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The Hausdorff distance between some sets of points

Tomislav Marošević*

Department of Mathematics, University of Osijek, Trg Ljudevita Gaja 6, HR-31 000 Osijek, Croatia

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Abstract. The Hausdorff distance can be used in various areas, where the problems of shape matching and comparison appear. We look at the Hausdorff distance between two hyperspheres in $\mathbb{R}^n$. With respect to different geometric objects, the Hausdorff distance between a segment and a hypersphere in $\mathbb{R}^n$ is given, too. Using the Mahalanobis distance, a modified Hausdorff distance between a segment and an ellipse in the plane, and generally between a segment and a hyper-ellipsoid in $\mathbb{R}^n$ is adopted. Finally, the modified Hausdorff distance between ellipses is obtained.

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Key words: Hausdorff distance, segment, circle, ellipse

1. Introduction

The Hausdorff distance (i.e., Pompeiu-Hausdorff distance, see [4, 6]) between some sets of points can be used in problems referring to image comparison, contour fitting, pattern recognition, computer vision and many various fields, where the problems of shape matching and comparison appear ([1, 3, 5]). We look at some simple curves in the plane, such as segments, circles, ellipses, and the question on corresponding Hausdorff distances between them. Some of these cases are generalized in the space $\mathbb{R}^n$, $n \geq 3$. By means of Mahalanobis distances, we consider the modified Hausdorff distance between ellipses in the plane.

The Hausdorff distance between two (closed and bounded) sets of points S_1, $S_2 \subset \mathbb{R}^2$ is defined by

$$d_H(S_1, S_2) = \max\{\max_{T\in S_1} \min_{P\in S_2} d(T,P), \max_{P\in S_2} \min_{T\in S_1} d(P,T)\}, \tag{1}$$

where d denotes the Euclidean distance (d could be any other metric, too) and the distance from a point T to a set S is defined in an obvious way: $d(T,S) = \min\limits_{P\in S} d(T,P)$.

For example, looking at two polygons in the plane ([1, 2]), and generally in $\mathbb{R}^k$, $k \geq 3$,

$\mathcal{P}_1 = \overline{T_1T_2} \cup \overline{T_2T_3} \cup \cdots \overline{T_{n-1}T_n} \cup \overline{T_nT_1}$,
$\mathcal{P}_2 = \overline{P_1P_2} \cup \overline{P_2P_3} \cup \cdots \cup \overline{P_{m-1}P_m} \cup \overline{P_mP_1}$,

*Corresponding author. *Email address:* `tmarosev@mathos.hr` (T. Marošević)

(i.e., a polygon as a union of its sides), it is not difficult to see that the following assertion holds.

The Hausdorff distance between two polygons $\mathcal{P}_1$, $\mathcal{P}_2$ in $\mathbb{R}^k$, $k \geq 2$ is given by

$$d_H(\mathcal{P}_1, \mathcal{P}_2) = \max\{d(T_1, \mathcal{P}_2), \dots, d(T_n, \mathcal{P}_2), d(P_1, \mathcal{P}_1), \dots, d(P_m, \mathcal{P}_1)\},$$

where

$$\begin{aligned} d(T_i, \mathcal{P}_2) &= \min\{d(T_i, \overline{P_1P_2}), \dots, d(T_i, \overline{P_{m-1}P_m}), d(T_i, \overline{P_mP_1})\}, \quad i = 1, \dots, n, \\ d(P_j, \mathcal{P}_1) &= \min\{d(P_j, \overline{T_1T_2}), \dots, d(P_j, \overline{T_{n-1}P_n}), d(P_j, \overline{T_nT_1})\}, \quad j = 1, \dots, m, \end{aligned}$$

and $d(T_i, \overline{P_1P_2})$, ...,$d(P_j, \overline{T_nT_1})$ denote the corresponding distances from a point to a segment.

