Math. Commun. **18**(2013), 209–217

Tubular surfaces in Galilean space

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Received September 12, 2011; accepted January 21, 2013

Abstract. In this paper, firstly, the definition of tubular surfaces in Galilean space is given. Then, differential properties of tubular surfaces are obtained. Consequently, we proved that tubular surfaces in Galilean space are Weingarten surfaces.

AMS subject classifications: 53A35,53Z05

Key words: differential geometry, Galilean space, tubular surfaces

1. Introduction

Tubular surfaces are among the surfaces which are easier to describe both analytically and kinematically. They are still under active investigation [3-5,8,10,13]. A large number of papers have been published in the literature which deal with tubular surfaces in both Minkowski space and Euclidean space. The purpose of this paper is to introduce, analyze and compare tubular surfaces between Galilean space and Euclidean space.

The geometry of Galilean Relativity acts like a "bridge" from Euclidean geometry to Special Relativity. Galilean space is the space of Galilean Relativity. More about Galilean space and pseudo-Galilean space may be found in [1-2,6-7,9,11-12].

The Galilean space $\mathbb{G}_3$ is a Cayley–Klein space equipped with the projective metric of signature $(0, 0, +, +)$, as in [12]. The absolute figure of the Galilean geometry consists of an ordered triple $\{\omega, f, I\}$, where ω is the real (absolute) plane, f the real line (absolute line) in ω and I the fixed elliptic involution of points of f. We introduce homogeneous coordinates in $\mathbb{G}_3$ in such a way that the absolute plane ω is given by $x_0 = 0$, the absolute line f by $x_0 = x_1 = 0$ and the elliptic involution by

$$(0 : 0 : x_2 : x_3) \to (0 : 0 : x_3 : -x_2). \tag{1}$$

A plane is called Euclidean if it contains f, otherwise it is called isotropic or i.e., planes $x = const.$ are Euclidean, and so is the plane ω. Other planes are isotropic. A vector $\mathbf{u} = (u_1, u_2, u_3)$ is said to be non-isotropic if $u_1 \neq 0$. All unit non-isotropic vectors are of the form $\mathbf{u} = (1, u_2, u_3)$. For isotropic vectors $u_1 = 0$ holds.

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Definition 1. *Let* $\mathbf{a} = (x, y, z)$ *and* $\mathbf{b} = (x_1, y_1, z_1)$ *be vectors in the Galilean space. The scalar product is defined by*

$$< \mathbf{a}, \mathbf{b} >= x_1 x. \tag{2}$$

The norm of a defined by $\|\mathbf{a}\| = |x|$, *and* $\mathbf{a}$ *is called a unit vector if* $\|\mathbf{a}\| = 1$.

The scalar product of two isotropic vectors $\mathbf{p} = (0, y, z)$ *and* $\mathbf{q} = (0, y_1, z_1)$ *in Galilean space is defined by*

$$< \mathbf{p}, \mathbf{q} >_1 = yy_1 + zz_1. \tag{3}$$

The orthogonality of these vectors, $\mathbf{p} \perp_1 \mathbf{q}$, *means that* $< \mathbf{p}, \mathbf{q} >_1 = 0$. *The norm of* $\mathbf{p}$ *defined by* $\|\mathbf{p}\|_1 = \sqrt{y^2 + z^2}$, *and* $\mathbf{p}$ *is called a unit isotropic vector if* $\|\mathbf{p}\|_1 = 1$ *[7].*

Definition 2. *If* $\mathbf{u} = (u_1, u_2, u_3)$ *and* $\mathbf{v} = (v_1, v_2, v_3)$ *are vectors in the Galilean space, we define the vector product of u and v as the following:*

$$\mathbf{u} \wedge \mathbf{v} = \begin{vmatrix} 0 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (0, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1). \tag{4}$$

Definition 3. *If an admissible curve c of the class* $C^r (r \geq 3)$ *is given by the parametrization*

$$r(u) = (u, y(u), z(u)),$$

then u is a Galilean invariant of the arc length on C.

