Local convergence analysis of two competing two-step iterative methods free of derivatives for solving equations and systems of equations

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Abstract. We present the local convergence analysis of two-step iterative methods free of derivatives for solving equations and systems of equations under similar hypotheses based on Lipschitz-type conditions. The methods are in particular useful for solving equations or systems involving non-differentiable terms. A comparison is also provided using suitable numerical examples.

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1. Introduction

Numerous problems in mathematics, computational sciences, engineering and related sciences using mathematical modeling [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17] can be reduced to locating a solution $x^*$ of the nonlinear equation in the form

$$F(x) = 0,$$

where $X, Y$ are Banach spaces, $D$ is nonempty, open, convex, and $F : D \subseteq X \to Y$ is Fréchet-differentiable.

Analytic solutions or closed form solutions are hard or impossible to find in general. That explains why researchers utilize iterative methods to generate a sequence approximating $x^*$.

In this study, we present the local convergence of two-step secant method (TSSM) and the two-step Kurchatov-type method (TSKM) defined, respectively, for each $n = 0, 1, 2, \ldots$ by

$$x_{n+1} = x_n - [x_n, y_n; F]^{-1}F(x_n)$$

(1)

$$y_{n+1} = x_{n+1} - [x_{n+1}, y_n; F]^{-1}F(x_{n+1})$$

$$x_{n+1} = x_{n+1} - [2y_n - x_n, y_n; F]^{-1}F(x_n)$$

(2)

$$y_{n+1} = x_{n+1} - [2y_n - x_n, x_n; F]F(x_{n+1}).$$
where \( x_0, y_0 \in D \) are initial points and \([.,.; F] : D \times D \rightarrow \mathcal{L}(X, Y)\) is a divided difference of order one [16, 15] for \( F \) on \( D \) satisfying
\[
[x, y; F](x - y) = F(x) - F(y)
\]
for each \( x, y \) with \( x \neq y \), and \( F'(x) = [x, x; F] \), if \( F \) is Fréchet-differentiable. TSSM uses two inverses and three function evaluations per complete step, whereas TSKM uses one inverse and four function evaluations.

The rest of the paper is structured as follows: Section 2 and Section 3 contain the local convergence of TSSM and TSKM, respectively, under similar Lipschitz-type hypotheses. The numerical examples in Section 4 conclude this paper.

2. Local convergence I

We present the local convergence analysis of TSSM based on scalar parameters and functions. Let \( 0 \leq \alpha, \beta \leq 0 \) and \( b > 0 \) with \( \alpha + \beta \neq 0 \). Define parameters \( \rho_0, \rho_1 \) and functions \( f \) and \( h_f \) on the interval \([0, \rho_0]\) by
\[
\rho_0 = \frac{1}{\alpha + \beta}, \quad \rho_1 = \frac{1}{\alpha + \beta + b},
\]
\[
f(t) = (b + \frac{abt}{1 - (\alpha + b)t} + \beta)t
\]
and
\[
h_f(t) = f(t) - 1.
\]
We have that \( h_f(0) = -1 \) and \( h_f(t) \rightarrow +\infty \) as \( t \rightarrow \rho_0^- \). The intermediate value theorem assures that equation \( h_f(t) = 0 \) has solutions on the interval \((0, \rho_0)\). Denote by \( \rho^* \) the smallest such solution. Notice that \( h_f(\rho_1) = 0 \), so \( \rho^* \leq \rho_1 \). Then, we have that for each \( t \in [0, \rho^*) \)
\[
0 \leq \frac{bt}{1 - (\alpha + \beta)t} < 1
\]
and
\[
0 \leq f(t) < 1.
\]
Let \( U(z, \lambda) \) and \( \bar{U}(z, \lambda) \) denote the open and closed balls in \( X \), respectively, where \( z \in X \) is the center and \( \lambda > 0 \) is the radius. The local convergence analysis of TSSM is also based on the hypotheses (H):

\[(h_1)\] \( F : D \subset X \rightarrow Y \) is a continuously Fréchet differentiable operator and \([.,.; F] : D \times D \rightarrow \mathcal{L}(X, Y)\) is a divided difference of order one.

\[(h_2)\] There exist parameters \( \alpha \geq 0, \beta \geq 0 \) with \( \alpha + \beta \neq 0 \), \( x^* \in D \) such that
\[
F(x^*) = 0, \quad F'(x^*)^{-1} \in \mathcal{L}(Y, X)
\]
and for each \( x, y \in D \)
\[
\|F'(x^*)^{-1}(x, y; F - F'(x^*))\| \leq \alpha \|x - x^*\| + \beta \|y - x^*\|.
\]
Set \( D_0 = D \cap U(x^*, \rho_0) \), where \( \rho_0 \) was defined previously.
(h₃) There exists $b > 0$ such that for each $x, y \in D₀$
\[
\|F'(x^*)^{-1}([x, y; F] - [x, x^*; F])\| \leq b\|y - x^*\|.
\]

(h₄) $U(x^*, \rho^*) \subset D$, where $\rho^*$ was defined previously.

