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Asymptotic analysis of a double integral occurring in the rough Bergomi model*

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Abstract. Recently, Forde et al. [The Rough Bergomi model as $H \to 0$ – skew flattening/blow up and non-Gaussian rough volatility; preprint] found an explicit expression for the third moment of the log-price in the rough Bergomi model, in terms of a double integral, whose integrand involves a hypergeometric function. One of the parameters of this financial market model, the Hurst parameter H, is observed to be small in practice. We analyse the third moment asymptotically as H tends to zero, using as our main tools hypergeometric transformation formulas and uniform asymptotic expansions for the incomplete gamma function.

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Key words: integral, asymptotics, hypergeometric function, incomplete gamma function

1. Introduction

The rough Bergomi model, introduced in [3], belongs to the family of rough volatility models, which are non-Markovian stochastic volatility models driven by fractional Brownian motion or related processes. For further information on the rough Bergomi model and many references on rough volatility, we refer to [6, 7, 9]. The smoothness of the paths, as well as the option prices the model delivers, depend crucially on the Hurst parameter $H \in (0, \frac{1}{2})$. As H appears to be close to zero in practice and handling rough volatility models numerically is challenging, asymptotic approximations for $H \downarrow 0$ are of interest. In [5], this question was investigated at the process level for a variant of the rough Bergomi model, and an expression for the third moment was obtained as a byproduct. To state the latter, define stochastic processes X^H, V^H, Z^H by

$$\begin{aligned} dX_t^H &= \sqrt{V_t^H}(\rho dB_t + \bar{\rho} dW_t), \\ V_t^H &= \exp\left(\gamma Z_t^H - \tfrac{1}{2}\gamma^2 \mathrm{Var}(Z_t^H)\right), \\ Z_t^H &= \int_0^t (t-s)^{H-1/2} dB_t, \end{aligned} \tag{1}$$

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where B and W are independent Brownian motions. The parameters are $H \in (0, \frac{1}{2})$, $\rho \in (-1, 1)$, $\bar{\rho}^2 = 1 - \rho^2$, and $\gamma > 0$. Then X^H is the log-price process of the variant of the rough Bergomi model we are going to study. It is important to note here that the standard parametrisation, as introduced in [3], includes a factor $\sqrt{2H}$ in (1), and then the rBergomi model tends weakly to the Black-Scholes model for $H \downarrow 0$ (see [5]). The present parametrisation is motivated by the rich asymptotic behavior, involving Gaussian multiplicative chaos, uncovered in [5]. The process Z^H is Gaussian and satisfies $\mathrm{Var}(Z_t^H) = t^{2H}/(2H)$. According to [5], we have

$$\mathbb{E}[(X_T^H)^3] = 3\gamma\rho \int_0^T \int_0^t \exp\Big(\tfrac{1}{2}\gamma^2\big(R_H(s,t) - s^{2H}/(8H)\big)\Big)(t-s)^{H-1/2} ds\, dt \tag{2}$$

for $T > 0$, where

$$R_H(s,t) := \int_0^{s\wedge t} (s-u)^{H-1/2}(t-u)^{H-1/2} du. \tag{3}$$

Consequently, the time derivative of the third moment is

$$\frac{\partial}{\partial t}\mathbb{E}[(X_t^H)^3] = 3\gamma\rho \int_0^t \exp\big(\tfrac{1}{2}\gamma^2(R_H(s,t) - s^{2H}/(8H))\big)(t-s)^{H-1/2} ds. \tag{4}$$

The purpose of this note is to establish logarithmic asymptotics for (2) and (4) as $H \downarrow 0$ for fixed $T > 0$ resp. $t > 0$; see Theorems 1 and 2 below. Results of this kind can be applied to compare the marginal distribution of the model to the observed risk-neutral distribution implied by quoted option prices. We will see that both (2) and (4) have the same first order exponential behavior. They decay or increase exponentially, according to whether γ is smaller or larger than the unique solution

$$\gamma_0 = 1.61710802076\ldots \tag{5}$$

in $(1, \infty)$ of the equation

$$-\frac{1}{4} - \frac{1}{2}\log\gamma + \frac{3\gamma^2}{16} = 0.$$

Indeed, the left hand side of this equation is the coefficient of the leading term $1/H$ in our logarithmic asymptotics. The constant γ_0 can be expressed by a branch of the Lambert W function (see [4]) as

$$\gamma_0 = \sqrt{-\tfrac{4}{3}W_{-1}\big(-3/(4e)\big)}.$$

We note that γ_0 is numerically close to the golden ratio, although this does not seem to have any significance. Since the variance

$$\mathrm{Var}[X_T^H] = \mathbb{E}[(X_T^H)^2] = T$$

of X_T^H does not depend on H, it is clear that Pearson's moment coefficient of skewness $\mathbb{E}[(X_T^H)^3]/\mathrm{Var}[X_T^H]^{3/2}$ has the same decay resp. growth behavior as $\mathbb{E}[(X_T^H)^3]$ for $H \downarrow 0$.

