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Weighted approximation by q-Ibragimov-Gadjiev operators

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Abstract. The present paper deals with a q-generalization of Ibragimov-Gadjiev operators, which were constructed by Ibragimov and Gadjiev in 1970. We study convergence properties on the interval $[0, \infty)$ and obtain the rate of convergence in terms of a weighted modulus of continuity. We also give some representation formulas of the operators using q-derivatives.
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1. Introduction

In 1970, İbragimov and Gadjiev [10] introduced the following general sequence of positive linear operators defined on the space $C[0, A]$:

$$L_n(f;x) = \sum_{\nu=0}^{\infty} f\left(\frac{\nu}{n^2\Psi_n(0)}\right)\left[\frac{\partial^\nu}{\partial u^\nu}K_n(x,t,u)\,|_{\substack{u=\alpha_n\Psi_n(t)\\ t=0}}\right]\frac{(-\alpha_n\Psi_n(0))^\nu}{\nu!} \tag{1}$$

which contains the well-known operators of Bernstein, Bernstein-Chlodowsky, Szász and Baskakov as a particular case. Over time, these operators, called Ibragimov-Gadjiev operators, have been deeply investigated and different generalizations have been obtained in numerous papers, see $[1-4, 8, 9, 11]$ and the literature cited therein.

In [9], an extension in q-Calculus of Ibragimov-Gadjiev operators was constructed. Here we briefly describe some details. First of all, we should mention some basic definitions of q-Calculus. We refer to [13] as a good guide. Let $q > 0$. For any natural number $n \in \mathbb{N} \cup \{0\}$, the q-integer $[n] = [n]_q$ is defined by:

$$[n] = \frac{1-q^n}{1-q}, \qquad [0] = 0,$$

and the q-factorial $[n]! = [n]_q!$ by

$$[n]! = [1]\,[2]\cdots[n], \qquad [0]! = 1\ .$$

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For integers $0 \le k \le n$, the q-binomial coefficient is defined by:

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[n-k]!\,[k]!}.$$

Clearly, for $q = 1$,

$$[n]_1 = n, \quad [n]_1! = n!, \quad \begin{bmatrix} n \\ k \end{bmatrix}_1 = \binom{n}{k}.$$

The q-derivative of a function $f : \mathbb{R} \to \mathbb{R}$ is defined by:

$$D_q f(x) = \begin{cases} \dfrac{f(x) - f(qx)}{(1-q)\,x}, & x \neq 0 \\ f'(0), & x = 0 \end{cases},$$

for functions which are differentiable at $x = 0$ and the higher q-derivatives are

$$D_q^0 f = f, \; D_q^n f = D_q\left(D_q^{n-1} f\right), \quad n = 1, 2, 3, \ldots$$

The product rule is

$$D_q\left(f(x)\,g(x)\right) = D_q f(x)\,g(qx) + D_q g(x)\,f(x). \tag{2}$$

The q-analogue of $(x-a)^n$ is the polynomial

$$(x-a)_q^n = \begin{cases} 1, & n = 0 \\ (x-a)(x-qa)\cdots\left(x - q^{n-1}a\right), & n \geq 1 \end{cases}.$$

The q-exponential functions are given as follows:

$$e_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{[k]!} = \frac{1}{(1-(1-q)\,x)_q^{\infty}}, \quad |x| < \frac{1}{1-q}, \quad |q| < 1, \tag{3}$$

$$E_q(x) = \sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}} \frac{x^k}{[k]!} = (1+(1-q)\,x)_q^{\infty}, \; x \in \mathbb{R}, \; |q| < 1. \tag{4}$$

It is clear from equations (3) and (4) that

$$e_q(x)\,E_q(-x) = 1.$$

The following q-generalization of Taylor's formula was introduced by Jackson (see [6]).

Theorem 1. *If the function f can be expanded in a convergent power series and if q is not a root of unity, then*

$$f(x) = \sum_{n=0}^{\infty} D_q^n f(a)\, \frac{(x-a)_q^n}{[n]!}.$$

In [9], the authors introduced a q-analogue of the Ibragimov-Gadjiev operators that are called the q-Ibragimov-Gadjiev operators as follows:

Let $0 < q < 1$ and let $A > 0$ be a given number and $\{\psi_n(t)\}$ a sequence of functions in $C[0,A]$ such that $\psi_n(t) > 0$ for each $t \in [0,A]$, $\lim\limits_{n\to\infty} \frac{1}{[n]^2\psi_n(0)} = 0.$ Also, let $\{\alpha_n\}$ be a sequence of positive real numbers such that $\lim\limits_{n\to\infty} \frac{\alpha_n}{[n]} = 1.$