(h₅) There exists $R^* \geq \rho^*$ such that
\[
R^* < \frac{1}{\beta}, \quad \beta \neq 0.
\]

Set $D₁ = D \cap U(x^*, R^*)$.

**Theorem 1.** Suppose that the hypotheses (H) hold. Then, sequences $\{xₙ\}$, $\{yₙ\}$ starting from $x₀, y₀ \in U(x^*, \rho^*) - \{x^*\}$ and generated by TSSM are well defined in $U(x^*, \rho^*)$ for each $n = 0, 1, 2, \ldots$, remain in $U(x^*, \rho^*)$ and converge to $x^*$. Moreover, the following estimates hold for each $n = 0, 1, 2, \ldots$
\[
\|x_{n+1} - x^*\| \leq \frac{b\|y_n - x^*\|}{1 - (\alpha\|x_n - x^*\| + \beta\|y_n - x^*\|)} \|x_n - x^*\| \leq \|x_n - x^*\| < \rho^*
\]
and
\[
\|y_{n+1} - x^*\| \leq \frac{b\|y_n - x^*\|}{1 - (\alpha\|x_{n+1} - x^*\| + \beta\|y_n - x^*\|)} \|x_{n+1} - x^*\|.
\]

Furthermore, the limit point $x^*$ is the only solution to equation $F(x) = 0$ in $D₁$, where $D₁$ is defined in (h₅).

**Proof.** Let $x, y \in U(x^*, \rho^*)$. Using (h₂), we have in turn that
\[
\|F'(x^*)^{-1}([x, y; F] - F'(x^*))\| \leq \alpha\|x - x^*\| + \beta\|y - x^*\| < (\alpha + \beta)\rho^* < 1.
\]

In view of (5) and the Banach lemma on invertible operators [5, 6, 7, 13], $[x, y; F]^{-1} \in L(Y, X)$ and
\[
\|([x, y; F]^{-1}F'(x^*))\| \leq \frac{1}{1 - (\alpha\|x - x^*\| + \beta\|y - x^*\|)}.
\]

In particular, $[x₀, y₀; F]^{-1} \in L(Y, X)$, since $x₀, y₀ \in U(x^*, \rho^*)$. By the first substep of TSSM, we can write
\[
x₁ - x^* = x₀ - x^* - [x₀, y₀; F]^{-1}F(x₀)
= [x₀, y₀; F]^{-1}([x₀, y₀; F] - [x₀, x^*; F])(x₀ - x^*).
\]

By (h₃), (6) for $x = x₀, y = y₀$ and (7), we get in turn
\[
\|x₁ - x^*\| \leq \|[x₀, y₀; F]^{-1}F'(x^*)\|\|F'(x^*)^{-1}([x₀, y₀; F] - [x₀, x^*; F])(x₀ - x^*)\|
\leq \frac{b\|y₀ - x^*\|}{1 - (\alpha\|x₀ - x^*\| + \beta\|y₀ - x^*\|)}\|x₀ - x^*\|
\leq \|x₀ - x^*\| < \rho^*.
\]
so (3) holds for \( n = 0 \) and \( x_1 \in U(x^*, \rho^*) \) and \([x_1, y_0; F]^{-1} \in L(Y, X)\). We also have by (6) that

\[
\|x_1, y_0; F\|^{-1} F'(x^*) \| \leq \frac{1}{1 - (\alpha \|x_1 - x^*\| + \beta \|y_0 - x^*\|)}.
\]

Moreover, by the second substep of TSSM, we can write that

\[
y_1 - x^* = x_1 - x^* - [x_1, y_0; F]^{-1} F(x_1)
\]

\[
= [x_1, y_0; F]^{-1} ([x_1, y_0; F] - [x_1, x^*; F])(x_1 - x^*),
\]

so

\[
\|y_1 - x^*\| \leq \frac{b \|y_0 - x^*\| \|x_1 - x^*\|}{1 - (\alpha \|x_1 - x^*\| + \beta \|y_0 - x^*\|)}
\]

\[
\leq \frac{b \rho^*}{1 - (\alpha + \beta) \rho^*} \|x_1 - x^*\| < \rho^*,
\]

