Math. Commun. **26**(2021), 197–213

A hybrid functions method for solving linear and non-linear systems of ordinary differential equations

Mohammadreza Doostdar[1], Alireza Vahidi[1]*,
Tayebeh Damercheli[1] and Esmail Babolian[2]

[1] *Department of Mathematics, College of Science, Yadegar-e-Imam Khomeini (RAH) Shahr-e-Rey Branch, Islamic Azad University, Persian Gulf Freeway, Tehran, 1815163111, Iran*
[2] *Faculty of Mathematical Sciences and Computer, Kharazmi University, 50 Taleghani Avenue, Tehran, 1561836314, Iran*

Received August 27, 2020; accepted March 27, 2021

Abstract. In the present paper, we use a hybrid method to solve linear or non-linear systems of ordinary differential equations (ODEs). By using this method, these systems are reduced to a linear or non-linear system of algebraic equations. In error discussion of the suggested method, an upper bound of the error is obtained. Also, to survey the accuracy and the efficiency of the present method, some examples are solved and comparisons between the obtained results with those of several other methods are carried out.
AMS subject classifications: 34A30, 34A34

Key words: block-pulse functions, hybrid functions, Legendre polynomials, system of ordinary differential equations

1. Introduction

Studies on differential equations and their systems have always been considered due to their widespread applications in physical systems, engineering sciences, medicine, economy, biology, population systems, etc. [1, 7]. For instance, the predator and prey problem in biological systems [6, 11, 18], the mathematical models for describing $CD4^+$ T cells and HIV interactions in medicine [9, 12, 17, 30, 31, 33, 34, 45], and stiff problems in biochemistry and life sciences [13], are expressed as linear or non-linear systems of ODEs. Many researchers have applied various methods for solving systems of ODEs. The Adomian decomposition method (ADM) [4, 37], the variational iteration method (VIM) [5, 10, 27], the homotopy perturbation method (HPM) [3, 26], the differential transform method [41, 43], and a collocation method [36, 46, 47] have been proposed for solving linear and non-linear systems of ODEs.

In recent years, using hybrid functions has been considered for solving various mathematical models [2, 15, 22–25, 28, 29, 42]. One of these functions is obtained by the combination of block-pulse functions (BPFs) and Legendre polynomials. Based on a hybrid of BPFs and Legendre polynomials, Hsiao [16] solved the Fredholm

*Corresponding author. *Email addresses:* mrdoostdar@yahoo.com (M. R. Doostdar), alrevahidi@yahoo.com (A. R. Vahidi), tdamercheli@gmail.com (T. Damercheli), babolian@khu.ac.ir (E. Babolian)

and Volterra integral equations; Singh et al. [40] introduced a numerical method for numerical evaluation of the Hankel transform; a Maleknejad and Hashemizadeh [20] presented an approach to solve Hammerstein integral equations of mixed type; Maleknejad and Ebrahimzadeh [19] proposed a method for solving optimal control of Volterra integral systems. Also, by using these hybrid functions Sahu and Saha Ray [39] solved a system of non-linear Fredholm-Hammerstein; Rafiei et al. [35] found the optimal solution of linear time delay systems; Maleknejad and Saeedipoor [21] solved Fredholm integral equation of the first kind; Hesameddini and Riahi [14] obtained a numerical solution of partial differential equations with non-local integral conditions, and finally Saha Ray and Singh [38] found the numerical solutions of stochastic Volterra-Fredholm integral equations. Here, considering the general form of the system of ODEs as

$$\begin{cases} \dfrac{\mathrm{d}u_1}{\mathrm{d}t} = f_1(t, u_1, u_2, \ldots, u_n), \\ \dfrac{\mathrm{d}u_2}{\mathrm{d}t} = f_2(t, u_1, u_2, \ldots, u_n), \\ \qquad \vdots \\ \dfrac{\mathrm{d}u_n}{\mathrm{d}t} = f_n(t, u_1, u_2, \ldots, u_n), \end{cases} \tag{1}$$

with the initial conditions $u_1(0) = \lambda_1$, $u_2(0) = \lambda_2$, $\ldots$, $u_n(0) = \lambda_n$, we apply the hybrid BPFs and Legendre polynomials method (HBPLM) to solve this system. By using this method, systems of ODEs are reduced to a linear or non-linear system of algebraic equations, which can be solved by the proper methods. In addition, since every ordinary differential equation can be converted into a system of differential equations of the first order, the HBPLM can be implemented to solve most differential equations of higher orders.

We arranged the remainder of the paper as follows: In Section 2, we review required definitions and some properties of hybrid BPFs and Legendre polynomials. In Section 3, we apply the HBPLM for linear and non-linear systems of ODEs. Error analysis of the presented method is studied in Section 4. Finally, in Section 5, to verify the effectiveness of the proposed method, a mathematical model for HIV interactions with CD4$^+$ T cells, the predator and prey problem, some examples of stiff systems, and two other examples will be solved and the obtained absolute errors are compared with those of several other methods.

2. Preliminaries

In this section, required definitions and some properties of hybrid BPFs and Legendre polynomials and operational matrices are reviewed.

2.1. Hybrid BPFs and Legendre polynomials

Considering the interval $[0, T_f)$ and by hybridization of BPFs and the well-known Legendre polynomials denoted by $L_q(t)$, on this interval, hybrid functions $\ell_{ij}, i =$