In all our statements, $\gamma > 0$ is a fixed parameter. Notation: we write

$$f(H) \asymp g(H) \tag{6}$$

if

$$f(H) = O(g(H)) \quad \text{and} \quad g(H) = O(f(H)), \quad H \downarrow 0.$$

We write C for various positive constants whose value is irrelevant. In Section 2, they may depend on t, which is a fixed parameter there. In Section 3, C is always independent of t. Moreover, define the function

$$\psi(H) := \exp(-\tfrac{1}{H\log(1/H)}), \tag{7}$$

which satisfies $\psi(H) = o(1)$ and $\psi(H)^H = 1 + O(1/\log\frac{1}{H})$ for $H \downarrow 0$.

2. Single integral

For $0 \le a \le b \le t$, define

$$I_H(a,b) := \int_a^b \exp\big(\tfrac{1}{2}\gamma^2(R_H(s,t) - s^{2H}/(8H))\big)(t-s)^{H-1/2}ds.$$

As $t > 0$ is fixed in this section, the dependence of the integrand on t is suppressed in the notation $I_H(a,b)$.

Theorem 1. *Fix $t > 0$. Then the integral in (4) satisfies*

$$I_H(0,t) = \exp\Big(-\frac{r(\gamma)}{H} + o\big(\frac{1}{H}\big)\Big)$$

as $H \downarrow 0$, where

$$r(\gamma) := \begin{cases} \gamma^2/16 & 0 < \gamma < 1, \\ \frac{1}{4} + \frac{1}{2}\log\gamma - 3\gamma^2/16 & \gamma \ge 1. \end{cases} \tag{8}$$

The rest of this section is devoted to the proof of this theorem, which will be divided into a sequence of lemmas.

Note that $r(\gamma) \le \gamma^2/16$, and that $r(\gamma)$ is negative for γ larger than its unique root (5), which makes the integral explode as $H \downarrow 0$ for such values of γ. We first recall some uniform asymptotic results for the incomplete gamma function, which will be used at several places. As usual, we write

$$\gamma(a,z) := \int_0^z e^{-w}w^{a-1}dw \tag{9}$$

for the lower incomplete gamma function, and

$$\Gamma(a,z) := \int_z^\infty e^{-w}w^{a-1}dw$$

for the upper incomplete gamma function.

Lemma 1. *For $z \uparrow \infty$, we have*

$$\gamma(a,z) = z^a \exp(-z + O(\log z)), \tag{10}$$

uniformly w.r.t. $a \geq z$. The upper incomplete gamma function satisfies

$$\Gamma(a,z) = z^a \exp(-z + O(\log z)) \tag{11}$$

as $z \uparrow \infty$, uniformly w.r.t. $a \leq z$.

Proof. This is a special case of the expansions in [10], to which we also refer for earlier references on the asymptotics of $\gamma(\cdot,\cdot)$ and $\Gamma(\cdot,\cdot)$. Indeed, (2.13) in [10] yields (10). Note that the expression in curly braces in (2.13) of [10] is of polynomial growth in z; polynomial expressions resp. bounds for c_k and R_n^- are explicitly given in [10], and the first formula on p. 328 of [10] yields the polynomial growth of d_k, because the exponential factor $e^{\chi^2/4}$ cancels with the exponential term in the expansion of the parabolic cylinder function D_{-k-1}, which is given on p. 1029 of [8]. Similarly, (11) follows from the formula above (2.15) in [10]. $\square$

We now express (3) in terms of the Gaussian hypergeometric function. Substituting $s-u=w$ and then $v=w/s$, we get

$$\begin{aligned} R_H(s,t) &= (t-s)^{H-1/2} s^{H+1/2} \int_0^1 v^{H-1/2}(1+\tfrac{s}{t-s}v)^{H-1/2} dv \\ &= (t-s)^{H-1/2} s^{H+1/2} \frac{\Gamma(H+\frac12)}{\Gamma(H+\frac32)}\, {}_2F_1\Big(\begin{matrix}\frac12 - H, H+\frac12 \\ H+\frac32\end{matrix}\Big| -\frac{s}{t-s}\Big) && (12) \\ &= t^{H-1/2} s^{H+1/2} \frac{1}{H+\frac12}\, {}_2F_1\Big(\begin{matrix}\frac12 - H, 1 \\ H+\frac32\end{matrix}\Big| \frac{s}{t}\Big). && (13) \end{aligned}$$

The second equality follows from Euler's integral representation of ${}_2F_1$, and the third from Pfaff's transformation identity (Theorems 2.2.1 and 2.2.5 in [2]). Theorem 1 is a consequence of the following four lemmas, and the (easily verified) fact that

$$r(\gamma) \leq \gamma^2/16, \quad \gamma > 0, \tag{14}$$

which is used to compare the decay rates of the exponential estimates obtained in the lemmas.

