfy normalization conditions:

$$\begin{aligned}
\xi(0,y,z) &= \xi(x,0,z)=\xi(x,y,0)=0,\\
\alpha(1,y,z) &= \alpha(x,1,z)=\alpha(x,y,1)=0,\\
\alpha(0,y,z) &= \alpha(x,0,z)=\alpha(x,y,0)=0,\\
\lambda(1,y,z) &= \lambda(0,y,z)=\lambda(x,0,z)=\lambda(x,y,0)=0,\\
\rho(x,y,1) &= \rho(0,y,z)=\rho(x,0,z)=\rho(x,y,0)=0.
\end{aligned}$$

The induced operations on $S_{\mathcal{A}}$ do not depend on the choice of sticks. We now investigate the effect of different choices of the stick (X_s,i_X) in the induced constraints on $S_{\mathcal{A}}$.

Proposition 1. *Let $\mathcal{S}$ and $\mathcal{S}'$ be reduced Ann-categories of $\mathcal{A}$ corresponding to the sticks (X_s,i_X) and (X'_s,i'_X), respectively. Then the structures $(\xi,\eta,\alpha,\lambda,\rho)$ of $\mathcal{S}$ and $(\xi',\eta',\alpha',\lambda',\rho')$ of $\mathcal{S}'$ satisfy the following relations:*

A_{13}. $\xi(x,y,z)-\xi'(x,y,z)=\tau(y,z)-\tau(x+y,z)+\tau(x,y+z)-\tau(x,y)$,

A_{14}. $\eta(x,y)-\eta'(y,x)=\tau(x,y)-\tau(y,x)$,

A_{15}. $\alpha(x,y,z)-\alpha'(x,y,z)=x\nu(y,z)-\nu(xy,z)+\nu(x,yz)-\nu(x,y)z$,

A_{16}. $\lambda(x,y,z)-\lambda'(x,y,z)=\nu(x,y+z)-\nu(x,y)-\nu(x,z)+x\tau(y,z)-\tau(xy,xz)$,

A_{17}. $\rho(x,y,z)-\rho'(x,y,z)=\nu(x+y,z)-\nu(x,z)-\nu(y,z)+\tau(x,y)z-\tau(xz,yz)$,

where $\tau,\nu:(\pi_0\mathcal{A})^2\to\pi_1\mathcal{A}$ are the functions satisfying the normalization conditions $\tau(0,y)=\tau(x,0)=0$ and $\nu(0,y)=\nu(x,0)=\nu(1,y)=\nu(x,1)=0$.

Two structures $(\xi,\eta,\alpha,\lambda,\rho)$ and $(\xi',\eta',\alpha',\lambda',\rho')$ of Ann-categories of type (R,M) are *cohomologous* if and only if they satisfy the relations $A_{13}-A_{17}$ in Proposition 1.

Note that two unit constraints of $\oplus$ and $\otimes$ in an Ann-category of type (R,M) are both strict. It is easy to prove the following lemma.

Lemma 3. *Two structures h and h' are cohomologous if and only if there exists an Ann-functor* $(F, \breve{F}, \widetilde{F}) : (R, M, h) \to (R, M, h')$, *where* $F = id_{(R,M)}$.

2.3. Mac Lane cohomology groups of rings and obstruction theory

Let R be a ring and M an R-bimodule. From the definition of Mac Lane cohomology of rings [6], we obtain the description of elements in the cohomology group $H^3_{MaL}(R, M)$.

The group $Z^3_{MaL}(R, M)$ of 3-cocycles of R with coefficients in M consists of the quadruples $(\sigma, \alpha, \lambda, \rho)$ of the maps:

$$\sigma : R^4 \to M; \quad \alpha, \lambda, \rho : R^3 \to M$$

satisfying the following conditions:

M_1. $x\alpha(y, z, t) - \alpha(xy, z, t) + \alpha(x, yz, t) - \alpha(x, y, zt) + \alpha(x, y, z)t = 0,$

M_2. $-\alpha(x, z, t) - \alpha(y, z, t) + \alpha(x + y, z, t) + \rho(xz, yz, t) - \rho(x, y, zt) + \rho(x, y, z)t = 0,$

M_3. $-\alpha(x, y, t) - \alpha(x, z, t) + \alpha(x, y + z, t) + x\rho(y, z, t) - \rho(xy, xz, t) - \lambda(x, yt, zt)$
$+\lambda(x, y, z)t = 0,$

M_4. $\alpha(x, y, z) + \alpha(x, y, t) - \alpha(x, y, z + t) + x\lambda(y, z, t) - \lambda(xy, z, t) + \lambda(x, yz, yt) = 0,$