In Section 2, we give a formula for the Hausdorff distance between two circles in the plane and between two hyperspheres in $\mathbb{R}^n$, $n \geq 3$. In Section 3, we deal with the Hausdorff distance between a segment and a circle in the plane, and generally between a segment and a hypersphere in $\mathbb{R}^n$, $n \geq 3$. In Section 4, we adopt the modified Hausdorff distance between a segment and an ellipse in the plane, and generally between a segment and hyper-ellipsoid in $\mathbb{R}^n$, $n \geq 3$, by means of the Mahalanobis distance. In Section 5, the modified Hausdorff distance between two ellipses is obtained as result of a few calculations. In Section 6, we give concluding remarks about Hausdorff distances between the geometric objects taken into account in the paper.

2. The Hausdorff distance between two hyperspheres

Comparing circles by using the Hausdorff distance appears e.g. in multiple circle detection problems ([10]). Let two circles $k_1 = (C_1(p_1, q_1); r_1)$ and $k_2 = (C_2(p_2, q_2); r_2)$ be given in the plane (with center C_i, radius r_i, $i = 1, 2$).

It is well known that the distance from a point T to a circle k_i is given by

$$d(T, k_i) = \min_{P \in k_i} d(T, P) = |d(T, C_i) - r_i|, \qquad i = 1, 2.$$

In accordance with (1), the Hausdorff distance between two circles k_1 and k_2 is defined by

$$d_H(k_1, k_2) = \max\{\max_{T \in k_1} |d(T, C_2) - r_2|, \max_{P \in k_2} |d(P, C_1) - r_1|\}. \tag{2}$$

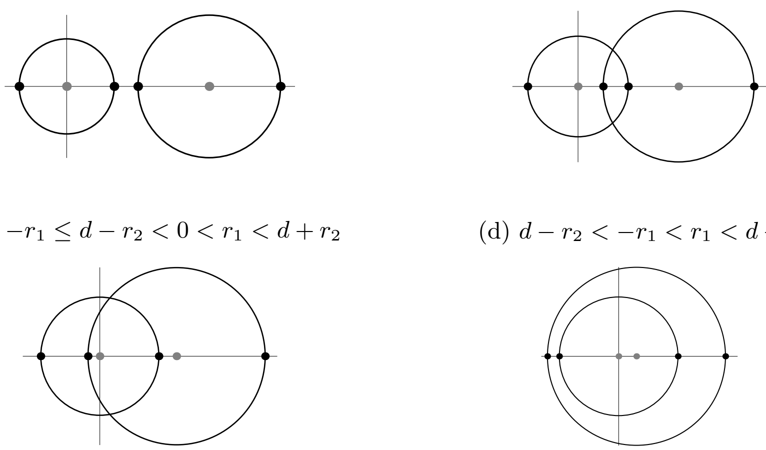
In [12], it is shown that the following formula holds.

Proposition 1. *For the Hausdorff distance between two circles k_1 and k_2 in the plane there holds*

$$d_H(k_1, k_2) = d(C_1, C_2) + |r_2 - r_1| = \sqrt{(p_2 - p_1)^2 + (q_2 - q_1)^2} + |r_2 - r_1|. \tag{3}$$

The proof can be seen in [12]. Formula (3) is proved by means of four possible locations of two circles (Figure 1). In each of these cases one can show that this formula holds.

Formula (3) can be generalized for hyperspheres in $\mathbb{R}^n$, $n \geq 3$.