Lemma 2. *For $H \downarrow 0$, we have*

$$I_H(0,H) \leq \exp\Big(-\frac{r(\gamma)}{H} + o\big(\frac{1}{H}\big)\Big). \tag{15}$$

Proof. By (13), there is a constant C such that $R_H(s,t) \leq C s^{H+1/2}$ for $0 \leq s \leq H$ with H sufficiently small. From this it easily follows that

$$\exp\big(\tfrac12 \gamma^2 R_H(s,t)\big)(t-s)^{H-1/2} \asymp 1, \quad H \text{ small}, 0 \leq s \leq H,$$

and thus (recall notation (6))

$$
\begin{aligned}
I_H(0,H) &\asymp \int_0^H \exp\Big(-\frac{\gamma^2 s^{2H}}{16H}\Big)ds \\
&= \frac{A^{-1/(2H)}}{2H}\int_0^{AH^{2H}} e^{-w}w^{1/(2H)-1}dw \\
&= \frac{A^{-1/(2H)}}{2H}\gamma(1/(2H), AH^{2H}),
\end{aligned} \tag{16}
$$

where we define $A = A(H) := \gamma^2/(16H)$, and $\gamma(\cdot,\cdot)$ is the lower incomplete gamma function (9). We have

$$
AH^{2H} = \frac{\gamma^2}{16H}\exp\Big(-2H\log\frac{1}{H}\Big) = \frac{\gamma^2}{16H}\big(1+o(1)\big).
$$

First, suppose that $\gamma^2 < 8$. Then, the first argument of $\gamma(\cdot,\cdot)$ in (16) is larger than the second one for small H. It then follows from (10) that

$$
\begin{aligned}
\gamma(1/(2H), AH^{2H}) &= z^a e^{-z}|_{a=1/(2H),\, z=AH^{2H}} \times e^{o(1/H)} \\
&= (AH^{2H})^{1/(2H)}\exp\big(-AH^{2H}+o(1/H)\big) \\
&= A^{1/(2H)}\exp\Big(-\frac{\gamma^2}{16H}\big(1+o(1)\big)\Big).
\end{aligned}
$$

Together with (16) and (14) this implies (15).

Now assume $\gamma^2 \geq 8$. From (16), we obtain

$$
I_H(0,H) \leq e^{o(1/H)}A^{-1/(2H)}\Gamma(1/(2H)). \tag{17}
$$

From Stirling's formula, $\Gamma(z) = z^z e^{-z+o(z)}$, we get

$$
\Gamma(1/(2H)) = H^{-1/(2H)}\exp\Big(\frac{1}{2}\big(\log\frac{1}{2}-1\big)\frac{1}{H}+o\big(\frac{1}{H}\big)\Big).
$$

We clearly have

$$
A^{-1/(2H)} = H^{1/(2H)}\exp\Big(-\frac{1}{2}\log\big(\frac{\gamma^2}{16}\big)\frac{1}{H}\Big),
$$

and inserting the latter two equations into (17) yields

$$
\begin{aligned}
I_H(0,H) &= \exp\Big(-\frac{1}{2}\log\big(\frac{\gamma^2}{16}\big)\frac{1}{H}+\frac{1}{2}\big(\log\frac{1}{2}-1\big)\frac{1}{H}+o\big(\frac{1}{H}\big)\Big) \\
&= \exp\Big(-\frac{1}{2}\big(1+\log\big(\frac{\gamma^2}{8}\big)\big)\frac{1}{H}+o\big(\frac{1}{H}\big)\Big).
\end{aligned}
$$

Now (15) follows from $r(\gamma) \leq \frac{1}{2}\big(1+\log(\gamma^2/8)\big)$. $\square$

Recall the transformation formula 15.3.7 in [1],

$$
\begin{aligned}
{}_2F_1\Big(\begin{matrix}a,b\\c\end{matrix}\Big|z\Big) &= \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}\,{}_2F_1\Big(\begin{matrix}a,a-c+1\\a-b+1\end{matrix}\Big|\frac{1}{z}\Big) \\
&\quad + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}\,{}_2F_1\Big(\begin{matrix}b,b-c+1\\b-a+1\end{matrix}\Big|\frac{1}{z}\Big).
\end{aligned}
$$

Combining this with (12), we get

$$R_H(s,t) = \frac{1}{2H} s^{2H} {}_2F_1\Big(\begin{matrix}\frac12 - H, -2H\\ 1-2H\end{matrix}\Big| -\frac{t-s}{s}\Big) + \frac{\Gamma(H+\frac32)\Gamma(-2H)}{(H+\frac12)\Gamma(\frac12 - H)}(t-s)^{2H}. \quad (18)$$

Lemma 3. *Fix $t > 0$. Then, with ψ defined in* (7)*,*

$$I_H\big(H, t-\psi(H)\big) \le \exp\Big(-\frac{\gamma^2}{16H}\big(1+o(1)\big)\Big).$$

Proof. Fix $0 < \varepsilon < t$. From (13), we have $R_H(s,t) \to R_0(s,t)$ uniformly w.r.t. s for $H \le s \le \varepsilon$. This easily implies that

$$I_H(H,\varepsilon) \le \exp\Big(-\frac{\gamma^2}{16H}\big(1+o(1)\big)\Big),$$

and so it remains to prove

$$I_H\big(\varepsilon, t-\psi(H)\big) \le \exp\Big(-\frac{\gamma^2}{16H}\big(1+o(1)\big)\Big).$$