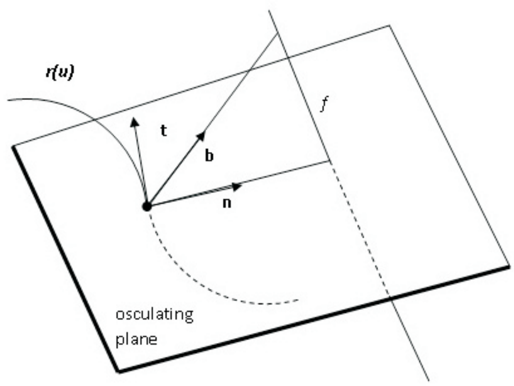

Figure 1:

In Figure 1, the associated invariant moving trihedron is given by

$$\begin{aligned}\mathbf{t} &= (1, y'(u), z'(u)),\\ \mathbf{n} &= \frac{1}{\kappa}(0, y''(u), z''(u)),\\ \mathbf{b} &= \frac{1}{\kappa}(0, -z''(u), y''(u))\end{aligned} \tag{5}$$

where $\kappa = \sqrt{y''(u)^2 + z''(u)^2}$ is the curvature and $\tau = \dfrac{1}{\kappa^2}\det[r'(u), r''(u), r'''(u)]$ is the torsion.

Frenet formulas may be written as

$$\frac{d}{du}\begin{bmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{bmatrix} = \begin{bmatrix}0 & \kappa & 0\\ 0 & 0 & \tau\\ 0 & -\tau & 0\end{bmatrix}\begin{bmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{bmatrix}. \tag{6}$$

Definition 4. *The equation of a surface in $\mathbb{G}_3$ is given by the parametrization*

$$\varphi = \varphi(v^1, v^2) = (x(v^1, v^2), y(v^1, v^2), z(v^1, v^2)), \quad v^1, v^2 \in \mathbb{R},$$

where $x(v^1, v^2), y(v^1, v^2), z(v^1, v^2) \in C^3$ [9].

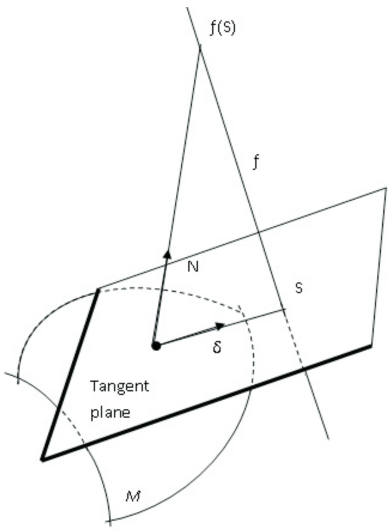

Figure 2:

In Figure 2, the isotropic unit normal vector N of the surface is defined by

$$N = \frac{\varphi_{,1} \wedge \varphi_{,2}}{\|\varphi_{,1} \wedge \varphi_{,2}\|_1}, \tag{7}$$

where $\varphi_{,1} = \dfrac{\partial \varphi(v^1, v^2)}{\partial v^1}$ and $\varphi_{,2} = \dfrac{\partial \varphi(v^1, v^2)}{\partial v^2}$.

Using (1) and $w = \|\varphi_{,1} \wedge \varphi_{,2}\|_1$, we obtain the isotropic unit vector δ in the tangent plane of the surface as

$$\delta = \frac{(0, y_{,1}x_{,2} - y_{,2}x_{,1}, z_{,1}x_{,2} - z_{,2}x_{,1})}{w}, \tag{8}$$

where

$$< N, \delta >_1 = 0, \quad \delta^2 = 1$$

by means of Galilean geometry. Observe that a straightforward computation shows that δ can be expressed by

$$\delta = \frac{x_{,2}\varphi_{,1} - x_1\varphi_{,2}}{w}, \tag{9}$$

where $x_{,1} = \dfrac{\partial x(v^1, v^2)}{\partial v^1}$ and $x_{,2} = \dfrac{\partial x(v^1, v^2)}{\partial v^2}$.