(a) $0 < r_1 \leq d - r_2 < d + r_2$ (b) $0 \leq d - r_2 < r_1 < d + r_2$

(c) $-r_1 \leq d - r_2 < 0 < r_1 < d + r_2$ (d) $d - r_2 < -r_1 < r_1 < d + r_2$

Figure 1: *Possible positions of circles k_1, k_2*

Proposition 2. *Let two hyperspheres $S_i^{n-1} = \{T \in \mathbb{R}^n \ : \ d(C_i, T) = r_i\} \subset \mathbb{R}^n$, $n \geq 3$, $i = 1, 2$, be given. The Hausdorff distance between these two hyperspheres has the same form as* (3)*:*

$$d_H(S_1^{n-1}, S_2^{n-1}) = d(C_1, C_2) + |r_2 - r_1|\,.$$

Proof. Let us look at the space $\mathbb{R}^3$. For any point T of the sphere k_1 one can look at the plane through T and centers C_1, C_2. This plane intersects two spheres in corresponding two (main) circles. In this way, the problem of the Hausdorff distance is reduced to the case in the plane, which is proved by Proposition 1. Further, one shows by induction that the formula holds in $\mathbb{R}^n$, too. □

3. The Hausdorff distance between a segment and a hypersphere

Let two different types of objects in the plane be given: a segment $l = \overline{T_1T_2}$ and a circle k with the center $C = (x_C, y_C)$ and the radius $r > 0$.

In accordance with (1), the Hausdorff distance between the segment l and the circle k is defined by

$$d_H(l, k) = \max\{\max_{T \in l} d(T, k), \max_{P \in k} d(P, l)\,\}. \tag{4}$$

Firstly, this gives rise to the following problem:

$$\max_{T \in l} |d(T, C) - r| = ?,$$

since $d(T, k) = |d(T, C) - r|$. It is not difficult to see that the maximum is attained at some of the following points of the segment l: at the endpoints T_1 or T_2, or at the orthogonal projection C' of the center C onto the line T_1T_2 provided that C' belongs to the segment (Figure 2).

The distance from the point C to C' has the form

$$d(C,C') = \|\overrightarrow{CC'}\| = \|\overrightarrow{CT_1} + \lambda_C \overrightarrow{T_1T_2}\|,$$

where

$$\lambda_C = \frac{\|\overrightarrow{T_1C}\| \cdot \cos\angle(\overrightarrow{T_1C}, \overrightarrow{T_1T_2})}{\|\overrightarrow{T_1T_2}\|} = \frac{\overrightarrow{T_1C} \cdot \overrightarrow{T_1T_2}}{\|\overrightarrow{T_1T_2}\|^2}.$$

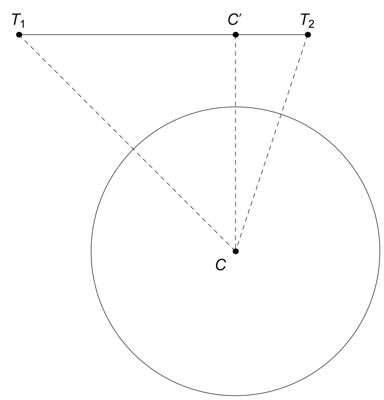


Figure 2: *Distances from points on the segment l to the circle k*

Therefore,

$$\max_{T\in l} |d(T,C) - r| = \max\{|d(T_1,C) - r|, |d(T_2,C) - r|, \left\{\begin{array}{cl} |d(C,C') - r|, & \text{if } \lambda_C \in [0,1] \\ 0, & \text{if } \lambda_C \notin [0,1] \end{array}\right\}\}. \quad (5)$$

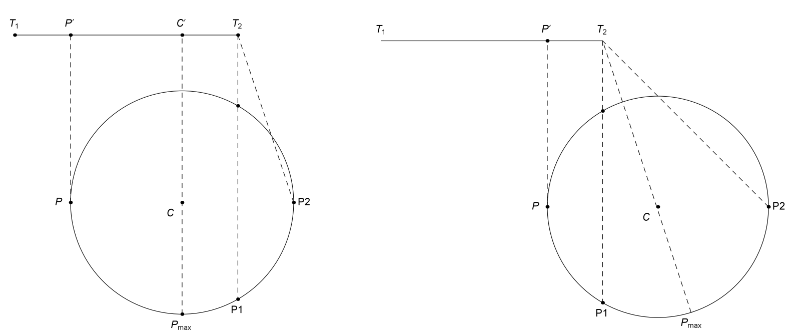


Figure 3: *Distances from points of the circle k to the segment l*

Secondly, with regard to (4), one has to find a maximum distance from points $P \in k$ to the segment l (Figure 3):