For $\varepsilon \le s \le t-\psi(H)$, $-(t-s)/s$ is bounded and bounded away from the singularity of ${}_2F_1$ at 1, and so (18) yields

$$\begin{aligned} R_H(s,t) &= \frac{1}{2H}\Big(s^{2H}(1+O(H)) - (t-s)^{2H}(1+O(H))\Big) \\ &= \frac{1}{2H}\big(s^{2H} - (t-s)^{2H}\big) + O(1), \quad \varepsilon \le s \le t-\psi(H). \end{aligned} \quad (19)$$

But for these s we have $s^{2H} = 1 + O(H)$ and

$$(t-s)^{2H} = 1 + O\Big(1/\log\frac{1}{H}\Big),$$

and thus

$$R_H(s,t) = O\Big(\frac{1}{H}\big(\log\frac{1}{H}\big)^{-1}\Big).$$

We conclude

$$\begin{aligned} &I_H\big(\varepsilon,\, t-\psi(H)\big) \\ &\quad \le \exp\Big(-\frac{\gamma^2}{16H} + O\Big(\frac{1}{H}\big(\log\frac{1}{H}\big)^{-1}\Big)\Big)\int_\varepsilon^t (t-s)^{H-1/2}ds \\ &\quad = \exp\Big(-\frac{\gamma^2}{16H}\big(1+o(1)\big)\Big). \end{aligned} \qquad \square$$

The following lemma identifies the main contribution to integral (4).

Lemma 4. *Fix $t > 0$. For any $c > 0$ sufficiently large,*

$$I_H\big(t-\psi(H), t-e^{-c/H}\big) = \exp\Big(-\frac{r(\gamma)}{H} + o\big(\frac{1}{H}\big)\Big).$$

Proof. It is easy to see that (19) holds for $s \geq t - \psi(H)$. Moreover, $s^{2H} \sim 1$ uniformly in the integration range, and so, with $B = B(H) = \gamma^2/(4H)$,

$$\begin{aligned}
&I_H\big(t-\psi(H), t-e^{-c/H}\big)\\
&\quad = \exp\Big(\frac{3\gamma^2}{16H} + o\big(\frac{1}{H}\big)\Big)\int_{t-\psi(H)}^{t-e^{-c/H}} \exp\Big(-\frac{\gamma^2}{4H}(t-s)^{2H}\Big)(t-s)^{H-1/2}ds\\
&\quad = \exp\Big(\frac{3\gamma^2}{16H} + o\big(\frac{1}{H}\big)\Big)B^{-1/(4H)}\int_{Be^{-2c}}^{B\exp(-2/\log\frac{1}{H})} e^{-w}w^{1/2+1/(4H)-1}dw \qquad (20)\\
&\quad = \exp\Big(\frac{3\gamma^2}{16H} + o\big(\frac{1}{H}\big)\Big)B^{-1/(4H)}\Big(\gamma\big(\tfrac12+\tfrac{1}{4H}, B\exp(-2/\log\tfrac{1}{H})\big)\\
&\qquad\qquad - \gamma\big(\tfrac12+\tfrac{1}{4H}, Be^{-2c}\big)\Big). \qquad (21)
\end{aligned}$$

First suppose that $\gamma \leq 1$. Then, by (10),

$$\gamma\big(\tfrac12+\tfrac{1}{4H}, B\exp(-2/\log\tfrac{1}{H})\big) = B^{1/(4H)}\exp\Big(-\frac{\gamma^2}{4H} + o\big(\frac{1}{H}\big)\Big),$$

and hence (21) yields

$$I_H\big(t-\psi(H), t-e^{-c/H}\big) \leq \exp\Big(-\frac{\gamma^2}{16H} + o\big(\frac{1}{H}\big)\Big).$$

Now let $\gamma > 1$ and $\gamma^2 e^{-2c} \leq 1$. The integral in (20) equals

$$\begin{aligned}
&\int_{Be^{-2c}}^{B\exp(-2/\log\frac{1}{H})} e^{-w}w^{1/2+1/(4H)-1}dw\\
&\quad = \Gamma\big(\tfrac12+\tfrac{1}{4H}\big) - \gamma\big(\tfrac12+\tfrac{1}{4H}, Be^{-2c}\big) - \Gamma\big(\tfrac12+\tfrac{1}{4H}, B\exp(-2/\log\tfrac{1}{H})\big). \qquad (22)
\end{aligned}$$

Stirling's formula yields

$$\Gamma\big(\tfrac12+\tfrac{1}{4H}\big) = B^{1/(4H)}\exp\Big(-\frac{1}{4H} - \frac{\log\gamma}{2H} + o\big(\frac{1}{H}\big)\Big). \qquad (23)$$