$1, 2, \ldots, N$ and $j = 0, 1, \ldots, M-1$ are defined as [16]

$$\ell_{ij}(t) = \begin{cases} L_j(\dfrac{2N}{T_f}t - 2i + 1), & t \in I_i, \\ 0, & \text{o. w}, \end{cases}$$

where N and M indicate the order of BPFs and the order of Legendre polynomials, respectively, and $I_i = [\frac{i-1}{N}T_f, \frac{i}{N}T_f)$. Without loss of generality, we let $T_f = 1$. Any function $f \in L^2[0, 1)$ can be expanded by the basis functions $\{\ell_{ij}\}$ as

$$f(t) = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} \alpha_{ij}\ell_{ij}(t), \tag{2}$$

where α_{ij} are hybrid coefficients. By truncating the infinite series in equation (2) at some values of N and M, it can be written as

$$f(t) \simeq f_{NM}(t) = \sum_{i=1}^{N}\sum_{j=0}^{M-1} \alpha_{ij}\ell_{ij}(t) = \mathbf{A}^T\mathbf{L}(t), \tag{3}$$

where $\mathbf{A}$ and $\mathbf{L}$(t) are defined as follows:

$$\mathbf{A} = [\alpha_{10}, \alpha_{11}, \ldots, \alpha_{1(M-1)}, \alpha_{20}, \alpha_{21}, \ldots, \alpha_{2(M-1)}, \ldots, \alpha_{N0}, \alpha_{N1}, \ldots, \alpha_{N(M-1)}]^T,$$

and

$$\mathbf{L}(t) = [\ell_{10}(t), \ldots, \ell_{1(M-1)}(t), \ell_{20}(t), \ldots, \ell_{2(M-1)}(t), \ldots, \ell_{N0}(t), \ldots, \ell_{N(M-1)}(t)]^T.$$

2.2. Operational matrices

Suppose that P denotes the operational matrix of the integration of the vector $\mathbf{L}$(t) defined above; then

$$\int_0^t \mathbf{L}(\eta)\, \mathrm{d}\eta \simeq P\mathbf{L}(t),$$

where P is an $NM \times NM$ matrix defined as [16]:

$$P = \begin{pmatrix} D & S & S & \ldots & S \\ \mathbf{0} & D & S & \ldots & S \\ \mathbf{0} & \mathbf{0} & D & \ldots & S \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots & D \end{pmatrix},$$

where the zero matrix $\mathbf{0}$, S and D are $M \times M$ matrices and

$$S = \frac{T_f}{N}\begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix},$$

$$D = \frac{T_f}{2N}\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{5} & 0 & \frac{1}{5} & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{7} & 0 & \frac{1}{7} & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{9} & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & \frac{1}{2M-9} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & \frac{-1}{2M-7} & 0 & \frac{1}{2M-7} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & \frac{-1}{2M-5} & 0 & \frac{1}{2M-5} & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \frac{-1}{2M-3} & 0 & \frac{1}{2M-3} \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \frac{-1}{2M-1} & 0 \end{pmatrix}.$$

To evaluate the product of $\mathbf{L}(t)$ and $\mathbf{L}^T(t)$, let

$$\mathbf{L}(t)\mathbf{L}^T(t)\mathbf{A} = \tilde{\mathbf{A}}\mathbf{L}(t),$$

where $\tilde{\mathbf{A}}$ is an $NM \times NM$ matrix defined in [16].

3. The HBPLM for linear and non-linear systems of ODEs

In this section, the HBPLM will be implemented to approximate the solutions of linear and non-linear systems of ODEs.

3.1. Implementation for linear systems of ODEs

We consider linear form of system (1) as

$$\begin{cases} \dfrac{\mathrm{d}u_1}{\mathrm{d}t} = g_1(t) + \displaystyle\sum_{k=1}^{n} h_{1,k}(t)u_k(t), \\ \dfrac{\mathrm{d}u_2}{\mathrm{d}t} = g_2(t) + \displaystyle\sum_{k=1}^{n} h_{2,k}(t)u_k(t), \\ \qquad\vdots \\ \dfrac{\mathrm{d}u_n}{\mathrm{d}t} = g_n(t) + \displaystyle\sum_{k=1}^{n} h_{n,k}(t)u_k(t), \end{cases} \tag{4}$$

where g_k, $h_{r,k}$ for $k, r = 1, 2, \dots, n$, are the known functions. From equation (3), $u_k(t)$ can be written as

$$u_k(t) \simeq \mathbf{A}_k^T\mathbf{L}(t), \qquad k = 1, 2, \dots, n, \tag{5}$$

where the unknown vector $\mathbf{A}_k$, which must be calculated, is considered as

$$\mathbf{A}_k = [a_{k,1} \quad a_{k,2} \quad \dots \quad a_{k,NM}]^T. \tag{6}$$

Now, we approximate $u'_k(t)$ as follows:

$$u'_k(t) \simeq \mathbf{A}'^T_k \mathbf{L}(t), \qquad k = 1, 2, \dots, n, \tag{7}$$

where $\mathbf{A}'_k \simeq (\mathbf{A}^T_k - u^T_{k(0)})P^{-1}$ and $u_{k(0)}$ can be obtained as follows:

$$u_{k(0)} = \Big(\underbrace{u_k(0)\ \overbrace{0\ 0\ \dots\ 0}^{M-1}\ u_k(0)\ \overbrace{0\ 0\ \dots\ 0}^{M-1}\ \dots\ u_k(0)\ \overbrace{0\ 0\ \dots\ 0}^{M-1}}_{NM}\Big)^T. \tag{8}$$

Considering equation (3), the known functions g_k and $h_{r,k}$, $r, k = 1, 2, \dots, n$, by the basis functions $\{\ell_{ij}\}$, can be written as

$$g_k(t) \simeq \mathcal{G}^T_k \mathbf{L}(t), \tag{9}$$

and

$$h_{r,k}(t) \simeq \mathcal{H}^T_{r,k} \mathbf{L}(t). \tag{10}$$

Also, the terms $h_{r,k}(t)u_k(t)$, $r, k = 1, 2, \dots, n$, can be expanded in terms of the vector $\mathbf{L}(t)$ as

$$h_{r,k}(t)u_k(t) \simeq (\mathcal{H}^T_{r,k}\mathbf{L}(t))(\mathbf{A}^T_k\mathbf{L}(t)) = \mathcal{H}^T_{r,k}\mathbf{L}(t)\mathbf{L}^T(t)\mathbf{A}_k = \mathcal{H}^T_{r,k}\tilde{\mathbf{A}}_k\mathbf{L}(t). \tag{11}$$