$$\max_{P\in k} d(P,l) \ = ?$$

Taking into account the formula for the distance from the point C to the segment l,

$$d(C,l) = \min_{T\in l} d(C,T) = \begin{cases} d(C,C'), & \text{if} \ \ \lambda_C \in [0,1] \\ \min\{d(C,T_1), d(C,T_2)\}, & \text{if} \ \ \lambda_C \notin [0,1] \end{cases}, \tag{6}$$

it is not difficult to see that the following holds:

a) if $C' \in l$ (i.e., if $\lambda_C \in [0,1]$), then $\max\limits_{P\in k} d(P,l) = d(C,C') + r$, where C' is an orthogonal projection of the center C onto the segment l;

b) if $C' \notin l$ (i.e., if $\lambda_C \notin [0,1]$), then $\max\limits_{P\in k} d(P,l) = \min\{d(C,T_1), d(C,T_2)\} + r$.

Therefore, it follows that

$$\max_{P\in k} d(P,l) = \begin{cases} d(C,C') + r, & \text{if } \lambda_C \in [0,1] \\ \min\{d(C,T_1), d(C,T_2)\} + r, & \text{if } \lambda_C \notin [0,1] \end{cases}. \tag{7}$$

So, the following proposition follows from (4), (5) and (7).

Proposition 3. *The Hausdorff distance between the segment l and the circle k is determined by the following expression:*

$$\begin{aligned} d_H(l,k) = \max\{&|d(T_1,C) - r|, |d(T_2,C) - r|, \\ &\left\{\begin{matrix} d(C,C') + r, & \lambda_C \in [0,1] \\ \min\{d(C,T_1), d(C,T_2)\} + r, & \lambda_C \notin [0,1] \end{matrix}\right\}\}. \end{aligned} \tag{8}$$

Furthermore, formula (8) also holds in the case of the space $\mathbb{R}^n$, $n \geq 3$.

Proposition 4. *The Hausdorff distance between the segment and the hypersphere in $\mathbb{R}^n$, $n \geq 3$, is determined by the same expression as* (8).

Proof. Since formulae for distances from a point to a segment (6) and from a point to a circle in the plane analogously hold for a segment and a hypersphere in $\mathbb{R}^n$, $n \geq 3$, formula (8) holds in $\mathbb{R}^n$, too. □

4. A modified Hausdorff distance between a segment and an ellipse by using the Mahalanobis distance

Let an ellipse be given by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{9}$$

(without loss of generality, assume that its center $C = O = (0,0)$ and half-axes are parallel to coordinate axes).

Remark 1. *Since the Euclidean distance from a point to an ellipse is more complicated than from a point to a circle, we are going to use here a Mahalanobis distance-like function* $d_M\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}_+$, *(see, e.g., [5, 13])*

$$d_M(u, v; S) = (u - v)^T S^{-1}(u - v), \tag{10}$$

where $S \in \mathbb{R}^{2\times 2}$ *is a positive definite symmetric matrix. Namely, in that case ellipse* (9) *is expressed as a transformed circle, i.e., in the form of a Mahalanobis circle (M-circle)*

$$E(C, 1; S) = \{u \in \mathbb{R}^2\colon d_M(u, C; S) = 1\}, \tag{11}$$

where radius $r = 1$, *center* $C = (0, 0)$, *and matrix* $S = \operatorname{diag}(a^2, b^2)$. *Notice that in this case* $S^{-1} = \operatorname{diag}(1/a^2, 1/b^2)$.