The other two terms in (22) can be treated by Lemma 1. The resulting estimates are

$$\gamma\big(\tfrac12+\tfrac{1}{4H}, Be^{-2c}\big) = B^{1/(4H)}\exp\Big(-\frac{c}{2H} - \frac{\gamma^2 e^{-2c}}{4H} + o\big(\frac{1}{H}\big)\Big)$$

and

$$\Gamma\big(\tfrac12+\tfrac{1}{4H}, B\exp(-2/\log\tfrac{1}{H})\big) = \exp\Big(-\frac{\gamma^2}{4H} + o\big(\frac{1}{H}\big)\Big).$$

As c is large, and $\gamma^2/4 \geq \frac14 + \frac12\log\gamma$, these are negligible compared to (23). We have thus shown that integral (22) can be replaced by (23) in (20), which yields the assertion. □

Lemma 5. *Fix $t > 0$. For any $c \geq \gamma^2/2$,*

$$I_H(t - e^{-c/H}, t) \leq \exp\Big(-\frac{\gamma^2}{16H}\big(1 + o(1)\big)\Big).$$

Proof. For any s, t with $0 \leq s \leq t \leq T$, we have the estimate

$$\begin{aligned} R_H(s,t) &\leq \int_0^s (s-u)^{2H-1} du \\ &= \frac{s^{2H}}{2H} \leq \frac{T^{2H}}{2H} = \frac{1}{2H}\big(1 + o(1)\big). \end{aligned} \tag{24}$$

Since $s^{2H}/(8H) \sim 1/(8H)$ in the integration range, we get

$$\begin{aligned} I_H(t - e^{-c/H}, t) &\leq \exp\Big(\frac{3\gamma^2}{16H}\big(1 + o(1)\big)\Big) \int_{t-e^{-c/H}}^t (t-s)^{H-1/2} ds \\ &= \exp\Big(\frac{3\gamma^2}{16H} - \frac{c}{2H} + o\Big(\frac{1}{H}\Big)\Big), \end{aligned}$$

which yields the assertion. $\square$

The proof of Theorem 1 is complete.

3. Double integral

We now show that (2) has the same first order exponential asymptotic behavior as (4).

Theorem 2. *Fix $T > 0$. Then the integral in (2) satisfies*

$$\int_0^T \int_0^t \exp\big(\tfrac{1}{2}\gamma^2(R_H(s,t) - s^{2H}/(8H))\big)(t-s)^{H-1/2} ds dt = \exp\Big(-\frac{r(\gamma)}{H} + o\big(\frac{1}{H}\big)\Big)$$

as $H \downarrow 0$, where $r(\gamma)$ is defined in (8).

We divide the integration domain into several parts depending on H, according to the following lemmas. The lower bound is established by Lemma 9, and the upper bound by Lemmas 6–12. We write C for various positive constants whose value is irrelevant. Unlike in the previous section, they do not depend on t. We write f for the integrand,

$$f(s,t,H) := \exp\big(\tfrac{1}{2}\gamma^2(R_H(s,t) - s^{2H}/(8H))\big)(t-s)^{H-1/2}.$$

Throughout this section, $T > 0$ is fixed.

Lemma 6. *For any $\hat{c} > 0$, there is $c > 0$ such that*

$$\int_0^{e^{-c/H}} \int_0^t f(s,t,H) ds dt \leq \exp\Big(-\frac{\hat{c}}{H}\big(1 + o(1)\big)\Big).$$

Proof. By (24),

$$\begin{aligned}\int_0^{e^{-c/H}}\int_0^t f(s,t,H)dsdt &\le \exp\Big(\frac{C}{H}\Big)\int_0^{e^{-c/H}}\int_0^t (t-s)^{H-1/2}dsdt\\ &= \exp\Big(\frac{C}{H}\big(1+o(1)\big)\Big)\int_0^{e^{-c/H}} t^{H+1/2}dt\\ &= \exp\Big(\frac{C}{H}-\frac{3c}{2H}+o\Big(\frac{1}{H}\Big)\Big).\end{aligned}$$

□

Lemma 7. *For any $\hat{c}>0$, there is $c>0$ such that*

$$\int_{e^{-c/H}}^T\int_0^{e^{-c/H}} f(s,t,H)dsdt \le \exp\Big(-\frac{\hat{c}}{H}\big(1+o(1)\big)\Big).$$

Proof. Again, using (24), we can estimate this integral by

$$\begin{aligned}&\exp\Big(\frac{C}{H}\Big)\int_{e^{-c/H}}^T\int_0^{e^{-c/H}}(t-s)^{H-1/2}dsdt\\ &\qquad = \exp\Big(\frac{C}{H}+o\Big(\frac{1}{H}\Big)\Big)\int_{e^{-c/H}}^T \big(t^{H+1/2}-(t-e^{-c/H})^{H+1/2}\big)dt.\end{aligned}$$