Assume that a function $K_{n,\nu}^q(x,t,u)$ $(x,t \in [0,A], -\infty < u < \infty)$ depending on the three parameters n,ν and q satisfies the following conditions:

(i) The function $K_{n,\nu}^q(x,t,u)$ is infinitely q-differentiable with respect to u for fixed $x,t \in [0,A]$,

(ii) For any $x \in [0,A]$ and for all $n \in \mathbb{N}$,

$$\sum_{\nu=0}^{\infty} q^{\frac{\nu(\nu-1)}{2}} \left[D_{q,u}^{\nu} K_{n,\nu}^q(x,t,u)\,|_{\substack{u=\alpha_n\psi_n(t)\\ t=0}} \right] \frac{(-q\alpha_n\psi_n(0))^{\nu}}{[\nu]!} = 1,$$

(iii) $\left\{ (-1)^{\nu} D_{q,u}^{\nu} K_{n,\nu}^q(x,t,u)\,|_{\substack{u=\alpha_n\psi_n(t)\\ t=0}} \right\} \geq 0 \ (\nu, n = 1,2,\ldots, x \in [0,A])$,

(This notation means that if the q-derivative with respect to u is taken ν times, then one sets $u = \alpha_n\psi_n(t)$ and $t = 0$.)

(iv) $D_{q,u}^{\nu} K_{n,\nu}^q(x,t,u)\,|_{\substack{u=\alpha_n\psi_n(t)\\ t=0}} = -[n]\,xq^{1-\nu} \left[D_{q,u}^{\nu-1} K_{n+m,\nu-1}^q(x,t,u)\,|_{\substack{u=\alpha_n\psi_n(t)\\ t=0}} \right]$ $(\nu, n = 1,2,\ldots, x \in [0,A])$, where m is a number such that $m+n$ is zero or a natural number.

According to these conditions, the q-Ibragimov-Gadjiev operators have the following form:

$$L_n(f;q;x) = \sum_{\nu=0}^{\infty} q^{\frac{\nu(\nu-1)}{2}} f\left(\frac{[\nu]}{[n]^2\psi_n(0)} \right) \left[D_{q,u}^{\nu} K_{n,\nu}^q(x,t,u)\,|_{\substack{u=\alpha_n\psi_n(t)\\ t=0}} \right] \frac{(-q\alpha_n\psi_n(0))^{\nu}}{[\nu]!} \tag{5}$$

for $x \in \mathbb{R}_+$ and any function f defined on $\mathbb{R}_+$.

By $\nu-$multiple application of property (iv), $L_n(f;q;x)$ can be reduced to the form:

$$\begin{aligned} L_n(f;q;x) = &\sum_{\nu=0}^{\infty} f\left(\frac{[\nu]}{[n]^2\psi_n(0)} \right) \frac{[n][n+m][n+2m]\cdots[n+(\nu-1)m]}{[\nu]!} \\ &\times (xq\alpha_n\psi_n(0))^{\nu} K_{n+\nu m,0}^q(x,0,\alpha_n\psi_n(0)). \end{aligned} \tag{6}$$

Note that for $q = 1$, the operators $L_n(f;q;.)$ are classical Ibragimov-Gadjiev operators. In [9], the authors showed that by taking $q_n \in (0,1)$ such that $\lim\limits_{n\to\infty} q_n = 1$, $L_n(f;q;.)$ satisfies the conditions of the Bohman-Korovkin theorem, and obtained the other convergence properties on the space $C[0,A]$. Furthermore, they showed that in special cases, this sequence of operators consist of q-positive linear operators. For instance, by choosing $K_{n,\nu}^{q}(x,t,u) = \left(1-\frac{q^{1-\nu}ux}{1+t}\right)_q^n$ and $\alpha_n = \frac{[n]}{q}$, $\psi_n(0) = \frac{1}{[n]}$, the operators defined by (5) are transformed into q-Bernstein polynomials which were introduced by Phillips [15], for $\alpha_n = \frac{[n]}{q}$, $\psi_n(0) = \frac{1}{[n]\,b_n}$ ($\lim\limits_{n\to\infty} b_n = \infty, \lim\limits_{n\to\infty} \frac{b_n}{[n]} = 0$) the operators become $q-$Chlodowsky polynomials defined by Karslı and Gupta [14]. For $K_{n,\nu}^{q}(x,t,u) = E_q\left(-[n]\left(t+q^{1-\nu}ux\right)\right)$, $\alpha_n = \frac{[n]}{q}$, $\psi_n(0) = \frac{1}{[n]b_n}$ $\left(\lim\limits_{n\to\infty} b_n = \infty, \lim\limits_{n\to\infty} \frac{b_n}{[n]} = 0\right)$, the $q-$analogue of the classical Szász-Mirakjan operators, which were defined by Aral [5], can be obtained.