In order to obtain an expression for the Hausdorff distance between a segment and an ellipse by using Mahalanobis distances (M-distances) (10), one should define an M-distance from the point $T \in \mathbb{R}^2$ to ellipse (11). Since the ellipse is an M-circle, the M-distance from a point $T \in \mathbb{R}^2$ to the M-circle $E(C, 1; S)$ is given by the expression (analogously to the Euclidean distance from a point to a circle)

$$\min_{P\in E} \sqrt{d_M(T, P; S)} = |\sqrt{d_M(T, C; S)} - 1|\,. \tag{12}$$

Then one can apply expressions analogous to the ones used in Section 3, by using the M-distance instead of the Euclidean distance.

In accordance with (4), the modified Hausdorff distance between the segment $l \equiv \overline{T_1T_2}$ and the ellipse, i.e., M-circle E is given by

$$D_H(l, E) = \max\{\max_{T\in l} |\sqrt{d_M(T, C; S)} - 1|, \max_{P\in E} \min_{T\in l} \sqrt{d_M(P, T; S)}\,\}. \tag{13}$$

Analogously to the case of a simple circle in Section 3, the maximum of distances $|\sqrt{d_M(T, C; S)} - 1|$ is attained at some of the following points of the segment l: endpoints T_1 or T_2, or at the projection of C' of the center C onto the line T_1T_2 provided that C' is situated on the segment. (It is supposed that $C = O = (0, 0)$.) Therefore, analogously to formula (5) we obtain

$$\begin{aligned}\max_{T\in l} |\sqrt{d_M(T, C; S)} - 1\,| = \max\{&|\sqrt{d_M(T_1, C; S)} - 1|, |\sqrt{d_M(T_2, C; S)} - 1|,\\ &\left\{\begin{array}{rl} |\sqrt{d_M(C, C'; S)} - 1|, & \lambda_C \in [0, 1] \\ 0, & \lambda_C \notin [0, 1] \end{array}\right\}\},\end{aligned} \tag{14}$$

where

$$\begin{aligned} d_M(C, C'; S) &= d_M(T_1 - C + \lambda_C(T_2 - T_1), O; S),\\ \lambda_C &= \frac{-\frac{1}{a^2}x_1(x_2 - x_1) - \frac{1}{b^2}y_1(y_2 - y_1)}{d_M(T_1, T_2; S)} = \frac{(C - T_1)^T \cdot S^{-1} \cdot (T_2 - T_1)}{d_M(T_1, T_2; S)}. \end{aligned}$$

With regard to the second part of (13), denoting the M-distance from a point $P \in E$ to the segment l by $\min_{T\in l} \sqrt{d_M(P, T; S)} = \sqrt{d_M(P, l)}$, one has to find a maximum:

$$\max_{P\in E} \sqrt{d_M(P, l)} = ?$$

Then one obtains (analogously to formula (7))

$$\max_{P\in E}\sqrt{d_M(P,l)}=\begin{cases}\sqrt{d_M(C,C';S)}+1, & \lambda_C\in[0,1]\\ \min\{\sqrt{d_M(C,T_1;S)},\sqrt{d_M(C,T_2;S)}\}+1, & \lambda_C\notin[0,1]\,.\end{cases}\tag{15}$$

So, from (13), (14) and (15) we obtain the following assertion (analogously to (8)).

Proposition 5. *The modified Hausdorff distance between the segment l and the M-circle E has the form:*

$$\begin{aligned}D_H(l,E)=\max\{&|\sqrt{d_M(T_1,C;S)}-1|,|\sqrt{d_M(T_2,C;S)}-1|,\\ &\left.\begin{cases}\sqrt{d_M(C,C';S)}+1, & \lambda_C\in[0,1]\\ \min\{\sqrt{d_M(C,T_1;S)},\sqrt{d_M(C,T_2;S)}\}+1, & \lambda_C\notin[0,1]\end{cases}\right\}\}.\end{aligned}\tag{16}$$

