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Ties in Worst-Case Analysis of the Euclidean Algorithm

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Abstract. We determine all pairs of positive integers below a given bound that require the most steps in the Euclidean algorithm. Also, we find asymptotic probabilities for a unique maximum pair or an even number of them. Our primary tools are continuant polynomials and the Zeckendorf representation using Fibonacci numbers.

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1. Introduction

When using the Euclidean algorithm to find the greatest common divisor of all pairs of integers each in the range from 1 to m, which pairs take the maximum number of steps? Table 1 shows the number of steps for all pairs bounded by 12; there is an 8-way tie for pairs that require the maximum four steps that arise in this range.

This worst-case analysis of the Euclidean algorithm famously involves the Fibonacci numbers, given by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. We give Knuth's version of the result.

Theorem 1. *For $n > 1$, let u and v be integers with $u > v > 0$ such that Euclid's algorithm applied to u and v requires exactly n steps, and such that u is as small as possible satisfying these conditions. Then $u = F_{n+2}$ and $v = F_{n+1}$.*

He remarks after the proof, "This theorem has the historical claim of being the first practical application of the Fibonacci sequence" [2, p. 360]. See [7] for the 19th century contributions of the French mathematicians Reynaud, Léger, Finck, Binet, and Lamé to what Shallit considers the first analysis of an algorithm in the modern sense.

Here we give a more nuanced analysis of this foundational algorithm: Given $m \geq 1$, what are all pairs (u, v) with $m \geq u \geq v > 0$ that require the maximum number of steps? The smallest such pair is provided by Theorem 1: Let $u = F_{n+2}$ be the greatest Fibonacci number less than or equal to m; then computing the greatest common divisor of $(u, v) = (F_{n+2}, F_{n+1})$ takes n steps, the maximum in this range. In the $m = 12$ example of Table 1, this is the pair $(F_6, F_5) = (8, 5)$, which requires

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$u \setminus v$	1	2	3	4	5	6	7	8	9	10	11	12
1	0	1	1	1	1	1	1	1	1	1	1	1
2	1	0	2	1	2	1	2	1	2	1	2	1
3	1	2	0	2	3	1	2	3	1	2	3	1
4	1	1	2	0	2	2	3	1	2	2	3	1
5	1	2	3	2	0	2	3	$\boxed{4}$	3	1	2	3
6	1	1	1	2	2	0	2	2	2	3	3	1
7	1	2	2	3	3	2	0	2	3	3	$\boxed{4}$	$\boxed{4}$
8	1	1	3	1	$\boxed{4}$	2	2	0	2	2	$\boxed{4}$	2
9	1	2	1	2	3	2	3	2	0	2	3	2
10	1	1	2	2	1	3	3	2	2	0	2	2
11	1	2	3	3	2	3	$\boxed{4}$	$\boxed{4}$	3	2	0	2
12	1	1	1	1	3	1	$\boxed{4}$	2	2	2	2	0

Table 1: *The number of steps required to find the greatest common divisor of u and v using the Euclidean algorithm, with occurrences of the maximum number indicated.*

four steps. Notice that the pairs $(11, 7)$, $(11, 8)$, and $(12, 7)$ also require four steps. How can we describe these worst-case ties in general?

The remaining sections of the paper are as follows. Section 2 gives an overview of continued fractions and continuant polynomials. Section 3 develops results on continuants which, along with Proposition 4, may be of independent interest. Section 4 presents our primary results in the form of a theorem and an algorithm: how to find all pairs (u, v) with $m \geq u \geq v > 0$ that require the maximum number of steps in the Euclidean algorithm, with two detailed examples. Section 5 establishes asymptotic results on the proportion of bounds that have certain numbers of pairs tying for the worst case. Finally, Section 6 discusses families of integer sequences that give rise to the entries of the pairs that tie for the worst case.