M_5. $-\lambda(x, z, t) - \lambda(y, z, t) + \lambda(x + y, z, t) + \rho(x, y, z) + \rho(x, y, t) - \rho(x, y, z + t)$
$+\sigma(xz, xt, yz, yt) = 0,$

M_6. $\lambda(r, x, y) + \lambda(r, z, t) - \lambda(r, x + z, y + t) - \lambda(r, x, z) - \lambda(r, y, t) + \lambda(r, x + y, z + t)$
$-r\sigma(x, y, z, t) + \sigma(rx, ry, rz, rt) = 0,$

M_7. $-\rho(x, y, r) - \rho(z, t, r) + \rho(x + z, y + t, r) + \rho(x, z, r) + \rho(y, t, r) - \rho(x + y, z + t, r)$
$-\sigma(xr, yr, zr, tr) + \sigma(x, y, z, t)r = 0,$

M_8. $-\sigma(r, s, u, v) - \sigma(x, y, z, t) + \sigma(r + x, s + y, u + z, v + t) + \sigma(r, s, x, y) + \sigma(u, v, z, t)$
$-\sigma(r + u, s + v, x + z, y + t) - \sigma(r, u, x, z) - \sigma(s, v, y, t)$
$+\sigma(r + s, u + v, x + y, z + t) = 0.$

These functions satisfy normalization conditions:

$$\begin{aligned}
\alpha(0, y, z) &= \alpha(x, 0, z) = \alpha(x, y, 0) = 0,\\
\lambda(0, y, z) &= \lambda(x, 0, z) = \lambda(x, y, 0) = 0,\\
\rho(0, y, z) &= \rho(x, 0, z) = \rho(x, y, 0) = 0,\\
\sigma(r, s, 0, 0) &= \sigma(0, 0, u, v) = \sigma(r, 0, u, 0) = \sigma(0, s, 0, v) = \sigma(r, 0, 0, v) = 0.
\end{aligned}$$

The 3-cocycle $h = (\sigma, \alpha, \lambda, \rho)$ belongs to the group $B^3_{MaL}(R, M)$ if and only if there exist the functions $\tau, \nu : R^2 \to M$ satisfying:

M_9. $\sigma(x, y, z, t) = \tau(x, y) + \tau(z, t) - \tau(x + z, y + t) - \tau(x, z) - \tau(y, t) + \tau(x + y, z + t)$

M_{10}. $\alpha(x, y, z) = x\nu(y, z) - \nu(xy, z) + \nu(x, yz) - \nu(x, y)z,$

M_{11}. $\lambda(x, y, z) = \nu(x, y + z) - \nu(x, y) - \nu(x, z) + x\tau(y, z) - \tau(xy, xz),$

M_{12}. $\rho(x,y,z) = \nu(x+y,z) - \nu(x,z) - \nu(y,z) + \tau(x,y)z - \tau(xz,yz)$,

where τ, ν satisfy the normalization conditions: $\tau(0,y) = \tau(x,0) = 0$ and $\nu(0,y) = \nu(x,0) = \nu(1,y) = \nu(x,1) = 0$.

The group $Z^2_{MaL}(R,M)$ consists of 2-cochains $g = (\tau,\nu)$ of the ring R with coefficients in the R-bimodule M satisfying

$$\partial g = 0.$$

The subgroup $B^2_{MaL}(R,M) \subset Z^2_{MaL}(R,M)$ of 2-coboundaries consists of the pairs (τ,ν) such that there exist the maps $t : R \to M$ satisfying $(\tau,\nu) = \partial_{MaL}t$, that is,

M_{13}. $\tau(x,y) = t(y) - t(x+y) + t(x)$,

M_{14}. $\nu(x,y) = xt(y) - t(xy) + t(x)y$,

where t satisfies the normalization condition, $t(0) = t(1) = 0$.

The group $Z^1_{MaL}(R,M)$ consists of 1-cochains t of the ring R with coefficients in the R-bimodule M satisfying

$$\partial t = 0.$$

The subgroup of 1-coboundaries, $B^1_{MaL}(R,M) \subset Z^1_{MaL}(R,M)$, consists of the functions t such that there exists $a \in R$ satisfying $t(x) = ax - xa$.

The quotient group

$$H^i_{MaL}(R,M) = Z^i_{MaL}(R,M)/B^i_{MaL}(R,M),\ i = 1,2,3,$$

is called the i^{th} *Mac Lane cohomology group* of the ring R with coefficients in the R-bimodule M.

Let us now recall some results on Ann-functors from [11]. Each Ann-functor $(F, \breve{F}, \widetilde{F}) : \mathcal{A} \to \mathcal{A}'$ induces one S_F between their reduced Ann-categories. Throughout this section, let $\mathcal{S}$ and $\mathcal{S}'$ be Ann-categories of types (R,M,h) and (R',M',h'), respectively.