Remark 2. *Furthermore, formula* (16) *can be generalized in $\mathbb{R}^n$, $n\geq 3$. Let a hyper-ellipsoid in $\mathbb{R}^n$ be given by equation*

$$\frac{x_1^2}{a_1^2}+\frac{x_2^2}{a_2^2}+\ldots+\frac{x_n^2}{a_n^2}=1\,.\tag{17}$$

A Mahalanobis distance-like function $d_M\colon\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}_+$ is defined by the expression analogous to (10)*. Analogously, in this case hyper-ellipsoid* (17) *is expressed in the form of a Mahalanobis hypersphere*

$$H(C,1;S)=\{u\in\mathbb{R}^n\colon d_M(u,C;S)=1\},\tag{18}$$

where radius $r=1$, center $C=(0,\ldots,0)$, and matrix $S^{-1}=\operatorname{diag}(1/a_1^2,1/a_2^2,\ldots,1/a_n^2)$.

So, the following proposition is obtained.

Proposition 6. *The modified Hausdorff distance between the segment and the hyper-ellipsoid in $\mathbb{R}^n$, $n\geq 3$, is determined by the expression analogous to* (16).

5. The modified Hausdorff distance between two ellipses by using the Mahalanobis distance

The problem of ellipse comparison appears e.g. in multiple ellipse detection problems ([9, 11]). With regard to Euclidean distances, ellipses are more complicated than circles. Therefore, an appropriate distance measure for comparison of two ellipses can be a modified Hausdorff distance by using the Mahalanobis distance, because an ellipse is an M-circle by means of the M-distance ([5, 7, 8]).

Given two ellipses (Figure 4), without loss of generality, assume that the first ellipse $E1$ has a center in the origin $C_1=(0,0)$ and its half-axes a_1, b_1 are parallel to coordinate axes. Then the ellipse $E1\equiv E(C_1,1;S_1)$ is expressed as a transformed circle, i.e., in the form of a Mahalanobis circle (11), where $S_1=\operatorname{diag}(a_1^2,b_1^2)$.

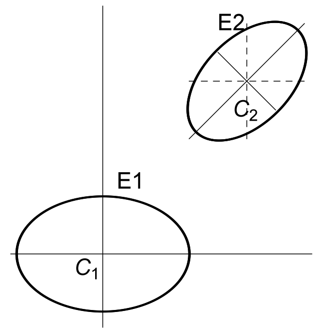


Figure 4: *Ellipses* $E1$ *and* $E2$

Let the second ellipse $E2$ be defined by its five parameters, i.e., the center $C_2 = (p,q)$, lengths of half-axes a_2, b_2, and the angle φ between the half-axis a_2 and the positive direction of the coordinate axis $0x$. So, the equation of the ellipse $E2$ is

$$\frac{[(x-p)\cos\varphi + (y-q)\sin\varphi]^2}{a_2^2} + \frac{[-(x-p)\sin\varphi + (y-q)\cos\varphi]^2}{b_2^2} = 1\,. \tag{19}$$

By means of Mahalanobis distance-like function (10), ellipse (19) is expressed in the form of an M-circle

$$E2 = E(C_2, 1; S_2) = \{u \in \mathbb{R}^2 \colon (u - C_2)^T S_2^{-1}(u - C_2) = 1\}, \tag{20}$$

with the radius equal to 1, where S_2 is a symmetric positive definite matrix

$$S_2 = U \begin{bmatrix} a_2^2 & 0 \\ 0 & b_2^2 \end{bmatrix} U^T \quad \text{and} \quad U = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}. \tag{21}$$

In order to obtain an expression for the modified Hausdorff distance between two ellipses (i.e., M-circles) by means of the M-distance, we use expression (12) for the M-distance from a point to an M-circle.

In accordance with (4) and (12), we define the modified Hausdorff distance between two M-circles $E1$ and $E2$:

$$D_H(E1, E2) = \max\{\max_{T\in E1} |\sqrt{d_M(T, C_2; S_2)} - 1|, \max_{P\in E2} |\sqrt{d_M(P, C_1; S_1)} - 1|\}. \tag{22}$$

Firstly, look at the problem: $\max_{P\in E2} |\sqrt{d_M(P, C_1; S_1)} - 1| = ?$.