2. Continued fractions and continuant polynomials

Suppose u and v are integers with $u \geq v > 0$. Using the Euclidean algorithm to find the greatest common divisor of u and v is essentially equivalent to determining the simple continued fraction for u/v. For example, $u = 11$ and $v = 8$ from Table 1 have

$$11 = 1 \cdot 8 + 3 \qquad \frac{11}{8} = 1 + \frac{3}{8} = 1 + \frac{1}{\frac{8}{3}}$$

$$8 = 2 \cdot 3 + 2 \qquad = 1 + \frac{1}{2 + \frac{2}{3}} = 1 + \frac{1}{2 + \frac{1}{\frac{3}{2}}}$$

$$3 = 1 \cdot 2 + 1 \qquad\qquad = 1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2}}}$$

$$2 = 2 \cdot 1 + 0.$$

Since numerators in regular continued fractions are all 1, the ordered list of partial denominators $[1, 2, 1, 2]$ completely describes $11/8$. Notice that the partial denominators are exactly the coefficients arising from the Euclidean algorithm, $1, 2, 1, 2$, respectively.

Euler developed continuants in his general work on continued fractions, working out continued fractions with variable entries:

$$x_1 + \frac{1}{x_2} = \frac{x_1x_2 + 1}{x_2},$$

$$x_1 + \cfrac{1}{x_2 + \cfrac{1}{x_3}} = \frac{x_1x_2x_3 + x_1 + x_3}{x_2x_3 + 1},$$

$$x_1 + \cfrac{1}{x_2 + \cfrac{1}{x_3 + \cfrac{1}{x_4}}} = \frac{x_1x_2x_3x_4 + x_1x_2 + x_1x_4 + x_3x_4 + 1}{x_2x_3x_4 + x_2 + x_4}.$$

Notice, for instance, that the denominator of $[x_1, x_2, x_3]$ is the previous numerator x_1x_2+1 with the indices increased by one; likewise the denominator of $[x_1, x_2, x_3, x_4]$ and the numerator of $[x_1, x_2, x_3]$.

A recursive definition of continuant polynomials is

$$K(x_1, \ldots, x_n) = \begin{cases} 1 & \text{if } n = 0, \\ x_1 & \text{if } n = 1, \\ x_1K(x_2, \ldots, x_n) + K(x_3, \ldots, x_n) & \text{if } n > 1. \end{cases}$$

The next terms are $K(x_1, x_2) = x_1x_2 + 1$, $K(x_1, x_2, x_3) = x_1x_2x_3 + x_1 + x_3$, and $K(x_1, x_2, x_3, x_4) = x_1x_2x_3x_4 + x_1x_2 + x_1x_4 + x_3x_4 + 1$ that arose above.

With this notation, the connection to continued fractions is

$$[a_1, \ldots, a_n] = \frac{K(a_1, \ldots, a_n)}{K(a_2, \ldots, a_n)}.$$

One can verify that $K(1, 2, 1, 2) = 11$ and $K(2, 1, 2) = 8$ in the example above. It is helpful to work out another pair from Table 1; $11/7 = [1, 1, 1, 3]$. Although $11 = K(1, 2, 1, 2) = K(1, 1, 1, 3)$, notice that removing the first entry of the second continuant leaves $K(1, 1, 3) = 7$ so that $K(1, 1, 1, 3)$ corresponds to the pair $(11, 7)$.

Among the many general properties of continuants, we will need two, which follow directly from the definition. See [2, pp. 356–359] for more details.

Proposition 1.

(*a*) $K(x_1, \ldots, x_n) = K(x_n, \ldots, x_1)$.

(*b*) *If* $x_n > 1$, *then* $K(x_1, \ldots, x_n) = K(x_1, \ldots, x_n - 1, 1)$.

3. Fibonacci numbers and continuants

We begin with some results on continuants with small positive integer entries. We use superscripts to denote repetition: c^a means c listed a times.

Proposition 2. *For integers $a, b \geq 0$ (sufficiently large for certain claims) and $m \geq 1$,*

(*a*) $K(1^a) = K(1^{a-2}, 2) = K(2, 1^{a-2}) = K(2, 1^{a-4}, 2) = F_{a+1}$.

(*b*) $K(1^a, m, 1^b) = (mF_{a+1} + F_a)F_{b+1} + F_{a+1}F_b$.

Proof. (a) From the definition of continuant polynomials, $K() = 1 = F_1$ and $K(1) = 1 = F_2$. For $a \geq 2$,

$$K(1^a) = K(1^{a-1}) + K(1^{a-2}) = F_a + F_{a-1} = F_{a+1}.$$

The other expressions follow from Proposition 1.