The assertion follows from

$$\begin{aligned}&\int_{e^{-c/H}}^T \big(t^{H+1/2}-(t-e^{-c/H})^{H+1/2}\big)dt\\ &= T^{H+3/2}-e^{-c(H+3/2)/H}-(T-e^{-c/H})^{H+3/2}\\ &= T^{H+3/2}-\exp\Big(-\frac{3c}{2H}+O(1)\Big)-T^{H+3/2}\exp\Big((H+\tfrac{3}{2})\log(1-T^{-1}e^{-c/H})\Big)\\ &= T^{H+3/2}-\exp\Big(-\frac{3c}{2H}+O(1)\Big)-T^{H+3/2}+O(e^{-c/H})\\ &\le \exp\Big(-\frac{c}{H}+O(1)\Big).\end{aligned}$$

□

Estimates very similar to Lemma 6 and Lemma 7, building on (24), work whenever the inner or the outer integral is taken over an exponentially small domain of size $e^{-c/H}$. Therefore, in what follows, we will omit these negligible integrals. For instance, in the following lemma we allow the inner integration to begin at $s=\frac{1}{2}e^{-c/H}$ instead of $s=0$ without further ado.

Lemma 8. *For any $c>0$, we have*

$$\int_{e^{-c/H}}^{\psi(H)}\int_{\frac{1}{2}e^{-c/H}}^{t/2} f(s,t,H)dsdt \le \exp\Big(-\frac{r(\gamma)}{H}+o\big(\frac{1}{H}\big)\Big).$$

Proof. Since s/t is bounded away from 1, the singularity of ${}_2F_1$, using (13) and $s\le t/2$, we get

$$R_H(s,t)\le Ct^{H-1/2}s^{H+1/2}\le Ct^{2H}=O(1) \tag{25}$$

and

$$(t-s)^{H-1/2} \le (t/2)^{H-1/2}.$$

Using these bounds and Fubini's theorem, we can bound the integral by

$$\begin{aligned}
C\int_{e^{-c/H}}^{\psi(H)}\int_{\frac{1}{2}e^{-c/H}}^{t/2} & t^{H-1/2}\exp\Big(-\frac{\gamma^2 s^{2H}}{16H}\Big)dsdt \\
&= C\int_{\frac{1}{2}e^{-c/H}}^{\frac{1}{2}\psi(H)}\exp\Big(-\frac{\gamma^2 s^{2H}}{16H}\Big)\int_{2s}^{\psi(H)} t^{H-1/2}dtds \\
&= \exp\Big(o\big(\frac{1}{H}\big)\Big)\int_{\frac{1}{2}e^{-c/H}}^{\frac{1}{2}\psi(H)}\exp\Big(-\frac{\gamma^2 s^{2H}}{16H}\Big)ds \\
&\le \exp\Big(o\big(\frac{1}{H}\big)\Big) I_H(0,H).
\end{aligned}$$

The statement now follows from Lemma 2. □

The following lemma proves the lower bound in Theorem 2 and another part of the upper bound.

Lemma 9. *For any $c>0$ sufficiently large, we have*

$$\int_{e^{-c/H}}^{\psi(H)}\int_{t/2}^{t-\frac{1}{2}e^{-c/H}} f(s,t,H)dsdt = \exp\Big(-\frac{r(\gamma)}{H}+o\big(\frac{1}{H}\big)\Big).$$

Proof. If we assume $s \ge t/2$, then $-(t-s)/s \in [-1,0]$ in (18) is bounded and bounded away from the singularity of ${}_2F_1$, and so, similarly to the proof of Lemma 3, we have

$$R_H(s,t) = \frac{1}{2H}\big(s^{2H}-(t-s)^{2H}\big)+O(1), \quad t/2 \le s \le t \le T. \tag{26}$$

Therefore, we see that the integral is

$$\asymp \int_{e^{-c/H}}^{\psi(H)}\int_{t/2}^{t-\frac{1}{2}e^{-c/H}}\exp\Big(\frac{\gamma^2}{4H}\big(s^{2H}-(t-s)^{2H}\big)-\frac{\gamma^2 s^{2H}}{16H}\Big)(t-s)^{H-1/2}dsdt. \tag{27}$$

We write $g(s,t,H)$ for the integrand in (27). From a drawing and Fubini's theorem, we see that the latter integral equals

$$\begin{aligned}
\int_{\frac{1}{2}e^{-c/H}}^{\frac{1}{2}\psi(H)}\int_{s+\frac{1}{2}e^{-c/H}}^{2s} & g(s,t,H)dtds \\
&+ \int_{\frac{1}{2}\psi(H)}^{\psi(H)-\frac{1}{2}e^{-c/H}}\int_{s+\frac{1}{2}e^{-c/H}}^{\psi(H)} g(s,t,H)dtds =: I^{(1)}(H)+I^{(2)}(H).
\end{aligned}$$