which shows (4) for \( n = 0 \) and \( y_1 \in U(x^*, \rho^*) \). The induction for (3) and (4) is completed analogously if \( x_0, y_0, x_1, y_1 \) are replaced by \( x_m, y_m, x_{m+1}, y_{m+1} \) in the preceding estimates, respectively. Then, from the estimates

\[
\|x_{m+1} - x^*\| \leq \mu_1 \|x_m - x^*\| < \rho^*
\]

and

\[
\|y_{m+1} - x^*\| \leq \mu_2 \|x_{m+1} - x^*\| < \rho^*,
\]

where \( \mu_1 = \frac{b \rho^*}{1 - (\alpha + \beta) \rho^*} \in (0, 1) \) and \( \mu_2 = f(\rho^*) \in (0, 1) \), we deduce that \( \lim_{m \to +\infty} x_m = \lim_{m \to +\infty} y_m = x^*, x_{m+1} \in U(x^*, \rho^*) \) and \( y_{m+1} \in U(x^*, \rho^*) \). The uniqueness part is shown by letting \( T = [x^*, y^*; F] \) for some \( y^* \in D \) with \( F(y^*) = 0 \). Using (h2) and (h5), we obtain in turn that

\[
\|F'(x^*) - F'(y^*)\| \leq \beta \|y^* - x^*\| \leq \beta R < 1,
\]

so \( T^{-1} \in L(Y, X) \). Finally, from the identity

\[
0 = F(x^*) - F(y^*) = [x^*, y^*; F](x^* - y^*),
\]

we conclude that \( x^* = y^* \).

\[
\Box
\]

3. Local convergence II

In this section, the local convergence of TSKM is presented in the way analogous to that shown in Section 2 for TSSM. Let \( a \geq 0, b_1 \geq 0, p \geq 0, q \geq 0, a + b_1 \neq 0 \) and \( c > 0 \) be given parameters. Define parameters \( r_0, r_1 \), functions \( g_1 \) and \( h_{g_1} \) on interval \([0, r_0]\) by

\[
r_0 = \frac{2}{a + \sqrt{a^2 + 16c}}, \quad r_1 = \frac{2}{a + b_1 + \sqrt{(x + b_1)^2 + 32c}}
\]

\[
g_1(t) = \frac{b_1 + 4ct}{1 - (a + 4ct)t^t}
\]
and
\[ h_{g_1}(t) = g_1(t) - 1. \]

Notice that \( h_{g_1}(r_1) = 0 \) and \( r_1 \) is the only solution to equation \( h_{g_1}(t) = 0 \) in \((0, r_0)\).
Moreover, define functions \( g_2 \) and \( h_{g_2} \) of the interval \([0, r_0)\) by
\[
g_2(t) = \frac{p_1 \left( \frac{(b_1 + 4ct) t}{1 - (a + 4ct) t} \right) + 1}{1 - (a + 4ct) t} t + q + 4ct \]
and
\[ h_{g_2}(t) = g_2(t) - 1. \]

We get \( h_{g_2}(0) = -1 < 0 \) and \( h_{g_2}(t) \to +\infty \) as \( t \to r_0^- \). Denote by \( r_2 \) the smallest solution to equation \( h_{g_2}(t) = 0 \) in \((0, r_1)\).
Define the radius of convergence \( r^* \) by
\[
r^* = \min\{r_1, r_2\}. \tag{8} \]

Then, we have that for each \( t \in [0, r^*) \),
\[ 0 \leq g_i(t) < 1, \ i = 1, 2. \]

The local convergence analysis of TSKM is based on hypotheses (A):

1. \((a_1) = h_1(1)\)

\((a_2)\) There exist \(a \geq 0, c \geq 0, x^* \in D\) such that \(F(x^*) = 0, F'(x^*)^{-1} \in L(Y, X)\) for each \(x, y \in D\)
\[
\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq a\|x - x^*\|
\]
and
\[
\|F'(x^*)^{-1}(2y - x; F - F'(y))\| \leq c\|y - x\|^2
\]
Set \(D_2 = D \cap \bar{U}(x^*, r_0)\), where \(r_0\) was defined previously.

\((a_3)\) There exists \(b \geq 0, p \geq 0, q \geq 0\) such that for each \(x, y \in D_2\)
\[
\|F'(x^*)^{-1}(x; y; F - [x, x^*; F])\| \leq b\|y - x^*\|
\]
and
\[
\|F'(x^*)^{-1}(x, x^*; F - F'(y))\| \leq p\|x - y\| + q\|y - x^*\|.
\]

\((a_4)\) \(\bar{U}(x^*, 3r^*) \subseteq D\), where \(r^*\) was defined previously.