A functor $F : \mathcal{S} \to \mathcal{S}'$ is called a functor of *type* (p,q) if

$$F(x) = p(x), \quad F(x,a) = (p(x), q(a)),$$

where $p : R \to R'$ is a ring homomorphism and $q : M \to M'$ is a group homomorphism such that

$$q(xa) = p(x)q(a), \quad x \in R, a \in M.$$

The group M' can be regarded as an R-module with the action $xa' = p(x)a'$, so q is an R-bimodule homomorphism. In this case, we say that (p,q) is a *pair of homomorphisms* and that the function

$$k = q_*h - p^*h' \tag{4}$$

is an *obstruction* of F, where p^*, q_* are canonical homomorphisms,

$$Z^3_{MacL}(R,M) \xrightarrow{q_*} Z^3_{MacL}(R,M') \xleftarrow{p^*} Z^3_{MacL}(R',M').$$

Proposition 2 ([11, Proposition 4.3]). *Every Ann-functor $F : \mathcal{S} \to \mathcal{S}'$ is a functor of type (p, q).*

Keeping in mind that γ is the map defined by Diagram 1, we state the following proposition.

Proposition 3 ([11, Proposition 4.1]). *Let $\mathcal{A}$ and $\mathcal{A}'$ be Ann-categories. Then every Ann-functor $(F, \breve{F}, \widetilde{F}) : \mathcal{A} \to \mathcal{A}'$ induces an Ann-functor $S_F : S_{\mathcal{A}} \to S_{\mathcal{A}'}$ of type (p, q), where*

$$p = F_0 : \pi_0\mathcal{A} \to \pi_0\mathcal{A}',\ [X] \mapsto [FX],$$
$$q = F_1 : \pi_1\mathcal{A} \to \pi_1\mathcal{A}',\ u \mapsto \gamma_{F0}^{-1}(Fu).$$

Further,

i) *F is an equivalence if and only if F_0, F_1 are isomorphisms.*

ii) *The Ann-functor S_F satisfies the commutative diagram*

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "F"] & \mathcal{A}' \arrow[d, "G'"] \\
S_{\mathcal{A}} \arrow[u, "H"] \arrow[r, "S_F"] & S_{\mathcal{A}'},
\end{tikzcd}$$

where H, G' are canonical Ann-equivalences defined by (1), (2), (3).

Since $\breve{F}_{x,y} = (\bullet, \tau(x, y))$, and $\widetilde{F}_{x,y} = (\bullet, \nu(x, y))$, we call $g_F = (\tau, \nu)$ a pair of functions *associated* to $(\breve{F}, \widetilde{F})$, and hence an Ann-functor $F : \mathcal{S} \to \mathcal{S}'$ can be regarded as a triple (p, q, g_F). It follows from the compatibility of F with the constraints that

$$q_*h - p^*h' = \partial(g_F), \tag{5}$$

Moreover, Ann-functors (F, g_F) and $(F', g_{F'})$ are homotopic if and only if $F' = F$, that is, they are of the same type (p, q), and there is a function $t : R \to M'$ such that $g_{F'} = g_F + \partial t$.

We write

$$\mathrm{Hom}^{Ann}_{(p,q)}[\mathcal{S}, \mathcal{S}']$$

for the set of homotopy classes of Ann-functors of type (p, q) from $\mathcal{S}$ to $\mathcal{S}'$.

Theorem 2 ([11, Theorem 4.4, 4.5]). *The functor $F : \mathcal{S} \to \mathcal{S}'$ of type (p, q) is an Ann-functor if and only if the obstruction $[k]$ vanishes in $H^3_{MacL}(R, M')$. Then, there exists a bijection*

$$\mathrm{Hom}^{Ann}_{(p,q)}[\mathcal{S}, \mathcal{S}'] \leftrightarrow H^2_{MacL}(R, M'). \tag{6}$$

3. Classification of Ann-categories

In order to prove the main result (Theorem 3) of the paper, we first prove that the set of cohomology classes of structures on (R, M) and the group $H^3_{MaL}(R, M)$ are coincident.

Lemma 4. *Each structure of an Ann-category of type (R, M) induces a 3-cocycle in $Z^3_{MaL}(R, M)$.*

Proof. Let $h = (\xi, \eta, \alpha, \lambda, \rho)$ be a structure of an Ann-category $\mathcal{S}$ of type (R, M). We define a function $\sigma : R^4 \to M$ by

$$\sigma(x, y, z, t) = \xi(x + y, z, t) - \xi(x, y, z) + \eta(y, z) + \xi(x, z, y) - \xi(x + z, y, t) \qquad (7)$$

This equation shows that σ is just the morphism

$$\mathbf{v} : (x + y) + (z + t) \to (x + z) + (y + t)$$

in an Ann-category of type (R, M).