Consequently, to simplify the presentation (9), we may use Einstein summation convention. Then, δ is

$$\delta = g^i \varphi_{,i} = g^1 \varphi_{,1} + g^2 \varphi_{,2},$$

where

$$g_1 = x_{,1}, \quad g_2 = x_{,2} \quad g_{ij} = g_i g_j \tag{10}$$

and

$$g^1 = \frac{x_{,2}}{w}, \quad g^2 = -\frac{x_{,1}}{w} \quad g^{ij} = g^i g^j. \tag{11}$$

The first fundamental form of the surface is defined by

$$\begin{aligned} I &= (ds)^2 \\ &= (g_i dv^i)^2 + \epsilon h_{ij} dv^i dv^j, \end{aligned} \tag{12}$$

where

$$h_{ij} =< \varphi_{,i}, \varphi_{,j} >_1 \tag{13}$$

and ϵ is

$$\epsilon = \begin{cases} 0, \ dv^1 : dv^2 \text{ non-isotropic} \\ \\ 1, \ dv^1 : dv^2 \text{ isotropic} \end{cases}.$$

The coefficients L_{ij} of the second fundamental form are given by

$$L_{ij} =< \frac{\varphi_{,ij}x_{,1} - x_{,ij}\varphi_{,1}}{x_{,1}}, N >_1 . \tag{14}$$

Corollary 1. *Let M be a surface in Galilean space. The Gauss curvature K and mean curvature H of the surface are defined as*

$$K = \frac{\det L_{ij}}{w^2} \tag{15}$$

and

$$2H = g^{ij} L_{ij}. \tag{16}$$

2. Tubular surfaces in Galilean space

The aim of this paper is to introduce a simple method for parametrization of tubular surfaces in Galilean space. Let us denote by ρ the vector connecting the point from the arc-length parametrized curve $r(u)$ with the point from the surface. Then, clearly, we have the position vector R of a point on the surface in the following form

$$R = r(u) + \rho. \tag{17}$$

On the other hand, since ρ lies in the Euclidean normal plane of the curve $r(u)$ shown in Figure 3, the points at a distance $a = \mathit{const}$ from a point of $r(u)$ form a Euclidean circle in Galilean space.

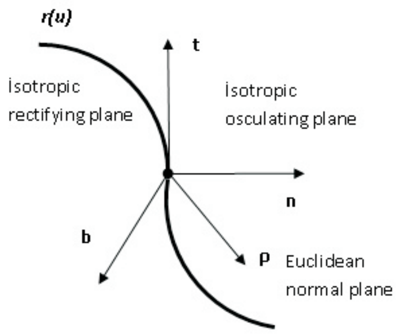

Figure 3:

We may define the Euclidean angle v between the isotropic vectors $\mathbf{n}$ and ρ. Then, as one can see immediately, we have

$$\rho = a(\cos v\mathbf{n} + \sin v\mathbf{b}). \tag{18}$$

Combining equations (17) and (18), the tubular surface at a distance a from $r(u)$ is described by means of parametrization as

$$X^a(u,v) = r(u) + a(\cos v\mathbf{n} + \sin v\mathbf{b}). \tag{19}$$

Using equations (19) and (5) implies that

$$X^a(u,v)=(u,y(u),z(u))+\frac{a}{\kappa}[\cos v(0,y''(u),z''(u))+\sin v(0,-z''(u),y''(u)]. \tag{20}$$

We denote partial derivatives of $X^a(u,v)$ with respect to u and v by $X^a_u(u,v)$ and $X^a_v(u,v)$. Then, from equations (19) and (6), we have

$$X^a_u(u,v) = \mathbf{t} + a\tau(\cos v\mathbf{b} - \sin v\mathbf{n}) \tag{21}$$

and

$$X^a_v(u,v) = a(\cos v\mathbf{b} - \sin v\mathbf{n}) \tag{22}$$

so that the vector cross product of these two vectors is given by

$$X_u^a(u,v) \wedge X_v^a(u,v) = -a(\sin v\mathbf{b} + \cos v\mathbf{n}). \tag{23}$$