\((a_5)\) There exists \(R_1^* \geq R^*\) such that
\[
R_1^* < \frac{2}{a}, a \neq 0.
\]

Set \(D_3 = D \cap \bar{U}(x^*, R_1^*)\).
Theorem 2. Suppose that the hypotheses (A) hold. Then, sequences \( \{x_n\}, \{y_n\} \) starting from \( x_0, y_0 \in U(x^*, r^*) - \{x^*\} \) and generated by TSKM are well defined in \( U(x^*, r^*) \) for each \( n = 0, 1, 2, \ldots \), remain in \( U(x^*, r^*) \), and converge to \( x^* \). Moreover, the following estimates hold for each \( n = 0, 1, 2 \ldots \)

\[
\|x_{n+1} - x^*\| \leq \frac{b\|y_n - x^*\| + c\|y_n - x_n\|^2}{1 - (a\|x_n - x^*\| + c\|y_n - x_n\|^2)} \leq \|x_n - x^*\| < r^* \tag{9}
\]

and

\[
\|y_{n+1} - x^*\| \leq \frac{b\|x_{n+1} - y_n\| + q\|y_n - x^*\| + c\|y_n - x_n\|^2}{1 - (a\|x_n - x^*\| + c\|y_n - x_n\|^2)} \|x_{n+1} - x^*\|. \tag{10}
\]

Furthermore, the limit point \( x^* \) is the only solution to equation \( F(x) = 0 \) in \( D_3 \).

Proof. Let \( x, y \in U(x^*, r^*) \) and set \( Q = [2y - x, x; F] \). Using \((a_2)\) and \((8)\), we have in turn that

\[
\|F'(x^*)^{-1}(F'(x^*) - Q)\| = \|F'(x^*)^{-1}(F'(x^*) - F'(y)) + (F'(y) - [2y - x, x; F])\|
\]

\[
\leq \|F'(x^*)^{-1}(F'(y) - F'(x^*))\| + \|F(x^*)^{-1}([2y - x, x; F] - F'(y))\|
\]

\[
\leq a\|y - x^*\| + c\|y - x\|^2
\]

\[
\leq ar^* + c(\|y - x^*\| + \|x^* - x\|^2)
\]

\[
\leq ar^* + 4c(r^*)^2 < 1,
\]

so \( Q^{-1} \in L(Y, X) \),

\[
\|Q^{-1}F(x^*)\| \leq \frac{1}{1 - (a\|y - x^*\| + c\|y - x\|^2)} \tag{11}
\]

and \([2y_0 - x_0, x_0; F]^{-1} \in L(Y, X) \) for \( x = x_0 \) and \( y = y_0 \). Hence, \( x_1 \) and \( y_1 \) are well defined by the first and the second substep of TSKM. Notice that condition \((a_1)\) guarantees that for \( x, y \in U(x^*, r^*) \) we have \( 2y - x \in U(x^*, r^*) \subseteq D \). By \((a_2)\) and \((a_3)\), we get in turn the estimate

\[
\|F'(x^*)^{-1}(Q - [x_0, x^*; F])\|
\]

\[
\leq \|F'(x^*)^{-1}([y_0, x^*; F] - F'(y_0)) + (F'(y_0) - [2y_0 - x_0, x_0; F])\|
\]

\[
\leq \|F'(x^*)^{-1}([y_0, x^*; F] - F'(y_0))\| + \|F'(x^*)^{-1}(F'(y_0) - [2y_0 - x_0, x_0; F])\|
\]

\[
\leq b\|y_0 - x^*\| + c\|y_0 - x_0\|^2. \tag{12}
\]

In view of the first substep of TSKM, \((8)\), \((11)\) and \((12)\), we obtain in turn from

\[
\begin{align*}
\|x_1 - x_0\| &= x_0 - x^* - Q^{-1}F(x_0) \\
&= Q^{-1}(Q - [x_0, x^*; F])(x_0 - x^*),
\end{align*}
\]

so

\[
\|x_1 - x_0\| \leq \mu_3\|x_0 - x^*\|
\]