Note that if $h_{r,k}(t)$ is a constant as c, then equation (11) can be written as

$$h_{r,k}(t)u_k(t) \simeq c\mathbf{A}^T_k\mathbf{L}(t), \qquad r, k = 1, 2, \dots, n.$$

Now, by substituting equations (5) - (11) into equation (4) and replacing $\simeq$ with $=$, we have

$$\begin{cases} \left((\mathbf{A}^T_1 - u^T_{1(0)})P^{-1} - \mathcal{G}^T_1 - \displaystyle\sum_{k=1}^{n} \mathcal{H}^T_{1,k}\tilde{\mathbf{A}}_k\right)\mathbf{L}(t) = 0, \\ \left((\mathbf{A}^T_2 - u^T_{2(0)})P^{-1} - \mathcal{G}^T_2 - \displaystyle\sum_{k=1}^{n} \mathcal{H}^T_{2,k}\tilde{\mathbf{A}}_k\right)\mathbf{L}(t) = 0, \\ \qquad\qquad\qquad\qquad\vdots \\ \left((\mathbf{A}^T_n - u^T_{n(0)})P^{-1} - \mathcal{G}^T_n - \displaystyle\sum_{k=1}^{n} \mathcal{H}^T_{n,k}\tilde{\mathbf{A}}_k\right)\mathbf{L}(t) = 0. \end{cases} \tag{12}$$

By collocating system (12) at the points $t = \frac{2\zeta-1}{2NM}$, $\zeta = 1, 2, \dots, NM$, and solving the obtained linear system of algebraic equations by a well-known numerical method, the vector $\mathbf{A}_k$ is obtained and $u_k(t)$ is consequently determined by using equation (5).

3.2. Implementation for non-linear systems of ODEs

A non-linear system of ODEs is considered as

$$\begin{cases} \dfrac{\mathrm{d}u_1}{\mathrm{d}t} = f_1(t, u_1, u_2, \ldots, u_n), \\ \dfrac{\mathrm{d}u_2}{\mathrm{d}t} = f_2(t, u_1, u_2, \ldots, u_n), \\ \quad\vdots \\ \dfrac{\mathrm{d}u_n}{\mathrm{d}t} = f_n(t, u_1, u_2, \ldots, u_n). \end{cases} \tag{13}$$

Here, $u_k(t)$ and $u'_k(t)$ are expanded as equations (5) and (7), respectively. If necessary, we use equation (9) to expand a known function. Also, non-linear terms can be expanded in terms of the vector $\mathbf{L}(t)$ as

$$u_r(t)u_k(t) \simeq (\mathbf{A}_r^T\mathbf{L}(t))(\mathbf{A}_k^T\mathbf{L}(t)) = \mathbf{A}_r^T\mathbf{L}(t)\mathbf{L}^T(t)\mathbf{A}_k = \mathbf{A}_r^T\tilde{\mathbf{A}}_k\mathbf{L}(t),\ r, k = 1, 2, \ldots, n,$$

and

$$u_k^3(t) = u_k^2(t).u_k(t) \simeq (\mathbf{A}_k^T\tilde{\mathbf{A}}_k\mathbf{L}(t))(\mathbf{A}_k^T\mathbf{L}(t)) = \mathbf{A}_k^T\tilde{\mathbf{A}}_k\mathbf{L}(t)\mathbf{L}^T(t)\mathbf{A}_k = \mathbf{A}_k^T\tilde{\mathbf{A}}_k^2\mathbf{L}(t),$$

and in the same way, $u_k^\alpha(t) \simeq \mathbf{A}_k^T\tilde{\mathbf{A}}_k^{\alpha-1}\mathbf{L}(t)$, where α is a positive integer. Suppose

$$f_k(t, u_1, u_2, \ldots, u_n) \simeq \bar{\mathbf{F}}_k^T\mathbf{L}(t), \qquad k = 1, 2, \ldots, n;$$

then system (13) can be written as the following non-linear algebraic system:

$$\left((\mathbf{A}_k^T - u_{k(0)}^T)P^{-1} - \bar{\mathbf{F}}_k^T\right)\mathbf{L}(t) = 0, \qquad k = 1, 2, \ldots, n. \tag{14}$$

Like the linear form, by collocating system (14) at the points $t = \frac{2\zeta-1}{2NM}$, $\zeta = 1, 2, \ldots, NM$, and solving the obtained non-linear system of algebraic equations by a proper method such as Newton's method, the vector $\mathbf{A}_k$ is obtained and $u_k(t)$ is consequently determined by using equation (5).

4. Error analysis

In this section, considering the Sobolev space and the associated norm, an upper bound of the error for the HBPLM is obtained. We consider the Sobolev norm on the interval $(-1, 1)$ as follows:

$$\|f\|_{H^d(-1,1)} = \left(\sum_{i=0}^{d} \|f^{(i)}\|^2_{L^2(-1,1)}\right)^{\frac{1}{2}},$$

where $d \geq 0$ is an integer.

Lemma 1. [8] *Let $\{L_q\}_{q=0}$ be the sequence of Legendre polynomials and $P_S(t) = \sum_{q=0}^{S} a_q L_q(t)$ the best polynomial approximation of degree S for $f \in L^2(-1,1)$. Then, for $d \geq 1$, there exists a constant $C_0 > 0$ such that*

$$\|f - P_S\|_{L^\infty(-1,1)} \leq C_0 S^{\frac{3}{4}-d} \|f\|_{H^d(-1,1)},$$

for all functions f in $H^d(-1,1)$.

Lemma 2. *Let $f \in H^d[0,1)$ and let f_{NM} be the polynomial approximation of f defined in equation (3). Then,*

$$\|f - f_{NM}\|_{L^\infty[0,1)} \leq C_0 (NM)^{\frac{3}{4}-d} \max_{1 \leq i \leq N} \|f\|_{H^d(I_i)},$$

where $I_i = [\frac{i-1}{N}, \frac{i}{N})$ and C_0 is a positive constant.