The aim of this paper is to establish Korovkin type theorems for continuous and unbounded functions defined on $[0,\infty)$ by the q-Ibragimov-Gadjiev operators. We also give some representations of these operators using the q-differences and divided differences.

2. Preliminary results

We define a suitable q-difference operator as follows:

$$\triangle_q^0 f_\nu = f_\nu,\ \Delta_q^r f_\nu = \Delta_q^{r-1} f_{\nu+1} - q^{r-1}\Delta_q^{r-1} f_\nu\ ,\ r\geq 1, \tag{7}$$

where $f_\nu = f\left(\frac{[\nu]}{[n]^2\psi_n(0)}\right)$, $\nu\in\mathbb{N}_0$.

Now, suppose that in addition to conditions $(i)-(iv)$ function $K_{n,\nu}^{q}(x,t,u)$ satisfies

(v) $K_{n,\nu}^{q}(x,t,u)$ is infinitely $q-$ differentiable with respect to x for fixed $t\in[0,A]$, $u\in\mathbb{R}$

$$D_{q,x}K_{n,\nu}^{q}(x,t,u)\,|_{\substack{u=\alpha_n\psi_n(t)\\ t=0}} = -[n]\,\alpha_n\psi_n(0)\,q^{1-\nu}K_{n+m,\nu-1}^{q}(x,0,\alpha_n\psi_n(0)),$$

where m is the natural number defined in (iv),

(vi) $K_{n,\nu}^{q}(0,0,u) = 1$ for all n,ν and $u\in\mathbb{R}$.

In what follows we take another look at our operators.

Theorem 2. *Let $q\in(0,1)$. The generalized q-Ibragimov-Gadjiev operators may be expressed in the form:*

$$\begin{aligned} L_n(f;q;x) = \sum_{\nu=0}^{\infty} \triangle_q^\nu f_0 \frac{[n][n+m]\cdots[n+(\nu-1)\,m]}{[\nu]!}(q\alpha_n\psi_n(0))^\nu \\ \times K_{n+\nu m,-\nu}^{q}(0,0,\alpha_n\psi_n(0))\,x^\nu, \end{aligned}$$

$x\in[0,A]$, where $\triangle_q^\nu f_0$ is defined as in (7).

Proof. Applying a q-derivative operator to (6) and taking into account product rule (2) and property (v), we have:

$$D_{q,x}L_n(f;q;x)$$
$$=\sum_{\nu=0}^{\infty}\triangle_q^1 f_\nu\frac{[n][n+m]\cdots[n+\nu m]}{[\nu]!}(q\alpha_n\psi_n(0))^{\nu+1}K^q_{n+(\nu+1)m,-1}(x,0,\alpha_n\psi_n(0))\,x^\nu$$

for $n\in\mathbb{N}$ and $x\in[0,A]$. By induction with respect to r we can prove:

$$D^r_{q,x}L_n(f;q;x)$$
$$=\sum_{\nu=0}^{\infty}\triangle_q^r f_\nu\frac{[n][n+m]\cdots[n+(\nu+r-1)\,m]}{[\nu]!}(q\alpha_n\psi_n(0))^{\nu+r}$$
$$\times K^q_{n+(\nu+r)m,-r}(x,0,\alpha_n\psi_n(0))\,x^\nu.$$

From the expansion of the above series by choosing $x=0$, we observe that all terms except the first one are zero,

$$D^r_{q,x}L_n(f;q;0)=\triangle_q^r f_0\frac{[n][n+m]\cdots[n+(r-1)\,m]}{[0]!}(q\alpha_n\psi_n(0))^r$$
$$\times K^q_{n+rm,-r}(0,0,\alpha_n\psi_n(0)),\,r=0,1,2,\dots$$