First, the normalized property of σ follows from the ones of ξ and η

$$\sigma(0, 0, z, t) = \sigma(x, y, 0, 0) = \sigma(0, y, 0, t) = \sigma(x, 0, z, 0) = \sigma(x, 0, 0, t) = 0.$$

We now show that the quadruple $\widehat{h} = (\sigma, \alpha, \lambda, \rho)$ satisfies the relations $M_1 - M_8$, and $\widehat{h}$ is therefore a 3-cocycle. The relation M_1 is just the relation A_{12}. The relations M_2, M_3, M_4, M_5 are just A_{11}, A_{10}, A_9, A_8, respectively.

According to the coherence theorem in an Ann-category of type (R, M), the following Diagrams 2 and 3 commute

$$\begin{tikzcd}
r[(x+y)+(z+t)] \arrow[r, "id\otimes\mathbf{v}"] \arrow[d, "\mathcal{L}"'] & r[(x+z)+(y+t)] \arrow[d, "\mathcal{L}"] \\
r(x+y)+r(z+t) \arrow[d, "\mathcal{L}\oplus\mathcal{L}"'] & r(x+z)+r(y+t) \arrow[d, "\mathcal{L}\oplus\mathcal{L}"] \\
(rx+ry)+(rz+rt) \arrow[r, "\mathbf{v}"] & (rx+rz)+(ry+rt)
\end{tikzcd}$$

Diagram 2

$$\begin{tikzcd}
{[(r+s)+(u+v)]+[(x+y)+(z+t)]} \arrow[r, "\mathbf{v}"] \arrow[d, "\mathbf{v}+\mathbf{v}"'] & {[(r+s)+(x+y)]+[(u+v)+(z+t)]} \arrow[d, "\mathbf{v}+\mathbf{v}"] \\
{[(r+u)+(s+v)]+[(x+z)+(y+t)]} \arrow[d, "\mathbf{v}"'] & {[(r+x)+(s+y)]+[(u+z)+(v+t)]} \arrow[d, "\mathbf{v}"] \\
{[(r+u)+(x+z)]+[(s+v)+(y+t)]} \arrow[r, "\mathbf{v}+\mathbf{v}"] & {[(r+x)+(u+z)]+[(s+y)+(v+t)]}
\end{tikzcd}$$

Diagram 3

These commutative diagrams imply the relations M_6, M_8. The relation M_7 follows from a commutative diagram which is analogous to Diagram 2, where r is tensored on the right-hand side. □

Lemma 5. *Each Mac Lane 3-cocycle* $(\sigma, \alpha, \lambda, \rho)$ *is induced by a structure* $(\xi, \eta, \alpha, \lambda, \rho)$ *of an Ann-category of type* (R, M).

Proof. Let $(\sigma, \alpha, \lambda, \rho)$ be an element in $Z^3_{MaL}(R, M))$. Set

$$\xi(x, y, z) = -\sigma(x, y, 0, z), \eta(x, y) = \sigma(0, x, y, 0),$$

we obtain a 5-tuple of functions $h = (\xi, \eta, \alpha, \lambda, \rho)$. The normalized properties of ξ, η follow from that of σ.

We now show that h is a structure of an Ann-category of type (R, M). First, the relations $A_{12} - A_9$ are just $M_1 - M_4$. The relation A_1 follows from M_8 when $u = 0 = x = y = z$. The relation A_3 follows from M_8 when $r = s = v = 0 = x = z = t$. The relations A_4 and A_5 follow from M_6 and M_7, respectively, when $x = t = 0$. The relations A_6 and A_7 follow from M_6 and M_7, respectively, when $z = 0$.

To prove the relation A_2, take $s = u = 0 = x = z = t$ in M_8 we obtain

$$-\xi(r, y, v) + \xi(r, v, y) - \eta(v, y) + \sigma(r, v, y, 0) = 0 \tag{8}$$

Now, take $r = u = 0 = y = z = t$ in M_8 we obtain

$$-\xi(x, s, v) + \eta(s, x) - \eta(s + v, x) + \sigma(s, v, x, 0) = 0.$$

In other words,

$$-\xi(y, r, v) + \eta(r, y) - \eta(r + v, y) + \sigma(r, v, y, 0) = 0 \tag{9}$$

Subtracting (9) from (8), we obtain the relation A_2.