Proof. It can be obviously concluded by using Lemma 1. □

Before we express the following theorem, we represent system (1) as

$$U(t) = U(0) + \int_0^t F(\tau, U(\tau))\, \mathrm{d}\tau, \tag{15}$$

where

$$U(t) = [u_1(t), u_2(t), \ldots, u_n(t)]^T,$$
$$F(\tau, U(\tau)) = \begin{pmatrix} f_1(\tau, U(\tau)) \\ f_2(\tau, U(\tau)) \\ \vdots \\ f_n(\tau, U(\tau)) \end{pmatrix}.$$

Theorem 1. *Let $U \in H^d[0,1)$ be the exact solution of equation (15) and $\tilde{U}(t) = U_{NM}(t)$ the approximate solution obtained by the HBPLM. Moreover, assume that $f_k(\tau, U(\tau))$, $k = 1, 2, \ldots, n$ is a continuous function for $0 \leq \tau \leq t < 1$ and that satisfies the Lipschitz condition*

$$|f_k(\tau, U(\tau)) - f_k(\tau, W(\tau))| \leq L_k \|U - W\|_\infty, \tag{16}$$

where $L_k > 0$, $k = 1, 2, \ldots, n$, is a Lipschitz constant. Then, there exists a constant $\delta > 0$ such that

$$\|U - \tilde{U}\|_\infty \leq \delta \max_{1 \leq i \leq N} \|u_p\|_{H^d(I_i)}.$$

Proof. Let $\tilde{u}_k(t) = u_{k(NM)}(t)$, $k = 1, 2, \ldots, n$, be the approximate solution of the system defined in equation (15), and let $e_k(t) = u_k(t) - \tilde{u}_k(t)$ be the error term. Then,

$$e_k(t) = u_k(t) - \tilde{u}_k(t) = \int_0^t \left(f_k(\tau, U(\tau)) - f_k(\tau, \tilde{U}(\tau)) \right) \mathrm{d}\tau.$$

Using equation (16) and $0 \le t < 1$, we obtain

$$|e_k(t)| \le L_k \, \|U - \tilde{U}\|_\infty = L_k \max_{1 \le k \le n} \|u_k - \tilde{u}_k\|_\infty.$$

Let $L = \max_{1 \le k \le n} L_k$; then

$$\|e_k\|_\infty \le L \, \|u_p - \tilde{u}_p\|_\infty, \qquad p \in \{1, 2, \ldots, n\}. \tag{17}$$

Now, by using Lemma 2 and equation (17), we have

$$\|e_k\|_\infty \le L \, \|u_p - \tilde{u}_p\|_\infty = L \, \|u_p - \tilde{u}_p\|_{L^\infty[0,1)} \le \delta \max_{1 \le i \le N} \|u_p\|_{H^d(I_i)}, \tag{18}$$

where $\delta = LC_0(NM)^{\frac{3}{4} - d}$.
If $e(t) = U(t) - \tilde{U}(t)$, then using (18), we get

$$\|e\|_\infty = \|U - \tilde{U}\|_\infty \le \delta \max_{1 \le i \le N} \|u_p\|_{H^d(I_i)}.$$

□

5. Numerical examples

In this section, the effectiveness of the present method is studied by applying the HBPLM to several examples of linear and non-linear systems of ODEs. As the first example, a mathematical model for HIV interactions with $CD4^+$ T cells is considered. Then, the predator-prey problem, two examples of linear and non-linear stiff systems, and two other examples are solved. All computations are performed using Matlab 2017a software package on a laptop with the Intel core i5-3210M CPU processor and 4GB RAM.

Example 1. As the first example, we consider a mathematical model of HIV interactions with $CD4^+$ T cells, which is described by a three-dimensional system of non-linear ODEs as [27, 32, 41, 45, 46]:

$$\begin{cases} T'(t) = s - \mu T + rT\left(1 - \dfrac{T + I}{T_{max}}\right) - \alpha V T, \\ I'(t) = \alpha V T - \beta I, \\ V'(t) = C\beta I - \gamma V, \end{cases} \tag{19}$$

where the initial conditions and parameters are as given

$$\begin{cases} T(0) = 0.1, & I(0) = 0, & V(0) = 0.1, \\ r = 3, & s = 0.1, & \mu = 0.02, & T_{max} = 1500, & \alpha = 0.0027, \\ \beta = 0.3, & C = 10, & \gamma = 2.4. \end{cases}$$

In this example, the HBPLM is implemented for the above model for $N = 2, M = 8$, which reduces it to a non-linear system of algebraic equations. Since system (19) has no exact solution, we prefer to use the solution of the classical forth-order Runge-Kutta method (RK4) as an acceptable solution for the purpose of comparison. Hence, numerical results of the suggested method and those of the modified variational iteration method (MVIM) [27], the Bessel collocation method (BCM) [46], the differential transform method (DTM) [41], and the Laplace Adomian decomposition method (LADM) [32] for $T(t)$, $I(t)$ and $V(t)$, are compared with the RK4 solutions. Tables 1a, 1b and 1c show these comparisons.

(a) Comparisons for $T(t)$

t	HBPLM	MVIM [27]	LADM [32]	BCM [46]	DTM [41]
0.0	0	0	0	0	0
0.2	1.80E-09	3.50E-09	8.10E-07	4.95E-03	2.84E-03
0.4	0.90E-09	2.56E-07	1.35E-04	2.59E-02	1.64E-02
0.6	0.90E-09	0.00E-00	3.28E-03	6.90E-02	5.35E-02
0.8	21.9E-09	0.00E-00	3.67E-02	1.38E-01	1.32E-01

(b) Comparisons for $I(t)$

t	HBPLM	MVIM [27]	LADM [32]	BCM [46]	DTM [41]
0.0	0	0	0	0	0
0.2	9.10E-14	4.99E-13	5.13E-12	2.15E-07	3.34E-07
0.4	1.90E-13	3.91E-11	8.21E-10	2.23E-07	8.34E-07
0.6	3.70E-13	4.75E-10	4.46E-08	8.71E-07	1.43E-06
0.8	5.60E-13	2.91E-09	1.08E-07	1.80E-06	3.11E-06