Hence, choosing $a=0$ in Theorem 1, we obtain

$$L_n(f;q;x)=\sum_{\nu=0}^{\infty}\triangle_q^\nu f_0\frac{[n][n+m]\cdots[n+(\nu-1)\,m]}{[\nu]!}(q\alpha_n\psi_n(0))^\nu$$
$$\times K^q_{n+(\nu)m,-\nu}(0,0,\alpha_n\psi_n(0))\,x^\nu, \tag{8}$$

which completes the proof. □

Now, we will give another representation of operators (5) in terms of divided differences, using the following theorem given in [16, p. 44]. Here, $[x_{0,}x_1,\dots,x_{k;};f]$ denotes the divided difference of the function f with respect to distinct points in the domain of f,

$$[x_0;f]=f(x_0),[x_0,x_1;f]=\frac{f(x_1)-f(x_0)}{x_1-x_0},\dots,$$
$$[x_0,x_1,\dots,x_k;f]=\frac{[x_1,\dots,x_k;f]-[x_0,\dots,x_{k-1};f]}{x_k-x_0}.$$

Theorem 3. *For all $j,k\geq 0$,*

$$f[x_j,x_{j+1},\dots,x_{j+k}]=\frac{\Delta_q^k f_j}{q^{k(2j+k-1)/2}[k]!},$$

where $x_j=[j]$ and $[k]!=[k][k-1]\cdots[1]$.

Corollary 1. *The q-Ibragimov-Gadjiev operators can be represented as:*

$$L_n\left(f;q;x\right)=\sum_{\nu=0}^{\infty}\frac{\left(q\alpha_n x\right)^{\nu}}{[n]^{2\nu}}\left[0,\frac{[1]}{[n]^2\psi_n\left(0\right)},\frac{[2]}{[n]^2\psi_n\left(0\right)},\ldots,\frac{[\nu]}{[n]^2\psi_n\left(0\right)};f\right]$$
$$\times\, q^{\frac{(\nu-1)\nu}{2}}[n][n+m]\cdots[n+\left(\nu-1\right)m]K^q_{n+\nu m,-\nu}\left(0,0,\alpha_n\psi_n\left(0\right)\right).$$

Proof. The proof is obvious from equality (8) and Theorem 3. □

Remark 1. *It will be noted that for $q=1$, this is the result obtained by Ibragimov and Gadjiev [10] for the operators defined by* (1).

Lemma 1. *For the operators defined by* (5)*, the following equalities hold:*

$$L_n\left(1;q;x\right)=1, \tag{9}$$

$$L_n\left(t;q;x\right)=\frac{q\alpha_n}{[n]}x, \tag{10}$$

$$L_n\left(t^2;q;x\right)=\left(\frac{q\alpha_n}{[n]}x\right)^2\frac{q\left[n+m\right]}{[n]}+\left(\frac{q\alpha_n}{[n]}x\right)\frac{1}{[n]^2\psi_n\left(0\right)}, \tag{11}$$

$$L_n\left(t^3;q;x\right)=\frac{q^6\alpha_n^3\left[n+m\right]\left[n+2m\right]x^3}{[n]^5}+\frac{\left(q^4+2q^3\right)\alpha_n^2\left[n+m\right]x^2}{[n]^5\psi_n\left(0\right)}+\frac{q\alpha_n x}{[n]^5\psi_n^2\left(0\right)}, \tag{12}$$

$$\begin{aligned}L_n\left(t^4;q;x\right)=&\frac{q^{10}\alpha_n^4\left[n+m\right]\left[n+2m\right]\left[n+3m\right]x^4}{[n]^7}\\&+\frac{\left(q^8+2q^7+3q^6\right)\alpha_n^3\left[n+m\right]\left[n+2m\right]x^3}{[n]^7\psi_n\left(0\right)}\\&+\frac{\left(q^5+3q^4+3q^3\right)\alpha_n^2\left[n+m\right]x^2}{[n]^7\psi_n^2\left(0\right)}+\frac{q\alpha_n x}{[n]^7\psi_n^3\left(0\right)}.\end{aligned} \tag{13}$$

Proof. For $s=0,1,2$, the equalities of $L_n\left(t^s;q;.\right)$ were obtained in view of the definition of the operators defined by (5) in [9]. So this part of proof is omitted. Hence for $s=3,4$, we will prove this lemma in terms of q-differences. Using equality (7) for $s=3$ one has

$$\begin{aligned}\triangle_q^0 f_0=&f_0=0\\ \triangle_q^1 f_0=&f_1-f_0=\left(\frac{1}{[n]^2\psi_n\left(0\right)}\right)^3\\ \triangle_q^2 f_0=&f_2-\left(1+q\right)f_1+qf_0=\left(\frac{[2]}{[n]^2\psi_n\left(0\right)}\right)^3-\left(1+q\right)\left(\frac{1}{[n]^2\psi_n\left(0\right)}\right)^3\\ \triangle_q^3 f_0=&f_3-\left(1+q+q^2\right)f_2+\left(q+q^2+q^3\right)f_1-q^3f_0\\ =&\left(\frac{[3]}{[n]^2\psi_n\left(0\right)}\right)^3-\left(1+q+q^2\right)\left(\frac{[2]}{[n]^2\psi_n\left(0\right)}\right)^3\\ &+\left(q+q^2+q^3\right)\left(\frac{1}{[n]^2\psi_n\left(0\right)}\right)^3.\end{aligned}$$