(b) We proceed by induction on a with two base cases. When $a = 0$,

$$K(m, 1^b) = mK(1^b) + K(1^{b-1}) = (mF_1 + F_0)F_{b+1} + F_1F_b.$$

When $a = 1$,

$$K(1, m, 1^b) = K(m, 1^b) + K(1^b) = (mF_2 + F_1)F_{b+1} + F_2F_b.$$

Assuming the claim for $a - 1$ and $a - 2$, we have

$$\begin{aligned} K(1^a, m, 1^b) &= K(1^{a-1}, m, 1^b) + K(1^{a-2}, m, 1^b) \\ &= [(mF_a + F_{a-1})F_{b+1} + F_aF_b] \\ &\quad + [(mF_{a-1} + F_{a-2})F_{b+1} + F_{a-1}F_b] \\ &= (mF_{a+1} + F_a)F_{b+1} + F_{a+1}F_b. \end{aligned}$$

□

Recall the Lucas numbers, given by $L_0 = 2$, $L_1 = 1$, $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$, and $L_{-n} = (-1)^n L_n$. We will use the well known identities $L_n = F_{n+1} + F_{n-1}$ and $5F_n = L_{n+1} + L_{n-1}$. Also, using the golden ratio $\varphi = (1+\sqrt{5})/2$ and $\psi = (1-\sqrt{5})/2$, we have $L_n = \varphi^n + \psi^n$ while $F_n = (\varphi^n - \psi^n)/\sqrt{5}$.

The following identity from Stone [9, Example 3] allows us to express the sum of certain products of Fibonacci numbers in terms of Lucas numbers.

Lemma 1. *For integers a, b, c, d such that $a + b = c + d$,*

$$F_aF_b + F_cF_d = \frac{2}{5}L_{a+b} + \frac{(-1)^{b+1}}{5}L_{a-c}L_{a-d}.$$

Now we show how to express continuants whose entries are ones and a single two in terms of the Fibonacci numbers.

Proposition 3. *Given positive integers n and a with $n \geq 3$ and $1 \leq a \leq n/2$,*

$$K(1^a, 2, 1^{n-a}) = \begin{cases} F_{n+2} + \displaystyle\sum_{k=0}^{(a-1)/2} F_{n-4k} & \text{for } a \text{ odd,} \\ F_{n+2} + \left[\displaystyle\sum_{k=0}^{(a-2)/2} F_{n-4k}\right] + F_{n+1-2a} & \text{for } a \text{ even.} \end{cases}$$

Proof. By Proposition 2(b),

$$\begin{aligned} K(1^a, 2, 1^{n-a}) &= (2F_{a+1} + F_a)F_{n-a+1} + F_{a+1}F_{n-a} \\ &= F_{n-a+1}(F_{a+1} + F_a) + F_{a+1}(F_{n-a+1} + F_{n-a}) \\ &= F_{n-a+1}F_{a+2} + F_{a+1}F_{n-a+2}. \end{aligned}$$

By Lemma 1, we have

$$F_{n-a+1}F_{a+2} + F_{a+1}F_{n-a+2} = \frac{2}{5}L_{n+3} + \frac{(-1)^a}{5}L_{n-2a} \tag{1}$$

since $L_{-1} = -1$.

Next, consider $\sum_{k=0}^{m} F_{n-4k}$ for $m \geq 0$. With φ and ψ as defined above,

$$\varphi^4 - 1 = (\varphi^2 - 1)(\varphi^2 + 1) = \varphi^2(\varphi - \psi) = \sqrt{5}\varphi^2$$

and, similarly, $\psi^4 - 1 = -\sqrt{5}\psi^2$. Therefore,

$$\begin{aligned} \sum_{k=0}^{m} F_{n-4k} &= \frac{1}{\sqrt{5}} \sum_{k=0}^{m} (\varphi^{n-4k} - \psi^{n-4k}) \\ &= \frac{\varphi^n}{\sqrt{5}} \left(\frac{1 - \varphi^{-4(m+1)}}{1 - \varphi^{-4}} \right) - \frac{\psi^n}{\sqrt{5}} \left(\frac{1 - \psi^{-4(m+1)}}{1 - \psi^{-4}} \right) \\ &= \frac{\varphi^{n-4m}(\varphi^{4m+4} - 1)}{5\varphi^2} + \frac{\psi^{n-4m}(\psi^{4m+4} - 1)}{5\psi^2} \\ &= \frac{1}{5}(\varphi^{n+2} + \psi^{n+2} - \varphi^{n-4m-2} - \psi^{n-4m-2}) \\ &= \frac{1}{5}(L_{n+2} - L_{n-4m-2}). \end{aligned}$$

To complete the proof, consider the parity of a. In each case, we use the identity $F_n = (L_{n-1} + L_{n+1})/5$.