Hence for small $a > 0$, we have

$$\|X_u^a(u,v) \wedge X_v^a(u,v)\|_1 = a. \tag{24}$$

Using equations (23) and (24), we obtain the isotropic normal vector of tubular surfaces as

$$N = -\cos v\mathbf{n} - \sin v\mathbf{b}. \tag{25}$$

On the other hand, from equations (25) and (8), it is easy to see that

$$\delta = -\sin v\mathbf{n} + \cos v\mathbf{b}. \tag{26}$$

By means of equations (10), (20) and (22), we obtain

$$g_1 = 1, \;\; g_2 = 0\,. \tag{27}$$

Using the projection of $X_u^a(u,v)$ and $X_v^a(u,v)$ vectors onto the Euclidean yz-plane, we have

$$h_{22} = a^2 \tag{28}$$

Substituting equations (27) and (28) into equation (12) consequently, we obtain the first fundamental form of tubular surfaces in Galilean space as

$$I = du^2 + \epsilon a^2 dv^2, \tag{29}$$

where ϵ is

$$\epsilon = \begin{cases} 0, \; du \neq 0 \\ 1, \; du = 0 \end{cases}. \tag{30}$$

From equations (21) and (22), we have

$$\begin{aligned} X_{uu}^a(u,v) &= (\kappa - a\tau' \sin v - a\tau^2 \cos v)\mathbf{n} + (a\tau' \cos v - a\tau^2 \sin v)\mathbf{b} \\ X_{uv}^a(u,v) &= -a\tau(\sin v\mathbf{b} + \cos v\mathbf{n}) \\ X_{vv}^a(u,v) &= -a(\cos v\mathbf{n} + \sin v\mathbf{b}). \end{aligned} \tag{31}$$

Equations (31) and (25) lead to the coefficients of the second fundamental form obtained by,

$$\begin{aligned} L_{11} &= -\kappa \cos v + a\tau^2 \\ L_{12} &= a\tau \\ L_{22} &= a, \end{aligned} \tag{32}$$

respectively.

Substituting equations (32) into equation (15) implies that

$$K = \frac{-\kappa \cos v}{a}. \tag{33}$$

From equations (11) and (27), we get

$$g^{11} = 0, g^{12} = 0, g^{22} = \frac{1}{a^2}. \tag{34}$$

Then, substituting equations (32) and (34) into equation (16), finally, we obtain the mean curvature of tubular surface as

$$2H = \frac{1}{a}. \tag{35}$$

Corollary 2. *Tubular surfaces are constant mean curvature surfaces in Galilean space.*

Consequently, we have the following theorem:

Theorem 1. *Tubular surfaces are Weingarten surfaces in Galilean space.*

Proof. Differentiating K and H with respect to u and v gives

$$K_v = \frac{\kappa \sin v}{a}, K_u = \frac{-\kappa' \cos v}{a} \tag{36}$$

and

$$H_v = H_u = 0. \tag{37}$$

By using (36) and (37), $X^a(u,v)$ satisfies identically the Jacobi equation

$$\Phi(H,K) = K_v H_u - H_v K_u = 0.$$

Therefore, $X^a(u,v)$ is a Weingarten surface.

$$H_v K_u - H_u K_v = 0.$$

□

Example 1. *Let α be a parametrized by*

$$r(u) = (u, \cos u, \sin u)$$

It is easy to see that the Frenet frame is

$$\begin{aligned} \mathbf{t} &= (1, -\sin u, \cos u), \\ \mathbf{n} &= (0, -\cos u, -\sin u), \\ \mathbf{b} &= (0, \sin u, -\cos u), \end{aligned}$$

where $\kappa = 1$ is the curvature and $\tau = 1$ is the torsion of the curve.

Hence for $a = 1$, we have a tubular surface shown in Figure 4, parametrized by

$$X(u,v) = (u, \cos u - \cos v \cos u + \sin v \sin u, \sin u - \cos v \sin u - \sin v \cos u).$$

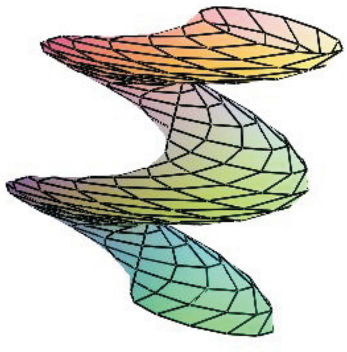

Figure 4:

Acknowledgement

The author would like to thank the referees for their helpful suggestions.

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