By using Theorem 2, we can write:

$$
\begin{aligned}
L_n\left(t^3;q;x\right) =&\triangle_q^0 f_0+\triangle_q^1 f_0\frac{[n]}{[1]!}q\alpha_n\psi_n\left(0\right)K^q_{n+m,-1}\left(0,0,\alpha_n\psi_n\left(0\right)\right)x+\\
&+\triangle_q^2 f_0\frac{[n][n+m]}{[2]!}(q\alpha_n\psi_n\left(0\right))^2K^q_{n+2m,-2}\left(0,0,\alpha_n\psi_n\left(0\right)\right)x^2+\\
&+\triangle_q^3 f_0\frac{[n][n+m][n+2m]}{[3]!}(q\alpha_n\psi_n\left(0\right))^3K^q_{n+3m,-3}\left(0,0,\alpha_n\psi_n\left(0\right)\right)x^3.
\end{aligned}
$$

Taking into account condition (vi), by a straightforward calculation, we obtain equality (12).

We now prove (13). For $s=4$, we have:

$$
\begin{aligned}
\triangle_q^0 f_0 =&f_0=0\\
\triangle_q^1 f_0 =&f_1-f_0=\left(\frac{1}{[n]^2\psi_n\left(0\right)}\right)^4\\
\triangle_q^2 f_0 =&f_2-\left(1+q\right)f_1+qf_0=\left(\frac{[2]}{[n]^2\psi_n\left(0\right)}\right)^4-\left(1+q\right)\left(\frac{1}{[n]^2\psi_n\left(0\right)}\right)^4\\
\triangle_q^3 f_0 =&f_3-\left(1+q+q^2\right)f_2+\left(q+q^2+q^3\right)f_1-q^3f_0\\
=&\left(\frac{[3]}{[n]^2\psi_n\left(0\right)}\right)^4-\left(1+q+q^2\right)\left(\frac{[2]}{[n]^2\psi_n\left(0\right)}\right)^4+\\
&+\left(q+q^2+q^3\right)\left(\frac{1}{[n]^2\psi_n\left(0\right)}\right)^4\\
\Delta_q^4 f_0 =&f_4-\left(1+q+q^2+q^3\right)f_3+\left(q+q^2+2q^3+q^4+q^5\right)f_2\\
&-\left(q^3+q^4+q^5+q^6\right)f_1+q^6f_0\\
=&\left(\frac{[4]}{[n]^2\psi_n\left(0\right)}\right)^4-\left(1+q+q^2+q^3\right)\left(\frac{[3]}{[n]^2\psi_n\left(0\right)}\right)^4\\
&+\left(q+q^2+2q^3+q^4+q^5\right)\left(\frac{[2]}{[n]^2\psi_n\left(0\right)}\right)^4-\left(q^3+q^4+q^5+q^6\right)\left(\frac{1}{[n]^2\psi_n\left(0\right)}\right)^4.
\end{aligned}
$$

Thus applying again Theorem 2 and using condition (vi), proceeding similarly, we get (13). $\square$

3. Weighted approximation

This section is devoted to obtaining weighted approximation properties of the q-Ibragimov-Gadjiev operators. Note that we will use a weighted Korovkin type theorem proved by Gadjiev [7]. When considering that theorem we should mention some notations: Let $\rho\left(x\right)=1+x^2$ be a weight function and $B_\rho\left[0,\infty\right)$ the set of all functions f defined on the semi-axis $\left[0,\infty\right)$ satisfying the condition $\left|f\left(x\right)\right|\leq M_f\rho\left(x\right)$, where M_f is a constant depending only on f. $C_\rho\left[0,\infty\right)$ denotes the subspace of all continuous functions belonging to $B_\rho\left[0,\infty\right)$ and $C^k_\rho\left[0,\infty\right)$ denotes the subspace of all functions $f \in C_\rho[0,\infty)$ with $\lim\limits_{|x|\to\infty} \frac{f(x)}{\rho(x)} = k_f < \infty$. Obviously $C_\rho^k[0,\infty)$ is a linear normed space with the $\rho-$norm:

$$\|f\|_\rho = \sup_{x\in[0,b_n)} \frac{|f(x)|}{\rho(x)},$$

where $\{b_n\}$ is a sequence of positive numbers, which has a finite or an infinite limit.