First, suppose a is odd and set $m = (a-1)/2$ in the previous computation. Then

$$F_{n+2} + \sum_{k=0}^{(a-1)/2} F_{n-4k} = F_{n+2} + \frac{1}{5}L_{n+2} - \frac{1}{5}L_{n-2a}$$

$$= \frac{1}{5}(L_{n+1} + L_{n+3}) + \frac{1}{5}L_{n+2} - \frac{1}{5}L_{n-2a}$$
$$= \frac{2}{5}L_{n+3} + \frac{-1}{5}L_{n-2a}.$$

Second, suppose a is even and set $m = (a-2)/2$. Then

$$\begin{aligned} F_{n+2} &+ \left[\sum_{k=0}^{(a-2)/2} F_{n-4k}\right] + F_{n+1-2a} \\ &= \frac{1}{5}(L_{n+1} + L_{n+3}) + \frac{1}{5}(L_{n+2} - L_{n+2-2a}) + \frac{1}{5}(L_{n-2a} + L_{n+2-2a}) \\ &= \frac{2}{5}L_{n+3} + \frac{1}{5}L_{n-2a}. \end{aligned}$$

In both cases, the expressions match (1). □

4. Main results

In light of Proposition 1(b), known as "decoupling the unit," we now write continuants with $x_n \geq 2$. This matches the convention of not having 1 as the final term in a regular continued fraction.

Here is the heuristic for our project. Given the bound m and the maximal $u = F_{n+2} \leq m$, we know $u = K(1^{n-1}, 2)$ is part of the minimal pair for which the Euclidean algorithm requires n steps. In fact, any n positive integers $a_1, \ldots, a_n$ with $a_n \geq 2$ give a $u = K(a_1, \ldots, a_n)$ that is part of a pair for which the Euclidean algorithm requires n steps. The challenge is to find the u which are less than $F_{n+3} = K(2, 1^{n-2}, 2)$ and also bounded by $m < F_{n+3}$.

The following proposition addresses the F_{n+3} bound.

Proposition 4. *Given positive integers $a_1, \ldots, a_n$ with $\sum_{i=1}^n a_i \geq n+3$ and $a_n \geq 2$, the continuant $K(a_1, \ldots, a_n) \geq F_{n+3}$.*

Proof. Let $\sum_{i=1}^n a_i = n+3$; continuant values of the required form cannot be less for greater sums. There are three possibilities for the unordered collection of a_i values. With guidance from [6], the entries giving minimal values are

- $K(2,2,2) = 12 > 8 = F_6$, $K(1,2,2,2) = 17 > 13 = F_7$, and, for $n \geq 5$,
$$K(1,2,1,2,1^{n-5},2) = F_{n+3} + F_{n-1} + F_{n-3} + F_{n-6}.$$
- $K(3,2) = 7 > 5 = F_5$ and $K(1,3,1^{n-3},2) = F_{n+3} + F_{n-2}$ for $n \geq 3$.
- $K(4) = 4 > 3 = F_4$, $K(1,4) = 5 = F_5$, and $K(1^{n-1},4) = K(1,3,1^{n-3},2)$ for $n \geq 3$.

We leave the details, similar to previous propositions, to the reader.

Note that $K(1,4) = F_5$ is the only case of equality here; in all other cases, $K(a_1, \ldots, a_n) > F_{n+3}$. □

Recall the Zeckendorf representation of a positive integer n: There is a unique collection of nonconsecutive Fibonacci numbers whose sum is n [10]. For instance, $49 = 34+13+2 = F_9+F_7+F_3$. Write $n = \sum \delta_i F_i$, where $\delta_i = 1$ if F_i is included in the Zeckendorf representation of n, otherwise $\delta_i = 0$. This establishes a correspondence between the Zeckendorf representation and a binary string without adjacent ones, e.g., $49 = 101000100_F$.