Now, defining $B = B(H) = \gamma^2/(4H)$, we have (similarly to the proof of Lemma 4)

$$\begin{aligned}I^{(1)}(H) &= \int_{\frac{1}{2}e^{-c/H}}^{\frac{1}{2}\psi(H)} \exp\Big(\frac{3\gamma^2 s^{2H}}{16H}\Big) \int_{s+\frac{1}{2}e^{-c/H}}^{2s} \exp\Big(-\frac{\gamma^2}{4H}(t-s)^{2H}\Big)(t-s)^{H-1/2} dtds \\ &= e^{o(\frac{1}{H})} B^{-1/(4H)} \int_{\frac{1}{2}e^{-c/H}}^{\frac{1}{2}\psi(H)} \exp\Big(\frac{3\gamma^2 s^{2H}}{16H}\Big) \\ &\qquad \Big(\gamma(\tfrac{1}{2}+\tfrac{1}{4H}, Bs^{2H}) - \gamma(\tfrac{1}{2}+\tfrac{1}{4H}, B(\tfrac{1}{2})^{2H}e^{-2c})\Big)ds \\ &=: I^{(1,1)}(H) - I^{(1,2)}(H).\end{aligned}$$

We now show that $I^{(1,2)}(H)$ is negligible, which is straightforward because the incomplete gamma factor does not depend on s. We again use the asymptotics $\gamma(a,z) = z^a e^{-z} e^{o(1/H)}$, for $a \geq z$ (see Lemma 1), which imply

$$\gamma(\tfrac{1}{2}+\tfrac{1}{4H}, B(\tfrac{1}{2})^{2H}e^{-2c}) = B^{1/(4H)} \exp\Big(-\frac{c}{2H} - \frac{\gamma^2 e^{-2c}}{4H} + o\big(\frac{1}{H}\big)\Big).$$

Hence,

$$I^{(1,2)}(H) \leq \exp\Big(\frac{3\gamma^2\psi(H)^{2H}}{16H} - \frac{c}{2H} - \frac{\gamma^2 e^{-2c}}{4H} + o\big(\frac{1}{H}\big)\Big).$$

Since $\psi(H)^H \sim 1$, we have arbitrarily fast exponential decay here by taking c large. Now we estimate $I^{(1,1)}(H)$. First assume $\gamma < 1$. Then, the factor $\gamma(a,z)$ in the integrand of $I^{(1,1)}(H)$ satisfies $a \geq z$ for any s in the integration range and small H. Evaluating its asymptotics $z^a e^{-z} e^{o(1/H)}$ yields

$$\gamma(\tfrac{1}{2}+\tfrac{1}{4H}, Bs^{2H}) = B^{1/(4H)} s^{H+1/2} \exp\Big(-\frac{\gamma^2 s^{2H}}{4H} + o\big(\frac{1}{H}\big)\Big),$$

and so, since $s^{H+1/2} \leq T^{H+1/2} = O(1)$,

$$\begin{aligned}I^{(1,1)}(H) &= e^{o(\frac{1}{H})} \int_{\frac{1}{2}e^{-c/H}}^{\frac{1}{2}\psi(H)} \exp\Big(-\frac{\gamma^2 s^{2H}}{16H}\Big) s^{H+1/2} ds \\ &\leq e^{o(\frac{1}{H})} I_H(0,H).\end{aligned}$$

Now proceed as in the proof of Lemma 2. The lower estimate follows from

$$\begin{aligned}I^{(1,1)}(H) &\geq e^{o(\frac{1}{H})} \int_{\frac{1}{3}\psi(H)}^{\frac{1}{2}\psi(H)} \exp\Big(-\frac{\gamma^2 s^{2H}}{16H}\Big) s^{H+1/2} ds \\ &\geq \exp\Big(-\frac{\gamma^2}{16H} + o\big(\frac{1}{H}\big)\Big).\end{aligned}$$

To estimate $I^{(1,1)}(H)$ for $\gamma \geq 1$, we have to split the integration in order to apply the correct asymptotics for $\gamma(\cdot,\cdot)$. The lower part $\frac{1}{2}e^{-c/H} \leq s \leq \gamma^{-1/H}$ is handled analogously to the case $\gamma < 1$; the upper bound $e^{o(\frac{1}{H})} I_H(0,H)$ suffices. To complete our analysis of $I^{(1)}(H)$, it remains to bound the portion $\gamma^{-1/H} \leq s \leq \frac{1}{2}\psi(H)$ of the integral $I^{(1,1)}(H)$ for $\gamma \geq 1$. We estimate $\gamma(\cdot,\cdot)$ by the complete gamma function $\Gamma(\cdot)$, which is also good enough for the lower bound, again by using (11) for the upper incomplete gamma function $\Gamma(\cdot,\cdot)$. By (23), the upper portion of $I^{(1,1)}(H)$ is

$$\begin{aligned} B^{-1/(4H)} e^{o(\frac{1}{H})} &\int_{\gamma^{-1/H}}^{\frac{1}{2}\psi(H)} \exp\Big(\frac{3\gamma^2 s^{2H}}{16H}\Big)\Gamma(\tfrac12 + \tfrac{1}{4H})ds \\ &= \exp\Big(-\big(\frac14 \log(\gamma^2) + \frac14\big)\frac{1}{H} + o\big(\frac1H\big)\Big)\int_{\gamma^{-1/H}}^{\frac12\psi(H)} \exp\Big(\frac{3\gamma^2 s^{2H}}{16H}\Big)ds \\ &= \exp\Big(-\big(\frac14 \log(\gamma^2) + \frac14\big)\frac{1}{H} + \frac{3\gamma^2}{16H} + o\big(\frac1H\big)\Big) \\ &= \exp\Big(-\frac{r(\gamma)}{H} + o\big(\frac1H\big)\Big). \end{aligned}$$