Theorem 4 (see [7])**.** *Let $\{T_n\}$ be a sequence of linear positive operators which are mappings from C_ρ into B_ρ satisfying the conditions:*

$$\lim_{n\to\infty} \left\|T_n(t^k;x) - x^k\right\|_\rho = 0, \ \ k = 0,1,2.$$

Then, for any function $f \in C_\rho^k$,

$$\lim_{n\to\infty} \|T_n f - f\|_\rho = 0.$$

In order to study convergence properties of operators (5) into the weighted approximation fields we consider the following modified form of the q-Ibragimov-Gadjiev operators.

Let $\{b_n\}$ be a sequence of positive numbers, which has a finite or an infinite limit and $\{\psi_n(t)\}$ be a sequence of functions in $C[0,b_n]$ such that $\psi_n(t) > 0$ for each $t \in [0,b_n]$, satisfying the condition:

$$\lim_{n\to\infty} \frac{1}{[n]^2\psi_n(0)} = 0. \tag{14}$$

Also, let $\{\alpha_n\}$ be a sequence of positive real numbers such that

$$\frac{\alpha_n}{[n]} = 1 + O\left(\frac{1}{[n]^2\,\psi_n(0)}\right). \tag{15}$$

Now, assume that a function $K_{n,\nu}^q(x,t,u)$ $(x,t\in[0,b_n],\ -\infty < u < \infty)$ depending on the three parameters n,ν and q satisfies the following conditions:

(i^o) The function $K_{n,\nu}^q(x,t,u)$ is infinitely q-differentiable with respect to u for fixed $x,t\in[0,b_n]$,

(ii^o) For any $x\in[0,b_n]$ and for all $n\in\mathbb{N}$,

$$\sum_{\nu=0}^{\infty} q^{\frac{\nu(\nu-1)}{2}} \left[D_{q,u}^{\nu} K_{n,\nu}^q(x,t,u)\,|_{\substack{u=\alpha_n\psi_n(t)\\ t=0}}\right] \frac{(-q\alpha_n\psi_n(0))^\nu}{[\nu]!} = 1,$$

(iii^o) $\left\{(-1)^\nu D_{q,u}^{\nu} K_{n,\nu}^q(x,t,u)\,|_{\substack{u=\alpha_n\psi_n(t)\\ t=0}}\right\} \geq 0 \ (\nu,n = 1,2,\ldots,\ x\in[0,b_n]),$

(iv^o) $D_{q,u}^{\nu} K_{n,\nu}^q(x,t,u)\,|_{\substack{u=\alpha_n\psi_n(t)\\ t=0}} = -[n]\,xq^{1-\nu}\left[D_{q,u}^{\nu-1} K_{n+m,\nu-1}^q(x,t,u)\,|_{\substack{u=\alpha_n\psi_n(t)\\ t=0}}\right]$ $(\nu,n = 1,2,\ldots,x\in[0,b_n])$, where m is a number such that $m+n$ is zero or a natural number.

Under above conditions we call operators (5) $L_n^*(f;q;.)$. It is clear that $L_n^*(f;q;.)$ holds for the equalities in Lemma 1. In the case $\lim\limits_{n\to\infty} b_n = A$ and $\lim\limits_{n\to\infty} \dfrac{\alpha_n}{[n]}$ =1 we obtain q-İbragimov-Gadjiev operators defined by (5).

Theorem 5. *Let $q = q_n$, $0 < q_n < 1$, satisfy the condition $(q_n) \to 1$ as $n \to \infty$. Then for each function $f \in C_\rho^k[0,\infty)$,*

$$\lim_{n\to\infty} \|L_n^*(f;q;.) - f\|_{\rho,[0,b_n]} = 0.$$

Proof. Making use of the Korovkin type theorem on weighted approximation, it is sufficient to verify the following three conditions:

$$\lim_{n\to\infty} \left\|L_n^*(t^k;q;x) - x^k\right\|_\rho = 0,\ k = 0,1,2. \tag{16}$$

From equality (9) clearly $\|L_n(1,q_n,x) - 1\|_\rho \to 0$ as $n \to \infty$ on $[0,b_n]$. From equalities (10) and (14) we have:

$$\begin{aligned}\sup_{x\in[0,b_n)} \frac{|L_n^*(t;q;x) - x|}{1+x^2} &\le \left|\frac{q\alpha_n}{[n]} - 1\right| \sup_{x\in[0,b_n)} \frac{x}{1+x^2} \\ &\le \left|\frac{q\alpha_n}{[n]} - 1\right|,\end{aligned}$$

hence the condition in (16) holds for $k = 1$.