Table 2 shows an example of the continuant values in increasing order relative to the a values.

a	$K(1^a, 2, 1^{n-a})$
1	$10\ 1000\ 0000\ 0000_F = 521$
3	$10\ 1000\ 1000\ 0000_F = 542$
5	$10\ 1000\ 1000\ 1000_F = 545$
6	$10\ 1000\ 1000\ 1001_F = 546$
4	$10\ 1000\ 1001\ 0000_F = 547$
2	$10\ 1001\ 0000\ 0000_F = 555$

Table 2: *Values for $K(1^a, 2, 1^{n-a})$ from Proposition 3 for $n = 12$ in increasing order. Spaces in the binary representation are for legibility.*

Part of Knuth's proof of Theorem 1 is that $K(x_1, \ldots, x_n)$, taking positive integer arguments with the last at least 2, is minimized with $x_1 = \cdots = x_{n-1} = 1$ and $x_n = 2$. These values have sum $n+1$. It is natural then to consider values satisfying $\sum_{i=1}^{n} x_i = n + 2$, again with $x_n \geq 2$.

Theorem 2. *Given a positive integer n, there are exactly n positive integer pairs (u, v) with $F_{n+2} \leq u < F_{n+3}$ and $v \leq u$ such that the Euclidean algorithm takes n steps to determine the greatest common divisor of u and v. The u values are $K(1^{n-1}, 2)$ and the results of adding one to a single argument of $K(1^{n-1}, 2)$ from the second to the nth, i.e.,*

$$K(1, 2, 1^{n-3}, 2), \ldots, K(1^{n-2}, 2, 2), K(1^{n-1}, 3).$$

In each case, the corresponding v is the continuant expression for u with the first argument removed.

Proof. Via the connection between continuants and the Euclidean algorithm, any $u = K(a_1, \ldots, a_n)$ with $a_n \geq 2$ corresponds to a positive integer pair (u, v), where $v = K(a_2, \ldots, a_n) \leq u$ for which the Euclidean algorithm takes n steps to determine the greatest common divisor of u and v. By Theorem 1, the smallest such is (F_{n+2}, F_{n+1}) corresponding to $u = K(1^{n-1}, 2)$. Note that this is the only allowed assignment of positive integers to the a_i with $\sum a_i = n + 1$. By Proposition 4, $K(a_1, \ldots, a_n) \geq F_{n+3}$ for any positive a_i with $\sum a_i \geq n + 3$.

It remains to consider positive a_i with $\sum a_i = n + 2$. The unordered collection of a_i values must be either 2 twos and $n-2$ ones or 1 three and $n-1$ ones. By the identity

$$K(1^a, 2, 1^{n-2-a}, 2) = K(1^a, 2, 1^{n-a}) = K(1^{n-a}, 2, 1^a) = K(1^{n-a}, 2, 1^{a-2}, 2),$$

Proposition 3 provides values for all u of the form $K(1^a, 2, 1^{n-2-a}, 2)$, where $2 \leq a \leq n-2$. Also,

$$K(1^{n-1}, 3) = K(1^{n-1}, 2, 1) = K(1, 2, 1^{n-1}) = K(1, 2, 1^{n-3}, 2).$$

Thus the u values occur in pairs except for $K(1^{n-1}, 2)$ and, when n is even, $K(1^\ell, 2, 1^{\ell-2}, 2)$, where $n = 2\ell$. However, the n expressions for the u values lead to n distinct pairs (u, v) since the corresponding continuants for v are not equal, i.e.,

$$K(1^{a-1}, 2, 1^{n-2-a}, 2) \neq K(1^{n-a-1}, 2, 1^{a-2}, 2)$$

and $K(1^{n-2}, 3) \neq K(2, 1^{n-3}, 2)$.