\[
\leq \|x_0 - x^*\| < r^*,
\]
where \( \mu_3 = \frac{b\|x_0 - x^*\| + c\|x_0 - y_0\|^2}{1 - (a\|x_0 - x^*\| + c\|y_0 - x_0\|^2)} \in [0, 1) \), which shows (9) for \( n = 0 \) and \( x_1 \in U(x^*, r^*) \). Similarly, from the second substep of TSKM, we can also write
\[
y_1 - x^* = x_1 - x^* - Q^{-1}F(x_1)
\]
so
\[
\|y_1 - x^*\| \\
\leq \frac{\|F'(x^*)^{-1}(2y_0 - x_0, x_0; F) - F'(y_0)\| + \|F'(x^*)^{-1}(F'(y_0) - [x_1, x^*; F])\|}{1 - (a\|x_0 - x^*\| + c\|y_0 - x_0\|^2)}
\]
\[\times \|x_1 - x^*\| \leq g_2(\|x_0 - x^*\|) \|x_1 - x^*\| \leq \|x_1 - x^*\| < r^*,
\]
which shows (10) for \( n = 0 \) and \( y_1 \in U(x^*, r^*) \). Then, from the estimates
\[
\|x_{m+1} - x^*\| \leq \mu_3\|x_n - x^*\| < r^*,
\]
and
\[
\|y_{n+1} - x^*\| \leq \mu_4\|x_{m+1} - x^*\| < r^*,
\]
where \( \mu_4 = g_2(\|x_0 - x^*\|) \in [0, 1) \), we obtain \( \lim_{m \to +\infty} x_m = \lim_{m \to +\infty} y_m = x^* \) and \( x_{m+1}, y_{m+1} \in U(x^*, r^*) \). As in Theorem 1, but using \( (a_2) \) and \( (a_5) \) for \( P = \int_0^1 F'(x^* + \theta(y^* - x^*))d\theta \), we obtain
\[
\|F'(x^*)^{-1}(P - F'(x^*))\| \leq \int_0^1 \theta\|y^* - x^*\|d\theta
\]
\[\leq \frac{a}{2}\|y^* - x^*\| \leq \frac{a}{2}R_1^* < 1,
\]
so \( P^{-1} \in L(Y, X) \). Then, from the identity
\[
0 = F(y^*) - F(x^*) = P(y^* - x^*),
\]
we derive that \( x^* = y^* \).
\[
\square
\]

**Remark 1.** Condition \((a_4)\) can be weakened if replaced by
\[
(a_4) \  \hat{U}(x^*, r^*) \subseteq D \text{ and for each } x, y \in D
\]
\[
2y - x \in D.
\] (13)

Condition (13) certainly holds if \( D = X \) (see also [1, 2, 3, 4, 5, 6, 7]).
4. Numerical examples

Let \( X = Y = \mathbb{R}^k, k \) be a positive integer equipped with the standard difference [13], and for

\[
\begin{align*}
x_m &= (x_m^{(1)}, x_m^{(2)}, \ldots, x_m^{(k)}) \\
y_m &= (y_m^{(1)}, y_m^{(2)}, \ldots, y_m^{(k)}),
\end{align*}
\]

there exists \( i = 1, 2, \ldots, k \) such that \( x_m^{(i)} = y_m^{(i)} \). Then, we cannot use TSSM or TSKM in the form (1) and (2). Assuming that \( x_0^{(i)} \neq y_0^{(i)}, y_0^{(i)} \neq x_1^{(i)} \) for each \( i = 1, 2, \ldots, k \), \([x_0, y_0; F]^{-1}\) and \([x_1, y_0; F]^{-1} \in L(Y, X)\), we can use a method similar to the TSSM method defined for each \( n = 0, 1, 2, \ldots \), by

\[
\begin{align*}
x_{n+1} &= x_n - [v_j, w_j; F]^{-1}F(x_n) \\
y_{n+1} &= x_{n+1} - [z_{j+1}, w_j; F]^{-1}F(x_{n+1}),
\end{align*}
\]

where \( j = 0, 1, 2, \ldots, n \) is the smallest index for which \( v_j^{(i)} \neq w_j^{(i)} \) and \( z_{j+1}^{(i)} \neq w_j^{(i)} \). Then, method (14) is always well defined and can be used to solve equations containing non-differentiable terms. Similarly, assume that \([2y_0 - x_0, x_0; F]^{-1}\) and \([2x_1 - y_0, y_0; F]^{-1} \in L(Y, X)\), \( x_0^{(i)} \neq y_0^{(i)} \) and \( y_0^{(i)} \neq x_1^{(i)} \) for each \( i = 1, 2, \ldots, k \). Then, the method corresponding to TSKM is defined by

\[
\begin{align*}
x_{n+1} &= x_n - [2w_j - v_j, v_j; F]^{-1}F(x_n) \\
y_{n+1} &= x_{n+1} - [2w_j - v_j, v_j; F]^{-1}F(x_{n+1}).
\end{align*}
\]

Clearly, methods (14) and (15) generalize methods (1) and (2) since they coincide with those for \( j = n \), respectively.