Similarly, using equality (11) for $k = 2$ we can write:

$$\begin{aligned}\sup_{x\in[0,b_n)} \frac{\left|L_n^*(t^2;q;x) - x^2\right|}{1+x^2} &\le \left|\left(\frac{q\alpha_n}{[n]}\right)^2 \frac{q[n+m]}{[n]} - 1\right| \sup_{x\in[0,b_n)} \frac{x^2}{1+x^2} \\ &\quad + \frac{q\alpha_n}{[n]} \frac{1}{[n]^2\psi_n(0)} \sup_{x\in[0,b_n)} \frac{x}{1+x^2} \\ &\le \left|\left(\frac{q\alpha_n}{[n]}\right)^2 \frac{q[n+m]}{[n]} - 1\right| + \frac{q\alpha_n}{[n]} \frac{1}{[n]^2\psi_n(0)},\end{aligned}$$

which implies that (16) also holds for $k = 2$. Therefore, the desired result follows from Theorem 4. □

Now, we will give an estimate concerning the rate of convergence for the q-Ibragimov-Gadjiev operators. As known, the first modulus of continuity $\omega(f;\delta)$ does not tend to zero as $\delta \to 0$ on the infinite interval. For this reason, we consider

$$\Omega(f;\delta) = \sup_{|h|\le\delta, x\ge 0} \frac{|f(x+h) - f(x)|}{(1+h^2)(1+x^2)}$$

a weighted modulus of smoothness associated to the space $C_\rho^k[0,\infty)$.

$\Omega(f;\delta)$ possesses the following properties (see [12]):

- $\Omega(f;\delta)$ is a monotonically increasing function of δ, $\delta \ge 0$,

- for every $f \in C_\rho^k[0,\infty)$, $\lim\limits_{\delta\to 0}\Omega(f;\delta)=0$,
- for each positive value of λ

$$\Omega(f;\lambda\delta) \le 2(1+\lambda)\left(1+\delta^2\right)\Omega(f;\delta). \tag{17}$$

From inequality (17) and the definition of $\Omega(f;\delta)$ we get

$$|f(t)-f(x)| \le 2\left(1+\frac{|t-x|}{\delta}\right)\left(1+\delta^2\right)\Omega(f;\delta)\left(1+x^2\right)\left(1+(t-x)^2\right), \tag{18}$$

for every $f \in C_\rho^k[0,\infty)$ and $x,t \in [0,\infty)$.

Theorem 6. *Let $q=q_n$, $0<q_n<1$ satisfies the condition $q_n \to 1$ as $n\to\infty$. If $f\in C_\rho^k[0,\infty)$, then the inequality*

$$\sup_{x\ge 0}\frac{|L_n^*(f;q;x)-f(x)|}{(1+x^2)^3} \le M\Omega\left(f;\left([n]^2\psi_n(0)\right)^{-1/2}\right)$$

is satisfied for a sufficiently large n, where M is a constant independent of α_n, $\psi_n(0)$.

Proof. From (18) we can write:

$$\begin{aligned}
|L_n^*(f;q;x)-f(x)| \le & 2\left(1+\delta_n^2\right)\Omega(f;\delta_n)\left(1+x^2\right)\sum_{\nu=0}^{\infty}P_{\nu,n}^q(x)\\
& \times\left(1+\frac{\left|\frac{[\nu]}{[n]^2\psi_n(0)}-x\right|}{\delta_n}\right)\left(1+\left(\frac{[\nu]}{[n]^2\psi_n(0)}-x\right)^2\right)\\
\le & 4\left(1+x^2\right)\Omega(f;\delta_n)\left\{1+\frac{1}{\delta_n}\sum_{\nu=0}^{\infty}\left|\frac{[\nu]}{[n]^2\psi_n(0)}-x\right|P_{\nu,n}^q(x)\right.\\
& +\sum_{\nu=0}^{\infty}\left(\frac{[\nu]}{[n]^2\psi_n(0)}-x\right)^2P_{\nu,n}^q(x)\\
& \left.+\frac{1}{\delta_n}\sum_{\nu=0}^{\infty}\left|\frac{[\nu]}{[n]^2\psi_n(0)}-x\right|\left(\frac{[\nu]}{[n]^2\psi_n(0)}-x\right)^2P_{\nu,n}^q(x)\right\},
\end{aligned}$$