Since the Fibonacci expressions given in Proposition 3 are all less than F_{n+3}, these n pairs have u values in the required range. $\square$

Table 3 shows the results for $n = 7$ given by Theorem 2.

u	v
$K(1,1,1,1,1,1,2) = 100000000_F = 34$	$K(1,1,1,1,1,2) = 21$
$K(1,1,1,1,1,1,3) = 101000000_F = 47$	$K(1,1,1,1,1,3) = 29$
$K(1,2,1,1,1,1,2) = 101000000_F = 47$	$K(2,1,1,1,1,2) = 34$
$K(1,1,1,1,2,1,2) = 101000100_F = 49$	$K(1,1,1,2,1,2) = 30$
$K(1,1,1,2,1,1,2) = 101000100_F = 49$	$K(1,1,2,1,1,2) = 31$
$K(1,1,2,1,1,1,2) = 101001000_F = 50$	$K(1,2,1,1,1,2) = 29$
$K(1,1,1,1,1,2,2) = 101001000_F = 50$	$K(1,1,1,1,2,2) = 31$

Table 3: *The seven pairs (u, v) satisfying $34 \leq u < 55$ that each require seven steps in the Euclidean algorithm.*

The interested reader should compare our results thus far to the work of Merkes and Meyers [4]. The current article can be considered a refinement and expansion of their results, using a different approach and alternative representations. Their concluding theorem counts $n + 1$ pairs; they allow $u = F_{n+3}$ and thus include the pair arising from $u = K(2, 1^{n-2}, 2)$, namely (F_{n+3}, F_{n+1}).

We can now address our motivating question: how to identify and count ties for the maximum number of steps in the Euclidean algorithm for positive integers in a specified range.

Algorithm 1. *Given a positive integer bound m, the following procedure gives all positive integer pairs (u, v) with $v \leq u \leq m$ for which the Euclidean algorithm takes the maximum number of steps to determine the greatest common divisor of u and v.*

1. *Write the Zeckendorf representation of m and let F_{n+2} be the greatest Fibonacci number in the sum.*

2. *Compare the length $n+2$ binary number for m to the Zeckendorf representation binary numbers of the values from Proposition 3 ordered as in Table 2. Each value at most m will be the u value for one or two pairs.*

3. *Write each u from the previous step as a continuant in one or two ways as specified in the proof of Theorem 2.*

4. *For each expression of $u = K(a_1, \ldots, a_n)$, compute the corresponding $v = K(a_2, \ldots, a_n)$.*

Here are two examples of the algorithm, each building on previous computations.

Example 1. *Consider the bound $m = 12 = 101001_F$ in Table 1. The u values for pairs requiring four steps are $K(1,1,1,2) = 8 = 100000_F$, $K(1,1,1,3) = K(1,2,1,2) = 11 = 101000_F$, and $K(1,1,2,2) = 12$. The corresponding four continuants for v are $K(1,1,2) = 5$, $K(1,1,3) = 7$, $K(2,1,2) = 8$, and $K(1,2,2) = 7$, respectively, giving $(8,5)$, $(11,7)$, $(11,8)$, and $(12,7)$, the locations of the four boxed numerals 4 in the lower triangular part of Table 1. Decreasing the bound to $m = 11$ would exclude $u = K(1,1,2,2) = 12$. Decreasing the bound to $m = 8, 9, 10$ would leave just the pair $(8,5)$.*

Example 2. *Recall the example $49 = 101000100_F$ and compare this to the binary numbers in Table 3. The continuant value $u = 50$ exceeds $m = 49$, so the five pairs $(34,21)$, $(47,29)$, $(47,34)$, $(49,30)$, $(49,31)$ tie for requiring the maximum seven steps. From the same table, the bounds $m = 34, \ldots, 46$ have only the pair $(34,21)$ requiring seven steps, $m = 47, 48$ have three pairs, and $m = 50, \ldots, 54$ have seven pairs (then $m = 55$ has the unique pair $(55,34)$ requiring eight steps).*

5. Asymptotic results

Table 4 gives the number of pairs bounded by m requiring the maximum number of steps in the Euclidean algorithm.

m	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
pairs	1	1	1	2	1	1	3	1	1	1	3	4	1	1	1
	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
	1	1	3	5	5	1	1	1	1	1	1	1	1	3	4

Table 4: *The number of positive integer pairs (u, v) with $v \le u \le m$ requiring the maximum number of steps.*

The sums in Proposition 3 give Zeckendorf representations and allow us to establish results on the occurrences of 1 and even numbers in Table 4.