Next, we shall show the convergence of method (14) under similar conditions. Let us consider hypotheses (H'):

1. (\( h'_1 \)) = (\( h_1 \))
2. (\( h'_2 \)) = (\( h_2 \))

(\( h'_3 \)) There exists \( \gamma \geq 0, \delta \geq 0 \) such that for each \( x, y, z \in D_0 \)

\[
\|F'(x^*)^{-1}([x, y; F] - [z, x^*; F])\| \leq \gamma \|x - z\| + \delta \|y - x^*\|.
\]

(\( h'_4 \)) \( \bar{U}(x^*, \bar{\rho}^*) \subset D \), where \( \bar{\rho}^* = \frac{1}{\alpha + \beta + \gamma + \delta} \).

(\( h'_5 \)) There exists \( \bar{R}^* \geq \bar{\rho}^* \) such that

\[
\bar{R}^* < \frac{1}{\beta}, \beta \neq 0.
\]

Let \( D_5 = D \cap \bar{U}(x^*, \bar{R}^*) \).
Theorem 3. Suppose that the hypotheses (H’) hold. Then, sequences \(\{x_n\}, \{y_n\}\) starting from \(x_0, y_0 \in U(x^*, \rho^*)\) and generated by method (14) are well defined in \(U(x^*, \rho^*)\), remain in \(U(x^*, \rho^*)\) for each \(n = 0, 1, 2\ldots\), and converge to \(x^*\). Moreover, the following estimates hold:

\[
\|x_{n+1} - x^*\| \leq \frac{\gamma \|v_j - x_n\| + \delta \|w_j - x^*\|}{1 - (\alpha \|v_j - x^*\| + \beta \|w_j - x^*\|)} \|x_n - x^*\| \\
\leq \frac{\gamma \|v_j - x^*\| + \|x_n - x^*\| + \delta \|w_j - x^*\|}{1 - (\alpha \|v_j - x^*\| + \beta \|w_j - x^*\|)} \|x_n - x^*\| \\
\leq \frac{(2\gamma + \delta)\bar{\rho}^*}{1 - (\alpha + \beta)\bar{\rho}^*} \|x_n - x^*\| \leq \|x_n - x^*\| < \rho^*,
\]

(16)

and

\[
\|y_{n+1} - x^*\| \leq \frac{\gamma \|z_{j+1} - x_{n+1}\| + \delta \|w_j - x^*\|}{1 - (\alpha \|z_{j+1} - x^*\| + \beta \|w_j - x^*\|)} \|x_{n+1} - x^*\| \\
\leq \frac{\gamma \|z_{j+1} - x^*\| + \|x_{n+1} - x^*\| + \delta \|w_j - x^*\|}{1 - (\alpha \|z_{j+1} - x^*\| + \beta \|w_j - x^*\|)} \|x_{n+1} - x^*\| \\
\leq \frac{(2\gamma + \delta)\bar{\rho}^*}{1 - (\alpha + \beta)\bar{\rho}^*} \|x_{n+1} - x^*\| \leq \|x_{n+1} - x^*\| < \hat{\rho}^*.
\]

(17)

Furthermore, the limit point \(x^*\) is the only solution to equation \(F(x) = 0\) in \(D_5\).

Proof. Use the proof of Theorem 1, the identities

\[
x_{n+1} - x^* = ([v_j, w_j; F]^{-1}F'(x^*)) \\
\times(F'(x^*)^{-1}([v_j, w_j; F] - [x_n, x^*; F]))(x_n - x^*)
\]

and

\[
y_{n+1} - x^* = ([z_{j+1}, w_j; F]^{-1}F'(x^*)) \\
\times(F'(x^*)^{-1}([z_{j+1}, w_j; F] - [x_{n+1}, x^*; F]))(x_{n+1} - x^*)
\]

to arrive at estimates (16) and (17), respectively. \(\square\)

The hypotheses (A’) are:

1. \((a_1')= (a_1)\)
2. \((a_2')= (h_2)\)
3. \((a_3')= (h_3)\)

\((a_4')\) \(\bar{U}(x^*, \bar{r}^*) \subset D\), where \(\bar{r}^* = \frac{1}{3\alpha + \beta + 4\gamma + \delta}\).