where $P_{\nu,n}^q(x)=q^{\frac{\nu(\nu-1)}{2}}\left[D_{q,u}^{\nu}K_{n,\nu}^q(x,t,u)\,|_{\substack{u=\alpha_n\psi_n(t)\\ t=0}}\right]\frac{(-q\alpha_n\psi_n(0))^{\nu}}{[\nu]!}$ and $\delta_n>0$. Applying the Cauchy-Schwarz inequality we obtain

$$|L_n^*(f;q;x)-f(x)| \le 4\left(1+x^2\right)\Omega(f;\delta_n)\left(1+\frac{2}{\delta_n}\sqrt{K_1}+K_1+\frac{1}{\delta_n}K_2\right), \tag{19}$$

where

$$K_1 = \sum_{\nu=0}^{\infty} \left(\frac{[\nu]}{[n]^2 \psi_n(0)} - x \right)^2 P_{\nu,n}^q(x)$$
$$K_2 = \sum_{\nu=0}^{\infty} \left(\frac{[\nu]}{[n]^2 \psi_n(0)} - x \right)^4 P_{\nu,n}^q(x).$$

Using Lemma 1, we have

$$\begin{aligned} K_1 &= L_n^*\left((t-x)^2; q; x\right) \\ &= \left(1 - 2q\left(\frac{\alpha_n}{[n]}\right) + \left(\frac{\alpha_n}{[n]}\right)^2 \frac{q^3[n+m]}{[n]}\right)x^2 + q\left(\frac{\alpha_n}{[n]}\right)\frac{1}{[n]^2\psi_n(0)}x \end{aligned}$$

and

$$\begin{aligned} K_2 =& L_n^*\left((t-x)^4; q; x\right) \\ =& \left(\left(\frac{\alpha_n}{[n]}\right)^4 \frac{q^{10}[n+m][n+2m][n+3m]}{[n]^3} - 4\left(\frac{\alpha_n}{[n]}\right)^3 \frac{q^6[n+m][n+2m]}{[n]^2}\right. \\ &\left. +6\left(\frac{\alpha_n}{[n]}\right)^2 \frac{q^3[n+m]}{[n]} - 4q\left(\frac{\alpha_n}{[n]}\right) + 1\right)x^4 \\ &+ \left(\left(\frac{\alpha_n}{[n]}\right)^3 \frac{\left(q^8+2q^7+3q^6\right)[n+m][n+2m]}{[n]^2}\frac{1}{[n]^2\psi_n(0)}\right. \\ &\left. -4\left(\frac{\alpha_n}{[n]}\right)^2 \frac{\left(q^4+2q^3\right)[n+m]}{[n]}\frac{1}{[n]^2\psi_n(0)} + 6q\left(\frac{\alpha_n}{[n]}\right)\frac{1}{[n]^2\psi_n(0)}\right)x^3 \\ &+ \left(\left(\frac{\alpha_n}{[n]}\right)^2 \frac{\left(q^5+3q^4+3q^3\right)[n+m]}{[n]}\frac{1}{\left([n]^2\psi_n(0)\right)^2} - 4q\left(\frac{\alpha_n}{[n]}\right)\frac{1}{\left([n]^2\psi_n(0)\right)^2}\right)x^2 \\ &+ q\left(\frac{\alpha_n}{[n]}\right)\frac{1}{\left([n]^2\psi_n(0)\right)^3}x. \end{aligned}$$

Using condition (15) we can write

$$K_1 = O\left(\frac{1}{[n]^2\psi_n(0)}\right)\left(x^2+x\right)$$

and

$$K_2 = O\left(\frac{1}{[n]^2\psi_n(0)}\right)\left(x^4+x^3+x^2+x\right),$$

thus by substituting these equalities (19) becomes

$$\begin{aligned} \left|L_n^*(f;q;x) - f(x)\right| \le& 4\left(1+x^2\right)\Omega(f;\delta_n)\left\{1 + \frac{2}{\delta_n}\sqrt{O\left(\frac{1}{[n]^2\psi_n(0)}\right)\left(x^2+x\right)} + \right. \\ &\left. +O\left(\frac{1}{[n]^2\psi_n(0)}\right)\left(x^2+x\right) + \frac{1}{\delta_n}O\left(\frac{1}{[n]^2\psi_n(0)}\right)\left(x^4+x^3+x^2+x\right)\right\}. \end{aligned}$$

By choosing $\delta_n = \left([n]^2 \psi_n(0)\right)^{-1/2}$, for sufficiently large n's, we obtain the desired result. $\square$

Remark 2. *It will be noted that, for $q = 1$, this is the result obtained by Gadjiev and Ispir [8] for Ibragimov-Gadjiev operators defined by* (1).

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