Finally, to prove the relation A_8, note that σ can be presented by ξ, η as in (7). Indeed, take $v = 0 = x = y = z$ in M_8 we obtain

$$\sigma(r, s, u, t) + \xi(r + u, s, t) - \xi(r + s, u, t) - \sigma(r, s, u, 0) = 0. \tag{10}$$

Now, take $v = s, y = u$ in (9) we obtain

$$\xi(r, u, s) - \xi(r, s, u) - \eta(s, u) + \sigma(r, s, u, 0) = 0. \tag{11}$$

Adding (10) to (11) and doing some appropriate calculations, we get (7).

Because of (7), M_8 becomes A_8. This means the 5-tuple of functions $h = (\xi, \eta, \alpha, \lambda, \rho)$ is a structure of an Ann-category of type (R, M). Further, this structure induces the 3-cocycle $\widehat{h} = (\sigma, \alpha, \lambda, \rho)$. □

Lemma 6. *The structures* h *and* h' *of the Ann-category of type* (R, M) *are cohomologous if and only if the corresponding 3-cocycles* $\widehat{h}, \widehat{h'}$ *are cohomologous.*

Proof. By Lemma 5, the structures h and h' induce elements $\widehat{h}$ and $\widehat{h'}$ in $Z^3_{MaL}(R,M)$, respectively. By Lemma 3, the functions $\alpha - \alpha'$, $\lambda - \lambda'$, $\rho - \rho'$ satisfy the relations $M_{10} - M_{12}$, where $\breve{F} = \tau, \widetilde{F} = \nu$. Besides, the following diagram commutes because of the coherence of a symmetric monoidal functor.

Note that $F = id$ and $\breve{F} = \tau$, so the above commutative diagram implies

$$\begin{aligned}\sigma(x,y,z,t) - \sigma'(x,y,z,t) &= \tau(x+y,z+t) + \tau(x,y) + \tau(z,t) - \tau(x+z,y+t) \\ &\quad -\tau(x,z) - \tau(y,t).\end{aligned}$$

That means $\sigma - \sigma'$ satisfies M_9. Thus, $\widehat{h}$ and $\widehat{h'}$ belong to the same cohomology class of $H^3_{MaL}(R,M)$.

Now, assume that $\widehat{h} - \widehat{h'} \in B^3_{MaL}(R,M)$. Then $\alpha - \alpha'$, $\lambda - \lambda'$, $\rho - \rho'$ satisfy $M_{10} - M_{12}$ which are just the relations $A_{15} - A_{17}$. By (7), the definition of σ and the normalized property of ξ, η, we have

$$\begin{aligned}\xi(x,y,z) &= -\sigma(x,0,y,z),\ \xi'(x,y,z) = -\sigma'(x,0,y,z),\\ \eta(x,y) &= \sigma(0,x,y,0),\ \eta'(x,y) = \sigma'(0,x,y,0).\end{aligned}$$

Therefore, A_{13}, A_{14} are obtained from M_9, and thus h, h' are cohomologous structures. $\square$

Let $\mathrm{Struct}[R,M]$ denote the set of cohomology classes of structures on (R,M). Then, Lemmas 4, 5 and 6 lead to the following result.

Proposition 4. *There exists a bijection*

$$\begin{aligned}\mathrm{Struct}[R,M] &\to H^3_{MacL}(R,M)\\ [h = (\xi,\eta,\alpha,\lambda,\rho)] &\mapsto [\widehat{h} = (\sigma,\alpha,\lambda,\rho)]\end{aligned}$$

By the above lemma, we regard each cohomology class $[h] = [(\xi,\eta,\alpha,\lambda,\rho)]$ as an element of the group $H^3_{MacL}(R,M)$.

Let **Ann** refer to the category whose objects are Ann-categories, and whose morphisms are their Ann-functors.

We determine the category $\mathbf{H^3_{Ann}}$ whose objects are triples $(R,M,[h])$, where $[h] \in H^3_{MacL}(R,M)$. A morphism $(R,M,[h]) \to (R',M',[h'])$ in $\mathbf{H^3_{Ann}}$ is a pair (p,q) such that there exists a function $g : R^2 \to M'$ so that $(p,q,g) : (R,M,h) \to (R',M',h')$ is an Ann-functor, that is, $[p^*h'] = [q_*h] \in H^3_{MacL}(R,M')$. The composition in $\mathbf{H^3_{Ann}}$ is defined by

$$(p',q') \circ (p,q) = (p'p, q'q).$$

Note that, *Ann-functors $F, F' : \mathcal{A} \to \mathcal{A}'$ are homotopic if and only if $F_i = F'_i, i = 0,1$ and $[g_F] = [g_{F'}]$ in $H^2_{MacL}(R,M)$.* Denote by

$$\mathrm{Hom}^{Ann}_{(p,q)}[\mathcal{A},\mathcal{A}']$$

the set of homotopy classes of Ann-functors from $\mathcal{A}$ to $\mathcal{A}'$ inducing the same pair (p,q), we prove the following classification result.