\((a_5')\) There exists \(\bar{R}_i^* \geq \bar{r}^*\) such that

\[
\bar{R}_i^* \leq \frac{1}{\beta}, \beta \neq 0.
\]

Let \(D_6 = D \cap \bar{U}(x^*, \bar{R}_i^*)\).
Theorem 4. Suppose that the hypotheses \( (A') \) hold. Then, sequences \( \{x_n\}, \{y_n\} \) starting from \( x_0, y_0 \in U(x^*, r^*) = \{x^*\} \) and generated by method \( (15) \) are well defined in \( U(x^*, r^*) \), remain in \( U(x^*, r^*) \) for each \( n = 0, 1, 2, \ldots \), and converge to \( x^* \). Moreover, the following estimates hold:

\[
\|x_{n+1} - x^*\| \leq \frac{\gamma \|2w_j - v_j - x_n\| + \delta \|v_j - x^*\|}{1 - (\alpha \|2w_j - v_j - x^*\| + \beta \|v_j - x^*\|)} \|x_n - x^*\|
\]

\[
\leq \frac{\gamma (\|2w_j - x^*\| + \|v_j - x^*\| + \|x_n - x^*\|) + \delta \|v_j - x^*\|}{1 - (\alpha (\|2w_j - x^*\| + \|v_j - x^*\|) + \beta \|v_j - x^*\|)} \|x_n - x^*\|
\]

\[
\leq \frac{(4\gamma + \delta)\overline{r}^*}{1 - (3\alpha + \beta)\overline{r}^*} \|x_n - x^*\| \leq \|x_n - x^*\| < \overline{r}^*, \tag{18}
\]

and

\[
\|y_{n+1} - x^*\| \leq \frac{\gamma \|2w_j - v_j - x_{n+1}\| + \delta \|v_j - x^*\|}{1 - (\alpha \|2w_j - v_j - x^*\| + \beta \|v_j - x^*\|)} \|x_{n+1} - x^*\|
\]

\[
\leq \frac{\gamma (\|2w_j - x^*\| + \|v_j - x^*\| + \|x_{n+1} - x^*\|) + \delta \|v_j - x^*\|}{1 - (\alpha (\|2w_j - x^*\| + \|v_j - x^*\|) + \beta \|v_j - x^*\|)} \|x_{n+1} - x^*\|
\]

\[
\leq \frac{(4\gamma + \delta)\overline{r}^*}{1 - (3\alpha + \beta)\overline{r}^*} \|x_{n+1} - x^*\| \leq \|x_{n+1} - x^*\| < \overline{r}^*. \tag{19}
\]

Furthermore, the limit point \( x^* \) is the only solution to equation \( F(x) = 0 \) in \( D_0 \).

Proof. Use the proof of Theorem 2, the identities

\[
x_{n+1} - x^* = ((2w_j - v_j, w_j; F) - [x_n, x^*; F])(x_n - x^*)
\]

and

\[
y_{n+1} - x^* = ((2w_j - v_j, v_j; F) - [x_{n+1}, x^*; F])(x_{n+1} - x^*)
\]

to arrive at estimates (18) and (19), respectively.

\[\square\]

Example 1. Let us consider the system for \( h = (h_1, h_2)^T \)

\[
f_1(h) = 3h_1^2h_2 + h_2^2 - 1 + |h_1 - 1| = 0
\]

\[
f_2(h) = h_1^4 + h_1h_2^3 - 1 + |h_2| = 0
\]

which can be written as \( F(h) = 0 \), where \( F = (f_1, f_2)^T \). Using the divided difference, \( ([a, b] F)_{ij}, \sum_{i=1}^{d} F_j(x) = (1, 1)^T, x_0 = (5, 5)^T \), we obtain by (2)

Hence, the solution \( p \) is given by \( p = (0.8945537334687, 0.3278626421746298)^T \).

Notice that mapping \( F \) is not differentiable, so the earlier results mentioned in the introduction of this study cannot be used.
Local convergence analysis of two competing two-step iterative methods

Table 1:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$x_n^{(1)}$</th>
<th>$x_n^{(2)}$</th>
<th>$|x_n - x_{n-1}|$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>0.909090909090909</td>
<td>0.363636363636364</td>
<td>3.0636E-01</td>
</tr>
<tr>
<td>3</td>
<td>0.894886945874111</td>
<td>0.329098638203090</td>
<td>1.271E-03</td>
</tr>
<tr>
<td>4</td>
<td>0.894655531991499</td>
<td>0.327827544745569</td>
<td>1.022E-06</td>
</tr>
<tr>
<td>5</td>
<td>0.894655373334793</td>
<td>0.327826521746906</td>
<td>6.089E-13</td>
</tr>
<tr>
<td>6</td>
<td>0.8946655373334687</td>
<td>0.327826521746298</td>
<td>2.710E-20</td>
</tr>
<tr>
<td>7</td>
<td>0.8946655373334687</td>
<td>0.327826421746298</td>
<td>2.710E-20</td>
</tr>
</tbody>
</table>