The estimate for $I^{(2)}(H)$ is similar and yields the same result as for $I^{(1)}(H)$. □

Lemma 10. *For any $c > 0$*

$$\int_{\psi(H)}^{T}\int_{e^{-c/H}}^{t/2} f(s,t,H)dsdt \leq \exp\Big(-\frac{r(\gamma)}{H} + o\big(\frac1H\big)\Big).$$

Proof. Since $s \leq t/2$, we can use bound (25). The resulting integral

$$\begin{aligned} \int_{\psi(H)}^{T}\int_{e^{-c/H}}^{t/2} &\exp\Big(-\frac{\gamma^2 s^{2H}}{16H}\Big)(t-s)^{H-1/2}dsdt \\ &= \int_{e^{-c/H}}^{\frac12\psi(H)} \exp\Big(-\frac{\gamma^2 s^{2H}}{16H}\Big)\int_{\psi(H)}^{T}(t-s)^{H-1/2}dtds \\ &\quad + \int_{\frac12\psi(H)}^{T/2} \exp\Big(-\frac{\gamma^2 s^{2H}}{16H}\Big)\int_{2s}^{T}(t-s)^{H-1/2}dtds \end{aligned}$$

is straightforward to bound, after explicitly evaluating the inner integrals and using Lemma 2 for the first outer integral. □

Lemma 11.

$$\int_{\psi(H)}^{T}\int_{t/2}^{t-\frac12\psi(H)} f(s,t,H)dsdt \leq \exp\Big(-\frac{\gamma^2}{16H}\big(1+o(1)\big)\Big).$$

Proof. We use (26) and observe that in this integration range we have

$$s^{2H} = 1 + o(1) \quad \text{and} \quad (t-s)^{2H} = 1 + o(1).$$

The cancellation in (26) then shows that $R_H(s,t) = O(1)$. Now the estimate

$$\begin{aligned} \exp\Big(-\frac{\gamma^2 s^{2H}}{16H}\Big) &\leq \exp\Big(-\frac{\gamma^2 s^{2H}}{16H}\Big)\Big|_{s=\frac12\psi(H)} \\ &= \exp\Big(-\frac{\gamma^2}{16H}\big(1+o(1)\big)\Big) \end{aligned}$$

easily implies the result. □

The following lemma completes the proof of Theorem 2; recall the remark after Lemma 7.

Lemma 12. *For any $c > 0$ sufficiently large, we have*

$$\int_{\psi(H)+e^{-c/H}}^{T} \int_{t-\frac{1}{2}\psi(H)}^{t-e^{-c/H}} f(s,t,H)dsdt \leq \exp\Big(-\frac{r(\gamma)}{H} + o\big(\frac{1}{H}\big)\Big).$$

Proof. By (26), the integrand can be estimated by the integrand of (27). From a drawing and Fubini's theorem, we obtain the upper bound (up to a constant factor)

$$\begin{aligned}
&\int_{\frac{1}{2}\psi(H)+e^{-c/H}}^{\psi(H)} \exp\Big(\frac{3\gamma^2 s^{2H}}{16H}\Big) \int_{\psi(H)+e^{-c/H}}^{s+\frac{1}{2}\psi(H)} \exp\Big(-\frac{\gamma^2}{4H}(t-s)^{2H}\Big)(t-s)^{H-1/2} dtds \\
&\quad + \int_{\psi(H)}^{T-\frac{1}{2}\psi(H)} \exp\Big(\frac{3\gamma^2 s^{2H}}{16H}\Big) \int_{s+e^{-c/H}}^{s+\frac{1}{2}\psi(H)} \exp\Big(-\frac{\gamma^2}{4H}(t-s)^{2H}\Big)(t-s)^{H-1/2} dtds \\
&\quad\quad + \int_{T-\frac{1}{2}\psi(H)}^{T-e^{-c/H}} \exp\Big(\frac{3\gamma^2 s^{2H}}{16H}\Big) \int_{s+e^{-c/H}}^{T} \exp\Big(-\frac{\gamma^2}{4H}(t-s)^{2H}\Big)(t-s)^{H-1/2} dtds.
\end{aligned}$$

Each of the inner integrals can be expressed by the incomplete gamma function as in the proof of Lemma 9. Then, we estimate the incomplete gamma function by the ordinary gamma function and use (23) to obtain the upper bound

$$\exp\Big(-\big(\frac{1}{4}\log(\gamma^2) + \frac{1}{4}\big)\frac{1}{H} + o\big(\frac{1}{H}\big)\Big),$$

which does not depend on s, for each of the inner integrals. It now suffices to use $s \leq T$ in the factor $\exp\Big(\frac{3\gamma^2 s^{2H}}{16H}\Big)$ to conclude the statement. $\square$

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