Theorem 3 (Classification Theorem). *There is a functor*

$$\begin{aligned} d : \mathbf{Ann} &\to \mathbf{H^3_{Ann}} \\ \mathcal{A} &\mapsto (\pi_0\mathcal{A}, \pi_1\mathcal{A}, [h_\mathcal{A}]) \end{aligned}$$

which has the following properties:

i) dF is an isomorphism if and only if F is an equivalence.

ii) d is surjective on objects.

iii) d is full, but not faithful. For $(p, q) : d\mathcal{A} \to d\mathcal{A}'$, there is a bijection

$$\overline{d} : \mathrm{Hom}^{Ann}_{(p,q)}[\mathcal{A}, \mathcal{A}'] \to H^2_{MacL}(\pi_0\mathcal{A}, \pi_1\mathcal{A}'). \tag{12}$$

Proof. In the Ann-category $\mathcal{A}$, for each stick (X_s, i_X) one can construct a reduced Ann-category $(\pi_0\mathcal{A}, \pi_1\mathcal{A}, h)$. If the choice of the stick is modified, then the 3-cocycle h changes to a cohomologous 3-cocycle h'. Therefore, $\mathcal{A}$ uniquely determines an element $[h] \in H^3(\pi_0\mathcal{A}, \pi_1\mathcal{A})$.

For Ann-functors

$$\mathcal{A} \xrightarrow{F} \mathcal{A}' \xrightarrow{F'} \mathcal{A}'',$$

it can be seen that $d(F' \circ F) = dF' \circ dF$, and $d(id_\mathcal{A}) = id_{d\mathcal{A}}$. Therefore, d is a functor.

i) According to Proposition 3.

ii) If $(R, M, [h])$ is an object of $\mathbf{H^3_{Ann}}$, then $\mathcal{S} = (R, M, h)$ is an Ann-category of type (R, M), and obviously $d\mathcal{S} = (R, M, [h])$.

iii) If (p, q) is a morphism in $\mathrm{Hom}_{\mathbf{H^3_{Ann}}}(d\mathcal{A}, d\mathcal{A}')$, then there is a function $g = (\tau, \nu)$, $\tau, \nu : (\pi_0\mathcal{A})^2 \to \pi_1\mathcal{A}'$ satisfying relation (5), and therefore

$$K = (p, q, g) : (\pi_0\mathcal{A}, \pi_1\mathcal{A}, h_\mathcal{A}) \to (\pi_0\mathcal{A}', \pi_1\mathcal{A}', h_{\mathcal{A}'})$$

is an Ann-functor. Thus, the composition $F = H'KG : \mathcal{A} \to \mathcal{A}'$ is an Ann-functor and $dF = (p, q)$. This shows that d is full.

In order to obtain the bijection (12), we prove that the correspondence

$$\begin{aligned} \Omega : \mathrm{Hom}^{\mathrm{Ann}}_{(p,q)}[\mathcal{A}, \mathcal{A}'] &\to \mathrm{Hom}^{\mathrm{Ann}}_{(p,q)}[S_\mathcal{A}, S_{\mathcal{A}'}] \\ [F] &\mapsto [S_F] \end{aligned} \tag{13}$$

is a bijection.

Clearly, if $F, F' : \mathcal{A} \to \mathcal{A}'$ are homotopic, then induced Ann-functors $S_F, S_{F'}$ are homotopic. Conversely, if S_F and $S_{F'}$ are homotopic, then the compositions $E = H'(S_F)G$ and $E' = H'(S_{F'})G$ are homotopic. Ann-functors E and E' are homotopic to F and F', respectively. So, F and F' are homotopic. This shows that Ω is an injection.

Now, if $K = (p, q, g) : S_\mathcal{A} \to S_{\mathcal{A}'}$ is an Ann-functor, then the composition

$$F = H'KG : \mathcal{A} \to \mathcal{A}'$$

is an Ann-functor with $S_F = K$, that is, Ω is surjective. Now, the bijection (12) is the composition of (13) and (6). $\square$

Based on Theorem 3, Ann-categories having the same first two invariants can be classified up to equivalence.

Let R be a ring with a unit, M an R-bimodule which is regarded as a ring with null-multiplication. We say that the Ann-category $\mathcal{A}$ has a *pre-stick of type* (R, M) if there is a pair of ring isomorphisms $\epsilon = (p, q)$

$$p : R \to \pi_0\mathcal{A}, \quad q : M \to \pi_1\mathcal{A}$$

which are compatible with the module action,

$$q(su) = p(s)q(u),$$

where $s \in R, u \in M$. The pair (p, q) is called a *pre-stick of type* (R, M) to the Ann-category $\mathcal{A}$.