(c) Comparisons for $V(t)$

t	HBPLM	MVIM [27]	LADM [32]	BCM [46]	DTM [41]
0.0	0	0	0	0	0
0.2	1.00E-10	6.54E-08	1.17E-07	7.52E-08	1.57E-07
0.4	2.00E-10	1.07E-06	1.84E-05	4.71E-08	1.41E-05
0.6	1.00E-10	5.74E-06	6.87E-04	2.31E-07	2.15E-04
0.8	1.00E-10	2.01E-05	4.71E-03	7.94E-07	1.53E-03

Table 1: *Comparisons between approximate solutions for (a) $T(t)$, (b) $I(t)$ and (c) $V(t)$ by* HBPLM, MVIM, LADM, BCM *and* DTM *with* RK4

Example 2. Consider the following system of ODEs that arises in the modeling of the two species predator and prey problem [7]:

$$\begin{cases} u'(t) = \big(2 - v(t)\big)u(t), \\ v'(t) = \big(u(t) - 1\big)v(t), \end{cases} \tag{20}$$

in which u denotes the prey population and v is the population of predators. With the passage of time, a permanently repeated cycle of interrelated falls and rises in the predator and prey populations occurs. The time period of these repeated cycles, which is indicated by the letter $\mathcal{T}$, has been calculated in [7], from which, with starting values $u(0) = 2$ and $v(0) = 2$, we see that the time period converges to $\mathcal{T} \simeq 4.61$. We solved system (20) by the HBPLM for $N = 2, M = 8$ and the RK4 method. Figure 1 shows the approximate solutions obtained by the HBPLM and RK4. Also, the obtained numerical results are shown in Table 2.

t	u			v		
	RK4	HBPLM	absolute error	RK4	HBPLM	absolute error
0.5	1.531714	1.529849	1.87E-03	3.036906	3.038620	1.71E-03
1.0	0.819587	0.819081	5.06E-04	3.274107	3.275590	1.48E-03
1.5	0.490400	0.491720	1.32E-03	2.715792	2.713292	2.50E-03
2.0	0.406754	0.404548	2.21E-03	2.047384	2.050809	3.43E-03
2.5	0.455341	0.449063	6.28E-03	1.533120	1.542198	9.08E-03
3.0	0.628130	0.627862	2.67E-04	1.212053	1.211689	3.64E-04
3.5	0.969317	0.969139	1.78E-04	1.087093	1.087698	6.05E-04
4.0	1.501637	1.502164	5.26E-04	1.215055	1.214361	6.94E-04
4.5	1.974847	1.973759	1.09E-03	1.784686	1.785414	7.27E-04
5.0	1.705620	1.704236	1.38E-03	2.828001	2.827758	2.43E-04

Table 2: *Numerical results for Example 2*

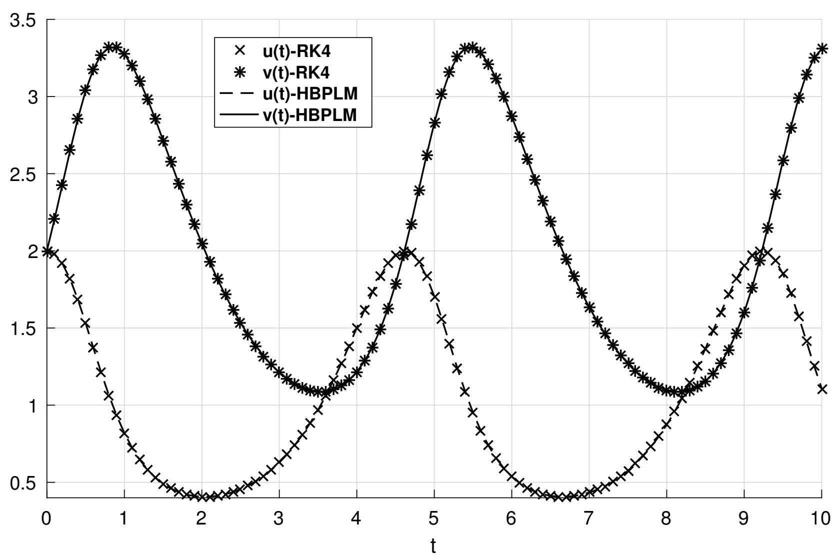

Figure 1: *Comparison between approximate solutions by the* HBPLM *and* RK4

Example 3. In this example, the following system is considered as a linear stiff system of ODEs [3]:

$$\begin{cases} u_1'(t) = -u_1(t) + 95\, u_2(t), \\ u_2'(t) = -u_1(t) - 97\, u_2(t). \end{cases}$$

For the initial conditions $u_1(0) = 1$, $u_2(0) = 1$, the exact solution is $u_1(t) = \frac{1}{47}(95e^{-2t} - 48e^{-96t})$, $u_2(t) = \frac{1}{47}(48e^{-96t} - e^{-2t})$. The absolute errors of u_1 and u_2 by using the HBPLM for $N = 10$, $M = 12$, the HPM [3], and the rational HPM (RHPM) [3] are shown in Table 3.

t	Absolute error of u_1			Absolute error of u_2		
	HBPLM	RHPM [3]	HPM [3]	HBPLM	RHPM [3]	HPM [3]
0.5	2.59E-16	1.98E-06	1.46E-06	5.07E-18	4.32E-05	6.83E-05
1.0	2.33E-16	8.71E-07	3.73E-05	8.82E-17	3.95E-05	1.54E-03
1.5	4.96E-18	1.22E-06	1.79E-04	1.36E-17	4.24E-05	6.56E-03
2.0	1.55E-15	8.19E-07	4.53E-04	3.09E-17	3.11E-07	1.45E-02
2.5	1.99E-15	1.54E-06	8.13E-04	4.82E-17	2.94E-05	2.25E-02
3.0	2.61E-17	7.77E-07	1.18E-03	5.52E-17	1.36E-05	2.75E-02
3.5	1.06E-16	4.38E-07	1.48E-03	1.27E-17	8.66E-06	2.79E-02
4.0	1.25E-16	1.39E-06	1.68E-03	2.03E-17	1.82E-05	2.41E-02
4.5	1.40E-16	1.87E-06	1.75E-03	2.54E-17	1.65E-05	1.73E-02
5.0	1.50E-16	1.93E-06	1.73E-03	2.98E-17	1.02E-05	9.20E-03