By means of a parametric equation of the ellipse $E2$

$$\begin{cases} x = p + a_2\cos\varphi\cos t - b_2\sin\varphi\sin t \\ y = q + a_2\sin\varphi\cos t + b_2\cos\varphi\sin t \end{cases}, \quad t \in [0, 2\pi],$$

it follows that

$$\begin{aligned} d_M(P, C_1; S_1) = {} & \frac{(p + a_2\cos\varphi\cos t - b_2\sin\varphi\sin t)^2}{a_1^2} \\ & + \frac{(q + a_2\sin\varphi\cos t + b_2\cos\varphi\sin t)^2}{b_1^2} = g(t). \end{aligned}$$

Function $g(t) \geq 0$ is continuous, differentiable and periodic.

Therefore, the continuous function $|\sqrt{g(t)} - 1| \geq 0$ attains the global maximum on $[0, 2\pi]$, at some of stationary points of the function $g(t) \neq 1$, or for $g(t) = 0$.

Stationary points of the function $g(t) \geq 0$ can be found as a solution of the equation

$$\begin{aligned} g'(t) = &\frac{(p + a_2 \cos\varphi \cos t - b_2 \sin\varphi \sin t) \cdot (-a_2 \cos\varphi \sin t - b_2 \sin\varphi \cos t)}{a_1^2} \\ &+ \frac{(q + a_2 \sin\varphi \cos t + b_2 \cos\varphi \sin t) \cdot (-a_2 \sin\varphi \sin t + b_2 \cos\varphi \cos t)}{b_1^2} = 0, \end{aligned} \tag{23}$$

$t \in [0, 2\pi]$.

Equation (23) can be shown in the form of an algebraic equation of the fourth order in the variable $\cos t = u$. So, by means of the formula of the Ferrari method, one can find its set of real solutions $\{t_{11}, \ldots, t_{1m_1}\}$. Following that, the global maximum on the interval $[0, 2\pi]$ can be easily found:

$$\max_{P \in E2} |\sqrt{d_M(P, C_1; S_1)} - 1| = \max_{t \in \{t_{11}, \ldots, t_{1m_1}\}} |\sqrt{g(t)} - 1|. \tag{24}$$

Secondly, with regard to (22), we look at the problem

$$\max_{T \in E1} |\sqrt{d_M(T, C_2; S_2)} - 1| = ?$$

By means of a parametric equation of the ellipse $E1$

$$\begin{cases} x = a_1 \cos t \\ y = b_1 \sin t \end{cases} \quad t \in [0, 2\pi],$$

it follows that

$$\begin{aligned} d_M(T, C_2; S_2) = &\frac{(a_1 \cos\varphi \cos t + b_1 \sin\varphi \sin t - p \cos\varphi - q \sin\varphi)^2}{a_2^2} \\ &+ \frac{(-a_1 \sin\varphi \cos t + b_1 \cos\varphi \sin t + p \sin\varphi - q \cos\varphi)^2}{b_2^2} = f(t). \end{aligned}$$

Function $f(t) \geq 0$ is continuous, differentiable and periodic.

Therefore, the continuous function $|\sqrt{f(t)} - 1| \geq 0$ attains the global maximum on $[0, 2\pi]$, at some of stationary points of the function $f(t) \neq 1$, or for $f(t) = 0$.

Stationary points of the function $f(t) \geq 0$ can be found as a solution of the equation $f'(t) = 0$, i.e.,

$$\begin{aligned} &\frac{(a_1 \cos\varphi \cos t + b_1 \sin\varphi \sin t - p \cos\varphi - q \sin\varphi) \cdot (-a_1 \cos\varphi \sin t + b_1 \sin\varphi \cos t)}{a_2^2} \\ &+ \frac{(-a_1 \sin\varphi \cos t + b_1 \cos\varphi \sin t + p \sin\varphi - q \cos\varphi) \cdot (a_1 \sin\varphi \sin t + b_1 \cos\varphi \cos t)}{b_2^2} = 0, \end{aligned} \tag{25}$$

where $t \in [0, 2\pi]$.