Example 2. We consider the boundary problem appearing in many studies of applied sciences [6] given by

\[
\varphi'' + \varphi^{1+\lambda} + \varphi^2 = 0, \quad \lambda \in [0, 1] \tag{20}
\]

\[
\varphi(0) = \varphi(1) = 0.
\]

Let $h = \frac{1}{l}$, where $l$ is a positive integer and set $s_i = ih, i = 1, 2, \ldots, l - 1$. The boundary conditions are then given by $\varphi_0 = \varphi_n = 0$. We shall replace the second derivative $\varphi''$ by the popular divided difference

\[
\varphi''(t) \approx \frac{[\varphi(t + h) - 2\varphi(t) + \varphi(t - h)]}{h^2} \tag{21}
\]

\[
\varphi''(s_i) = \frac{\varphi_{i+1} - 2\varphi_i + \varphi_{i-1}}{h^2}, \quad i = 1, 2, \ldots, l - 1.
\]

Using (20) and (21), we obtain the system of equations defined by

\[
\begin{align*}
2\varphi_1 - h^2\varphi_1^{1+\lambda} - h^2\varphi_2^2 - \varphi_2 &= 0 \\
-\varphi_{i-1} + 2\varphi_i - h^2\varphi_i^{1+\lambda} - h^2\varphi_i^2 - \varphi_{i+1} &= 0 \\
-\varphi_{i-2} + 2\varphi_{i-1} - h^2\varphi_{i-1}^{1+\lambda} - h^2\varphi_{i-1}^2 &= 0.
\end{align*}
\]

Define operator $F : \mathbb{R}^{l-1} \rightarrow \mathbb{R}^{l-1}$ by

\[
F(\varphi) = M(x) - h^2f(\varphi),
\]

where

\[
M = \begin{bmatrix}
2 & -1 & 0 & \ldots & 0 \\
-1 & 2 & -1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 2
\end{bmatrix}
\]

and

\[
f(\varphi) = [\varphi_1^{1+\lambda} + \varphi_2^{1+\lambda} + \varphi_2^2 + \ldots, \varphi_{l-1}^{1+\lambda} + \varphi_{l-1}^2]^T.
\]
Then, the Fréchet-derivative $F'$ of operator $F$ is given by

$$
F'(\phi) = M - (1 + \lambda)h^2 \begin{bmatrix}
\varphi_1^2 & 0 & 0 & \ldots & 0 \\
0 & \varphi_2^2 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \varphi_{l-1}^2
\end{bmatrix} - 2h^2 \begin{bmatrix}
\varphi_1 & 0 & 0 & \ldots & 0 \\
0 & \varphi_2 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \varphi_{l-1}
\end{bmatrix}.
$$

(22)

We shall use a special case of method (2) given by

$$
\begin{align*}
\psi_n^{(1)} &= \psi_n - F'(\psi_n)^{-1} F(\psi_n) \\
\psi_n^{(2)} &= \psi_n^{(1)} - F'(\psi_n)^{-1} F(\psi_n^{(1)}) \\
&\vdots \\
\psi_n^{(k)} &= \psi_n^{(k-1)} - F'(\psi_n)^{-1} F(\psi_n^{(k-1)}) \\
\psi_{n+1} &= \psi_n^{(k)}.
\end{align*}
$$

(23)

Let $\lambda = \frac{1}{2}$, $k = 3$ and $l = 10$. In this way, we obtain a $9 \times 9$ system. A good initial approximation is $10 \sin \pi t$ since a solution to (20) vanishes at the end points and is positive at the interior. This approximation gives the vector

$$
\xi = \begin{bmatrix}
3.0901699423 \\
5.877852523 \\
8.090169944 \\
9.510565163 \\
10 \\
9.510565163 \\
8.090169944 \\
5.877852523 \\
3.090169923
\end{bmatrix},
$$

which by using (23) leads to

$$
\psi_0 = \begin{bmatrix}
2.396257294 \\
4.698040582 \\
6.677432200 \\
8.038726637 \\
8.526409945 \\
8.038726637 \\
6.677432200 \\
4.698040582 \\
2.396257294
\end{bmatrix}.
$$
Using vector $\psi_0$ as the initial vector in (23), we get the solution $\psi^*$ given by

$$
\begin{bmatrix}
2.394640795 \\
4.694882371 \\
6.672977547 \\
8.033409359 \\
8.520791424 \\
8.033409359 \\
6.672977547 \\
4.694882371 \\
2.394640795 \\
\end{bmatrix}
$$

Notice that the operator $F'$ given in (22) is not Lipschitz.

References