Table 3: *Absolute errors for Example 3*

Example 4. As a non-linear stiff system of ODEs, we consider the following system [3]:

$$\begin{cases} u_1'(t) = -1002\, u_1(t) + 1000\, u_2^2(t), \\ u_2'(t) = u_1(t) - u_2(t) - u_2^2(t), \end{cases}$$

with the initial conditions $u_1(0) = 1$, $u_2(0) = 1$, and the exact solution $u_1(t) = e^{-2t}$, $u_2(t) = e^{-t}$. The absolute errors of u_1 and u_2 using the HBPLM for $N = 4$, $M = 12$, the HPM [3] and the RHPM [3] are shown in Table 4.

t	Absolute error of u_1			Absolute error of u_2		
	HBPLM	RHPM [3]	HPM [3]	HBPLM	RHPM [3]	HPM [3]
0.5	2.54E-12	3.00E-10	0.00E+00	1.09E-15	1.26E-08	0.00E+00
1.0	6.30E-13	1.50E-08	0.00E+00	5.36E-16	1.49E-07	2.80E-09
1.5	6.12E-14	2.75E-07	0.00E+00	5.52E-18	1.70E-07	1.75E-06
2.0	2.05E-13	4.61E-07	2.80E-09	3.11E-16	1.80E-07	1.66E-04
2.5	6.07E-13	3.69E-07	9.69E-08	7.05E-16	4.51E-08	5.62E-03
3.0	1.61E-14	6.80E-08	1.75E-06	3.87E-16	1.75E-07	9.93E-02
3.5	7.18E-15	3.99E-07	2.01E-05	2.83E-16	4.54E-08	1.12E+00
4.0	3.70E-15	8.09E-08	1.66E-04	7.72E-16	1.91E-07	9.09E+00
4.5	3.74E-15	5.19E-07	1.07E-03	3.88E-16	2.27E-07	5.75E+01
5.0	4.26E-14	1.19E-06	5.62E-03	2.12E-16	7.20E-07	2.98E+02

Table 4: *Absolute errors for Example 4*

Example 5. Consider a system of ODEs of order two as [5]

$$\begin{cases} u_1''(t) - u_2''(t) + u_1(t) - 4u_2(t) = 0, \\ \qquad\qquad u_1'(t) + u_2'(t) = \cos t + 2\cos 2t, \end{cases} \tag{21}$$

with the initial conditions $u_1(0) = 0$, $u_2(0) = 0$, $u_1'(0) = 1$, $u_2'(0) = 2$, and the exact solution $u_1(t) = \sin t$, $u_2(t) = \sin 2t$. We convert system (21) into a system of ODEs of order one and solve it using the HBPLM. To do this, we define functions $w_1(t)$, $w_2(t)$, $w_3(t)$, and $w_4(t)$ as

$$w_1(t) = u_1(t), \qquad w_2(t) = u_1'(t), \qquad w_3(t) = u_2(t), \qquad w_4(t) = u_2'(t).$$

Now, the following system is obtained:

$$\begin{cases} w_1'(t) = w_2(t), \\ w_2'(t) = -\dfrac{1}{2}w_1(t) + 2w_3(t) - \dfrac{1}{2}(\sin t + 4\,\sin 2t), \\ w_3'(t) = w_4(t), \\ w_4'(t) = \dfrac{1}{2}w_1(t) - 2w_3(t) - \dfrac{1}{2}(\sin t + 4\,\sin 2t), \end{cases} \tag{22}$$

where the initial conditions are $w_1(0) = 0$, $w_2(0) = 1$, $w_3(0) = 0$, and $w_4(0) = 2$. In Table 5a, the absolute errors of u_1 and u_2, using the HBPLM for $N = 2$, $M = 8$, and VIM [5] are compared. We point out that according to the definition of the functions w_2 and w_4, the approximate solutions of u_1' and u_2' are also obtained by solving system (22). By considering $u_1'(t) = \cos t$, $u_2'(t) = 2\cos 2t$, as the exact solutions of the derivatives, the absolute errors of u_1' and u_2' are shown in Table 5b. Compared to the errors obtained for u_1 and u_2 by the HBPLM, the errors for u_1' and u_2' are larger. The fact is that the numerical errors are usually larger for the derivatives.

Example 6. As the last example, we consider Duffing's equation as [44]

$$u''(t) + u'(t) + u(t) + u^3(t) = \cos^3 t - \sin t, \qquad u(0) = 1,\ u'(0) = 0. \tag{23}$$

Considering $w_1(t) = u(t)$ and $w_2(t) = u'(t)$, system (23) is converted into the following system of ODEs:

$$\begin{cases} w_1'(t) = w_2(t), \\ w_2'(t) = \cos^3 t - \sin t - w_1(t) - w_2(t) - w_1^3(t), \end{cases} \tag{24}$$

with the initial conditions $w_1(0) = 1$ and $w_2(0) = 0$. The HBPLM is used for solving system (24) for $N = 2$, $M = 8$. Then, the results obtained for the solutions of (24) and consequently for equation (23) with the exact solution $u(t) = \cos t$ are presented in Table 6.