Corollary 1. *Given the bound m and the maximal $u = F_{n+2} \le m$, there is exactly one pair (u, v) with $v \le u \le m$ requiring the maximum n steps in the Euclidean algorithm if and only if $m < F_{n+2} + F_n$. Asymptotically, this is the case with probability $1/\varphi \sim 0.618$, where φ is the golden ratio.*

Proof. The pair (F_{n+2}, F_{n+1}) from Theorem 1 requires the maximum n steps in the Euclidean algorithm. By our results, the first tie occurs with $u = K(1,2,1^{n-3},2) = K(1^{n-1},3) = F_{n+2} + F_n$. So over the range $F_{n+2} \le m < F_{n+3}$ of length F_{n+1}, there is a single pair requiring the maximum number of steps for the F_n values of m with $F_{n+2} \leq m < F_{n+2} + F_n$, giving the asymptotic ratio

$$\lim_{n\to\infty} \frac{F_n}{F_{n+1}} = \frac{1}{\varphi}. \qquad \square$$

Corollary 2. *Given the bound m and the maximal $u = F_{n+2} \leq m$ with n odd, there is an odd number of pairs (u, v) with $v \leq u \leq m$ requiring the maximum n steps in the Euclidean algorithm. For n even, say $n = 2\ell$, there is an odd number of pairs if $m < K(1^\ell, 2, 1^{\ell-2}, 2)$, else an even number. Asymptotically, there is an even number of pairs with probability $1/(3\varphi + 1) \sim 0.171$, where φ is the golden ratio.*

Proof. Every m has the one pair (F_{n+2}, F_{n+1}) requiring the maximum n steps. For increasingly larger values of m, as explained in Algorithm 1, the u values usually come with two continuant expressions, leading to two pairs. The exception is when $n = 2\ell$ and $m \geq K(1^\ell, 2, 1^{\ell-2}, 2)$ as $K(1^\ell, 2, 1^{\ell-2}, 2)$ has just one continuant expression, leading to one additional pair which gives an even total number of pairs.

For the asymptotic result, for $n = 2\ell$, consider m values in the interval from F_{n+1} to F_{n+3}. There is an even number of pairs requiring the maximum number of Euclidean algorithm steps only for m values from $K(1^\ell, 2, 1^{\ell-2}, 2)$ to F_{n+3}, where, by the proof of Proposition 3, $K(1^\ell, 2, 1^{\ell-2}, 2) = 2F_{\ell+1}F_{\ell+2}$. The identity $F_{a+b} = F_aF_{b-1} + F_{a+1}F_b$ allows us to write

$$F_{2\ell+3} = F_{(\ell+1)+(\ell+2)} = F_{\ell+1}F_{\ell+1} + F_{\ell+2}F_{\ell+2} = F_{\ell+1}^2 + F_{\ell+2}^2.$$

Thus, the number of m values from F_{n+1} to F_{n+3} with an even number of pairs is

$$F_{2\ell+3} - 2F_{\ell+1}F_{\ell+2} = F_{\ell+1}^2 + F_{\ell+2}^2 - 2F_{\ell+1}F_{\ell+2} = (F_{\ell+2} - F_{\ell+1})^2 = F_\ell^2.$$

The ratio of m values with an even number of pairs is then

$$\frac{F_{2\ell+3} - 2F_{\ell+1}F_{\ell+2}}{F_{2\ell+2}} = \frac{F_\ell^2}{F_{\ell+1}(F_{\ell+1} + 2F_\ell)} = \frac{1}{\frac{F_{\ell+1}}{F_\ell}(\frac{F_{\ell+1}}{F_\ell} + 2)}.$$

Since $F_{\ell+1}/F_\ell \to \varphi$ as $\ell \to \infty$ and $\varphi^2 = \varphi + 1$, the last expression converges to

$$\frac{1}{\varphi(\varphi + 2)} = \frac{1}{3\varphi + 1}. \qquad \square$$

Other results of this type can be found. Sungkon Chang claims that the probability of exactly three pairs requiring the maximum number of steps approaches $(2 + 3\varphi)/(13 + 21\varphi) \sim 0.146$. Interestingly, this same proportion arises in [1]; there may be additional connections between this current project and his work.