Analogously to the previous case, equation (25) can be shown in the form of an algebraic equation of the fourth order in the variable $\cos t = u$. So, by means of the formula of the Ferrari method, one can find its set of real solutions $\{t_{21}, \ldots, t_{2m_2}\}$. Following that, the global maximum on the interval $[0, 2\pi]$ can be easily found:

$$\max_{T \in E1} |\sqrt{d_M(T, C_2; S_2)} - 1| = \max_{t \in \{t_{21}, \ldots, t_{2m_2}\}} |\sqrt{f(t)} - 1|. \tag{26}$$

So, from (22), (24) and (26) we obtain the following proposition.

Proposition 7. *The modified Hausdorff distance between two ellipses by means of M-distances has the form:*

$$D_H(E1, E2) = \max\{\max_{t \in \{t_{21}, \ldots, t_{2m_2}\}} |\sqrt{f(t)} - 1|, \max_{t \in \{t_{11}, \ldots, t_{1m_1}\}} |\sqrt{g(t)} - 1|\}.$$

Example 1. *In order to illustrate the proposed modified Hausdorff distance between two ellipses $E1$ and $E2$, three particular cases are given in Table 1. The corresponding quantities $D_H(E1, E2)$ are obtained. In the last column, a* normalized modified Hausdorff distance *between two ellipses is also calculated. Modeled on the so-called normalized similarity measure for pairs of ellipses ([8]), we adopt a normalized modified Hausdorff distance between two ellipses by the following expression:*

$$e^{-D_H(E1,E2)}. \tag{27}$$

Let us note that the normalized (modified) Hausdorff distances have got the values in the interval $[0, 1]$, and the value closer to 1 describes the larger similarity, i.e., the smaller difference between the ellipses (or in general, between two geometric objects that are considered).

centers of ellipses	half-axes of ellipses	φ	$D_H(E1, E2)$	$e^{-D_H(E1,E2)}$
$C_1 = (0, 0)$, $C_2 = (0.01, 0.02)$,	$a_1 = 4$, $b_1 = 3$ $a_2 = 4.02$, $b_2 = 3.01$	$\varphi_1 = \pi/180$	0.0128222	0.98726
$C_1 = (0, 0)$, $C_2 = (0.1, 0.2)$,	$a_1 = 4$, $b_1 = 2$ $a_2 = 4.2$, $b_2 = 2.1$	$\varphi_1 = \pi/180$	0.149691	0.860974
$C_1 = (0, 0)$, $C_2 = (1.5, 1.2)$,	$a_1 = 4$, $b_1 = 2$ $a_2 = 5$, $b_2 = 3$	$\varphi_1 = \pi/3$	1.78109	0.168454

Table 1: *Modified Hausdorff distance between two ellipses*

6. Conclusion

We consider the Hausdorff distance between some sets of points in $\mathbb{R}^n$, $n \geq 2$. Firstly, the formula for the Hausdorff distance between two hyperspheres in $\mathbb{R}^n$, $n \geq 2$, is given.

Looking at a segment and a circle in the plane, and generally a segment and a hypersphere in $\mathbb{R}^n$, $n \geq 3$, we obtain the expression of the Hausdorff distance for that case. Then, in the case of a segment and an ellipse in the plane, and generally between a segment and a hyper-ellipsoid in $\mathbb{R}^n$, $n \geq 3$, we adopt the modified Hausdorff distance between them by means of the Mahalanobis distance.

Finally, using Mahalanobis distances and taking into account that ellipses are M-circles, we give expressions which have a few calculations for obtaining the modified Hausdorff distances between two ellipses.

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