(a) **(b)**

t	u_1		u_2		HBPLM	
	HBPLM	VIM [5]	HBPLM	VIM [5]	u_1'	u_2'
0.1	5.85E-10	4.10E-09	3.45E-10	7.40E-09	1.01e-08	1.10e-08
0.2	7.36E-10	9.60E-09	6.52E-10	6.60E-09	4.02e-09	4.23e-09
0.3	7.10E-10	2.50E-08	6.95E-10	1.20E-08	6.48e-09	6.59e-09
0.4	5.29E-10	2.50E-08	6.78E-10	2.30E-08	1.29e-08	1.37e-08
0.5	2.50E-10	2.70E-08	1.39E-10	2.90E-08	3.89e-08	4.04e-08
0.6	6.25E-10	4.00E-08	1.17E-09	3.40E-08	8.00e-09	8.03e-09
0.7	7.63E-10	3.50E-08	9.83E-10	2.80E-08	2.56e-09	2.62e-09
0.8	7.12E-10	4.50E-08	7.91E-10	3.90E-08	8.13e-09	8.11e-09
0.9	5.03E-10	6.10E-08	2.67E-10	6.10E-08	1.52e-08	1.52e-08
1.0	4.12E-10	1.50E-08	4.15E-10	1.70E-07	2.12e-08	2.12e-08

Table 5: *Absolute errors for* **(a)** u_1, u_2 *and* **(b)** u_1', u_2' *in Example 5*

t	Exact	HBPLM	Absolute error
0.1	0.9950041652	0.9950041641	1.09E-09
0.2	0.9800665778	0.9800665793	1.48E-09
0.3	0.9553364891	0.9553364875	1.58E-09
0.4	0.9210609940	0.9210609952	1.27E-09
0.5	0.8775825618	0.8775825614	4.77E-10
0.6	0.8253356149	0.8253356156	7.00E-10
0.7	0.7648421872	0.7648421864	8.50E-10
0.8	0.6967067093	0.6967067101	7.72E-10
0.9	0.6216099682	0.6216099677	5.30E-10
1.0	0.5403023058	0.5403023063	4.51E-10

Table 6: *Numerical results for Example 6*

6. Conclusion

This paper applied a hybrid functions method based on the combination of BPFs and Legendre polynomials to solve the systems of ODEs in linear and non-linear forms. By using the present method, a system of ODEs was reduced to a linear or non-linear system of algebraic equations. An upper bound of the error was obtained for the presented method. Various examples such as problem of HIV interactions with $CD4^+$ T cells, the predator and prey problem, and two examples of linear and non-linear stiff systems were solved. In addition, a system of ODEs of order two and Duffing's equation were converted into the systems of ODEs of order one and then solved. Simple application and the desired accuracy of the results obtained by this method show the efficiency of our proposed method.

Acknowledgement

The authors are grateful to the editor and the anonymous referees for their valuable comments and suggestions, which have significantly improved this paper.

References

[1] M. L. Abell, J. P. Braselton, *Introductory Differential Equations*, 5th ed., Academic Press, London, 2018.

[2] B. Asady, M. Tavassoli Kajani, A. Hadi Vencheh, A. Heydari, *Solving second kind integral equations with hybrid Fourier and Block-pulse functions*, Appl. Math. Comput. **160**(2005), 517–522.

[3] J. Biazar, M. A. Asadi, F. Salehi, *Rational homotopy perturbation method for solving stiff systems of ordinary differential equations*, Appl. Math. Model. **390**(2015), 1291–1299.

[4] J. Biazar, E. Babolian, R. Islam, *Solution of the system of ordinary differential equations by Adomian decomposition method*, Appl. Math. Comput. **147**(2004), 713–719.

[5] J. Biazar, H. Ghazvini, *He's variational iteration method for solving linear and non-linear systems of ordinary differential equations*, Appl. Math. Comput. **191**(2007), 287–297.

[6] J. Biazar, R. Montazeri, *A computational method for solution of the prey and predator problem*, Appl. Math. Comput. **163**(2005) 841–847.

[7] J. C. Butcher, *Numerical Methods for Ordinary Differential Equations*, 2nd ed., John Wiley & Sons, 2008.

[8] C. Canuto, M. Y. Hussaini, A. Quarteroni, T. A. Zang, *Spectral methods in fluid dynamics*, Springer-Verlag, Berlin, 1988.

[9] R. V. Culshaw, S. Ruan, *A delay-differential equation model of HIV infection of $CD4^+$ T cells*, Math. Biosci. **165**(2000), 27–39.

[10] M. T. Darvishi, F. Khani, A. A. Soliman, *The numerical simulation for stiff systems of ordinary differential equations*, Comput. Math. Appl. **54**(2007), 1055–1063.

[11] B. Dubey, B. Das, J. Hussain, *A predator-prey interaction model with self and cross diffusion*, Ecol. Model. **141**(2001), 67–76.

[12] P. Essunger, A. S. Perleson, *Modeling HIV infection of $CD4^+$ T-cell subpopulation*, J. Theoret. Biol. **170**(1994), 367–391.

[13] R. J. Field, R. M. Noyes, *Oscillations in chemical systems. IV. Limit cycle behavior in a model of a real chemical reaction*, J. Chem. Phys. **60**(1974), 1877–1884.

[14] E. Hesameddini, M. Riahi, *Hybrid Legendre Block-pulse functions method for solving partial differential equations with non-local integral boundary conditions*, J. Inf. Optim. Sci. **40**(2019), 1391–1403.

[15] E. Hesameddini, M. Shahbazi, *Hybrid Bernstein Block-pulse functions for solving system of fractional integro-differential equations*, Int. J. Comput. Math. **95**(2018), 2287–2307.

[16] C. H. Hsiao, *Hybrid function method for solving Fredholm and Volterra integral equations of the second kind*, J. Comput. Appl. Math. **230**(2009), 59–68.

[17] D. Kirschner, *Using mathematics to understand HIV immune dynamics*, Notices Amer. Math. Soc. **43**(1996), 191–202.

[18] A. J. Lotka, *Elements of Physical Biology*, William and Wilkins, Baltimore, 1925. (Reissued as Elements of Mathematical Biology, Dover, New York, 1956)

[19] K. Maleknejad, A. Ebrahimzadeh, *An efficient hybrid pseudo-spectral method for solving optimal control of Volterra integral systems*, Math. Commun. **19**(2014), 417–435.

[20] K. Maleknejad, E. Hashemizadeh, *Numerical solution of the dynamic model of a chemical reactor by hybrid functions*, Procedia. Comput. Sci. **3**(2011), 908–912.