A *morphism* between two Ann-categories $\mathcal{A}, \mathcal{A}'$ having pre-sticks of type (R, M) (with their pre-sticks are $\epsilon = (p, q)$ and $\epsilon' = (p', q')$, respectively) is an Ann-functor $(F, \breve{F}, \widetilde{F}) : \mathcal{A} \to \mathcal{A}'$ such that the following diagrams commute

$$\begin{tikzcd} \pi_0\mathcal{A} \arrow[rr, "F_0"] & & \pi_0\mathcal{A}' \\ & R, \arrow[ul, "p"] \arrow[ur, "p'"'] & \end{tikzcd} \qquad \begin{tikzcd} \pi_1\mathcal{A} \arrow[rr, "F_1"] & & \pi_1\mathcal{A}' \\ & M, \arrow[ul, "q"] \arrow[ur, "q'"'] & \end{tikzcd}$$

where (F_0, F_1) is a pair of homomorphisms induced by $(F, \breve{F}, \widetilde{F})$.

Clearly, it follows from the definition of an Ann-functor that F_0, F_1 are isomorphisms, therefore F is an equivalence.

Denote by

$$\mathbf{Ann}[R, M]$$

the set of equivalence classes of Ann-categories whose pre-sticks are of type (R, M). One can prove the following result based on Theorem 3.

Theorem 4. *There is a bijection*

$$\begin{aligned} \Gamma : \mathbf{Ann}[R, M] &\to H^3_{MacL}(R, M) \\ [\mathcal{A}] &\mapsto q_*^{-1} p^*[h_{\mathcal{A}}] \end{aligned}$$

Proof. By Theorem 3, each Ann-category $\mathcal{A}$ determines a unique element $[h_{\mathcal{A}}] \in H^3_{MacL}(\pi_0\mathcal{A}, \pi_1\mathcal{A})$, and hence an element

$$\epsilon[h_{\mathcal{A}}] = q_*^{-1} p^*[h_{\mathcal{A}}] \in H^3_{MacL}(R, M).$$

Now if $F : \mathcal{A} \to \mathcal{A}'$ is a functor between Ann-categories whose pre-sticks are of type (p, q), then the induced Ann-functor $S_F = (p, q, g_F)$ satisfies the relation (5), and therefore

$$p^*[h_{\mathcal{A}'}] = q_*[h_{\mathcal{A}}].$$

One can check that

$$\epsilon'[h_{\mathcal{A}'}] = \epsilon[h_{\mathcal{A}}].$$

This means Γ is a map. Moreover, it is an injection. Indeed, if $\Gamma[\mathcal{A}] = \Gamma[\mathcal{A}']$, then

$$\epsilon(h_{\mathcal{A}}) - \epsilon'(h_{\mathcal{A}'}) = \partial g.$$

Thus, there exists an Ann-functor J of type (id, id) from $\mathcal{I} = (R, M, \epsilon(h_{\mathcal{A}}))$ to $\mathcal{I}' = (R, M, \epsilon'(h_{\mathcal{A}'}))$. The composition

$$\mathcal{A} \xrightarrow{G} S_{\mathcal{A}} \xrightarrow{\epsilon^{-1}} \mathcal{I} \xrightarrow{J} \mathcal{I}' \xrightarrow{\epsilon'} S_{\mathcal{A}'} \xrightarrow{H'} \mathcal{A}'$$

shows that $[\mathcal{A}] = [\mathcal{A}']$, and Γ is an injection. Obviously, Γ is surjective. $\square$

In [9], the author proved that each structure of a regular Ann-category of type (R, M) (that is, a structure satisfies the *regular* condition, $\eta(x, x) = 0$) is an element in the group $Z^3_{Sh}(R, M)$ of Shukla 3-cocycles. From Classification Theorem 4.4 [9] and Theorem 3, the following result is obtained.

Corollary 1. *There is an injection*

$$H^3_{Sh}(R, M) \hookrightarrow H^3_{MacL}(R, M).$$

Appendix: A categorical ring which is not an Ann-category

Below, we construct a categorical ring which is not an Ann-category.