6. Related integer sequences

Consider the sequence of m values in Table 4, where the number of pairs increases, i.e.,

$$4, 7, 11, 12, 18, 19, 29, 30, 31, 47, 49, 50, 76, 79, 80, 81, 123, 128, 129, \ldots. \qquad (2)$$

The proof of Corollary 2 shows that the only increases in Table 4 are by one or two, with increases of one at each $2F_{\ell+1}F_{\ell+2}$ for $\ell \geq 1$, i.e., the subsequence $4, 12, 30, 80, 208, 546, \ldots$ of (2). We conclude the article by describing how to determine the complete sequence (2), including the values where the number of pairs increases by two.

Looking at Table 1, the first tie occurs at $(4, 3)$, which, like $(3, 2)$, requires two steps in the Euclidean algorithm. This comes from $3 = K(1, 3)$. Then $K(1, 1, 3) = 7$ gives the pair $(7, 4)$, which, like $(5, 3)$, requires three steps, and $K(1, 1, 1, 3)$ gives $(11, 7)$, as discussed above. By the propositions, $K(1^{n-1}, 3) = F_{n+2} + F_n = L_{n+1}$, the Lucas numbers starting from 3.

The first tie not described by consecutive Fibonacci or Lucas numbers is $(7, 5)$ from $7 = K(1, 2, 2)$. The propositions show that $K(1^{n-2}, 2, 2) = F_{n+2} + F_n + F_{n-3}$, the sequence [8, A001060] starting from 5.

The first eleven such sequences are given in Table 5. The formulas come from the propositions. The first pair arising from each sequence has the form (L_{n+1}, F_{n+2}) from the initial terms of each formula.

first pair	formula
$(4,3)$	$F_{n+2} + F_n$
$(7,5)^*$	$F_{n+2} + F_n + F_{n-3}$
$(11,8)$	$F_{n+2} + F_n + F_{n-4}$
$(18,13)^*$	$F_{n+2} + F_n + F_{n-4} + F_{n-7}$
$(29,21)$	$F_{n+2} + F_n + F_{n-4} + F_{n-8}$
$(47,34)^*$	$F_{n+2} + F_n + F_{n-4} + F_{n-8} + F_{n-11}$
$(76,55)$	$F_{n+2} + F_n + F_{n-4} + F_{n-8} + F_{n-12}$
$(123,89)^*$	$F_{n+2} + F_n + F_{n-4} + F_{n-8} + F_{n-12} + F_{n-15}$
$(199,144)$	$F_{n+2} + F_n + F_{n-4} + F_{n-8} + F_{n-12} + F_{n-16}$
$(322,233)^*$	$F_{n+2} + F_n + F_{n-4} + F_{n-8} + F_{n-12} + F_{n-16} + F_{n-19}$
$(521,377)$	$F_{n+2} + F_n + F_{n-4} + F_{n-8} + F_{n-12} + F_{n-16} + F_{n-20}$

Table 5: *First eleven sequences producing pairs that tie for requiring the maximum number of steps in the Euclidean algorithm.*

The sequence (2) of m values where the number of pairs increases is the union of the infinite family of sequences given by initial values $A_0 = F_{n+2}$, $A_1 = L_{n+1}$, for each $n \geq 2$, and the recurrence $A_k = A_{k-1} + A_{k-2}$, for $k \geq 2$. The first few terms are shown in Table 6. One can see the pairs for the $m = 12$ example in the columns headed by 5 and 8, and the pairs for the $m = 49$ example among those in the columns headed by 21 and 34.

Interestingly, after some values, the starred sequences in Table 5 coincide with sequences related to the second largest, third largest, etc., terms in rows of Stern's diatomic triangle [3], in particular, [8, A244472–A244476], respectively. In fact, Paulin [5] uses continuants with small entries similar to ours to verify two conjectures of Lansing [3]. Perhaps there are other connections between Stern's diatomic triangle and the Euclidean algorithm to be found.

1	1	2	3	5	8	13	21	34	55
		3	4	7	11	18	29	47	76
			5	7	12	19	31	50	81
				8	11	19	30	49	79
					13	18	31	49	80
						21	29	50	79
							34	47	81
								55	76

Table 6: *Sequences giving entries in the pairs requiring the maximum number of steps in the Euclidean algorithm. The Fibonacci numbers in the first row are followed by rows given by the sequences described in Table 5.*

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