[21] K. Maleknejad, E. Saeedipoor, *An efficient method based on hybrid functions for Fredholm integral equation of the first kind with convergence analysis*, Appl. Math. Comput. **304**(2017), 93–102.

[22] H. R. Marzban, S. M. Hoseini, M. Razzaghi, *Solution of Volterras population model via Block-pulse functions and Lagrange-interpolating polynomials*, Math. Methods Appl. Sci. **32**(2009), 127–134.

[23] H. R. Marzban, M. Razzaghi, *Solution of multi-delay systems using hybrid of Block-pulse functions and Taylor series*, J. Sound. Vib. **292**(2006), 954–963.

[24] H. R. Marzban, H. R. Tabrizidooz, M. Razzaghi, *Hybrid functions for nonlinear initial-value problems with applications to Lane-Emden type equations*, Phys. Lett. A. **372**(2008), 5883–5886.

[25] S. Mashayekhi, M. Razzaghi, *Numerical solution of distributed order fractional differential equations by hybrid functions*, J. Comput. Phys. **315**(2016), 169–181.

[26] M. Merdan, *Homotopy perturbation method for solving a model for HIV infection of $CD4^+$ T cells*, İstanbul Ticaret Üniversitesi Fen Bilimleri Dergisi Yil, **12**(2007), 39–52.

[27] M. Merdan, A. Gökdoğan, A. Yildirim, *On the numerical solution of the model for HIV infection of $CD4^+$ T cells*, Comput. Math. Appl. **62**(2011), 118–123.

[28] F. Mirzaee, S. Alipour, N. Samadyar, *Numerical solution based on hybrid of Block-pulse and parabolic functions for solving a system of nonlinear stochastic Ito-Volterra integral equations of fractional order*, J. Comput. Appl. Math. **349**(2019), 157–171.

[29] F. Mohammadi, L. Moradi, D. Baleanu, A. Jajarmi, *A hybrid functions numerical scheme for fractional optimal control problems: Application to non-analytic dynamic systems*, J. Vib. Control. **24**(2017), 5030–5043.

[30] M. Nowak, C. R. M. Bangham, *Population dynamics of immune responses to persistent viruses*, Science **272**(1996), 74–79.

[31] M. Nowak, R. May, *Mathematical biology of HIV infections: Antigenic variation and diversity threshold*, Math. Biosci. **106**(1991), 1–21.

[32] M. Y. Ongun, *The Laplace Adomian decomposition method for solving a model for HIV infection of $CD4^+$ T cells*, Math. Comput. Model. **53**(2011), 597–603.

[33] A. S. Perelson, *Modeling the interaction of the immune system with HIV*, in:*Mathematical and Statistical Approaches to AIDS Epidemiology*, (C. Castillo-Chavez, Ed.), *Lecture Notes in Biomathematics*, Vol. 83, Springer, Berlin, Heidelberg, 1990, 350–370.

[34] A. S. Perelson, D. E. Kirschner, R. D. Boer, *Dynamics of HIV infection $CD4^+$ T cells*, Math. Biosci. **114**(1993), 81–125.

[35] Z. Rafiei, B. Kafash, S. M. Karbasi, *State-control parameterization method based on using hybrid functions of Block-pulse and Legendre polynomials for optimal control of linear time delay systems*, Appl. Math. Model. **45**(2017), 1008–1019.

[36] J. Rahimi, D. D. Ganji, M. Khaki, K. Hosseinzadeh, *Solution of the boundary layer flow of an Eyring-Powell non-Newtonian fluid over a linear stretching sheet by collocation method*, Alex. Eng. J. **56**(2017), 621–627.

[37] A. Saad Mahmood, L. Casasus, W. Al-Hayani, *The decomposition method for stiff systems of ordinary differential equations*, Appl. Math. Comput. **167**(2005), 964–975.

[38] S. Saha Ray, S. Singh, *Numerical solution of stochastic Volterra-Fredholm integral equations by hybrid Legendre Block-pulse functions*, Int. J. Nonlinear Sci. Numer. Simul. **19**(2018), 1–9.

[39] P. K. Sahu, S. Saha Ray, *Hybrid Legendre Block-pulse functions for the numerical solutions of system of nonlinear Fredholm-Hammerstein integral equations*, Appl. Math. Comput. **270**(2015), 871–878.

[40] V. K. Singh, R. K. Pandey, S. Singh, *A stable algorithm for Hankel transforms using hybrid of Block-pulse and Legendre polynomials*, Comput. Phys. Commun. **181**(2010), 1–10.

[41] V. K. Srivastava, M. K. Awasthi, S. Kumar, *Numerical approximation for HIV infection of $CD4^+$ T cells mathematical model*, Ain Shams Engineering Journal **5**(2014), 625–629.

[42] M. TAVASSOLI KAJANI, H. VENCHEH, *Solving second kind integral equations with hybrid Chebyshev and Block-pulse functions*, Appl. Math. Comput. **163**(2005), 71–77.

[43] M. THONGMOON, S. PUSJUSO, *The numerical solutions of differential transform method and the Laplace transform method for a system of differential equations*, Nonlinear Anal. Hybrid Syst. **4**(2010), 425–431.

[44] A. R. VAHIDI, E. BABOLIAN, Z. AZIMZADEH, *An improvement to the homotopy perturbation method for solving nonlinear Duffings equations*, Bull. Malays. Math. Sci. Soc. **41**(2018), 1105–1117.

[45] L. WANG, M. Y. LI, *Mathematical analysis of the global dynamics of a model for HIV infection of $CD4^+$ T cells*, Math. Biosci. **200**(2006), 44–57.

[46] S. YÜZBAŞI, *A numerical approach to solve the model for HIV infection of $CD4^+$ T cells*, Appl. Math. Model. **36**(2012), 5876–5890.

[47] S. YÜZBAŞI, *An exponential collocation method for the solutions of the HIV infection model of $CD4^+$ T cells*, Int. J. Biomath. **9**(2016), 1–15.