Let R be a ring with a unit and A an R-bimodule. Then, one constructs a categorical ring $\mathcal{R}$ as follows. First, $\mathcal{R}$ is a category defined as in Section 2. The objects of $\mathcal{R}$ are elements of R, the morphisms in $\mathcal{R}$ are automorphisms $(r, a) : r \to r$, $r \in R$, $a \in A$. Composition is the addition on A. Operations $\oplus, \otimes$ on $\mathcal{R}$ are given by

$$\begin{aligned} &r \oplus s = r + s, \quad (r, a) \oplus (s, b) = (r + s, a + b), \\ &r \otimes s = rs, \quad (r, a) \otimes (s, b) = (rs, rb + as). \end{aligned}$$

Suppose that the system $(\mathcal{R}, \oplus, \otimes)$ has a left distributivity constraint

$$\lambda_{r,s,t} : r(s + t) \to rs + rt$$

given by $\lambda_{r,s,t} = (\bullet, \lambda(r, s, t))$, where $\lambda : R^3 \to A$, and other constraints are strict. Then, the commutative diagrams in the axioms of a categorical ring are equivalent to the equations

R_1. $r\lambda(s,t,u) - \lambda(rs,t,u) + \lambda(r,st,su) = 0$,
R_2. $\lambda(r,s,t)u - \lambda(r,su,tu) = 0$,
R_3. $\lambda(1,s,t) = 0$,
R_4. $\lambda(r,s+t,u+v)+\lambda(r,s,t)+\lambda(r,u,v) = \lambda(r,s+u,t+v)+\lambda(r,s,u)+\lambda(r,t,v)$,
R_5. $\lambda(r+r',s,t) = \lambda(r,s,t) + \lambda(r',s,t)$.

Let R be the ring of dual numbers on $\mathbb{Z}$, $R = \{a + b\epsilon \mid a,b \in \mathbb{Z}, \epsilon^2 = 0\}$ and $A = \mathbb{Z} \cong R/(\epsilon)$. Then, A is an R-bimodule with the natural actions

$$(a + b\epsilon)k = ak = k(a + b\epsilon).$$

The function $\lambda : R^3 \to A$, defined by

$$\lambda(a_r + b_r\epsilon, a_s + b_s\epsilon, a_t + b_t\epsilon) = b_r(a_s + a_t),$$

satisfies the equations $R_1 - R_5$, so that $\mathcal{R}$ is a categorical ring.

It is clear that if $b_r \neq 0$ and $a_s \neq 0$, then $\lambda(r,0,s) \neq 0$. Thus, by Theorem 1, $\mathcal{R}$ is not an Ann-category.

One can deduce that:

1. Since the function λ is not normalized, $\widehat{h} = (0,\lambda,0,0) \notin Z^3_{MacL}(R,A)$. This means that the classification theorem in [2] is wrong.

2. The condition (U) in the following theorem is necessary.

Theorem 5 (see [10]). *Each categorical ring $\mathcal{R}$ is an Ann-category if and only if it satisfies the following condition;*

(U): *Each of pairs $(L^A, \hat{L}^A)$, $(R^A, \hat{R}^A)$, $A \in \mathcal{R}$, is an $\oplus$-functor which is compatible with the unit constraint $(0, \mathbf{g}, \mathbf{d})$ with respect to the operation $\oplus$.*

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References

[1] A. Fröhlich, C. T. C Wall, *Graded monoidal categories*, Compositio Math. **28**(1974), 229–285.
[2] M. Jibladze, T. Pirashvili, *Third Mac Lane cohomology via categorical rings*, J. Homotopy Relat. Struct. **2**(2007), 187–216.
[3] M. M. Kapranov, V. A. Voevodsky, *2-Categories and Zamolodchikov Tetrahedra Equations*, Proc. Sympos. Pure Math. **56**(1994), 177–259.
[4] M. L. Laplaza, *Coherence for distributivity,* Lecture Notes Math. **281**(1972), 29–65.
[5] M. L. Laplaza, *Coherence for Categories with Group Structure: an alternative approach,* J. Algebra **84**(1983), 305–323.

[6] S. Mac Lane, *Extensions and obstruction for rings,* Illinois J. Math. **2**(1958), 316–345.

[7] C. T. K. Phung, N. T. Quang, N. T. Thuy, *Relation between Ann-categories and ring categories*, Comm. Korean Math. Soc. **25**(2010), 523–535.

[8] N. T. Quang, *Introduction to Ann-categories,* J. Math. Hanoi **4**(1987), 14–24.

[9] N. T. Quang, *Structure of Ann-categories and Mac Lane-Shukla cohomology,* East-West J. Math. **5**(2003), 51–66.

[10] N. T. Quang, D. D. Hanh, N. T. Thuy, *On the Axiomatics of Ann-categories,* JP J. Algebra Number Theory Appl. **11**(2008), 59–72.

[11] N. T. Quang, D. D. Hanh, *Homological classification of Ann-functors*, East-West J. Math. **11**(2009), 195–210.

[12] N. Saavedra Rivano, *Catégories Tannakiennes,* Lecture notes in mathematics, Springer-Verlag, Berlin, 1972.

[13] U. Shukla, *Cohomologie des algebras associatives,* Ann. Sci. Ecole Norm. Sup. **78**(1961), 163–209.

[14] H. X. Sinh, *Gr-catégories*, Thése de doctorat, Université Paris VII, 1975.