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Parabolically induced Banach space representation of p-adic groups

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Abstract. The paper expresses the dual of the parabolically induced p-adic Banach space representation of a p-adic group in terms of the tensor product.
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1. Introduction

In 1967, Prof. Robert Langlands introduced the Langlands program. It is about the connection between number theory and geometry. As a part thereof, he introduced the Local Langlands conjecture which is related to the n-dimensional complex representations. There are many different groups and many different fields for which these conjectures can be stated. One version is the p-adic Langlands correspondence. The p-adic Langlands correspondence for $GL_2(\mathbb{Q}_p)$ states:

$$\left\{\begin{array}{c}\text{2-dim } p\text{-adic representations}\\ \text{of } \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}p\text{-adic Banach space representations}\\ \text{of } GL_2(\mathbb{Q}_p)\end{array}\right\}$$

In 2002, Schneider and Teitelbaum developed the theory of p-adic Banach space representations of p-adic groups, which are important to study the p-adic Langlands program. In this paper, we study the p-adic Banach space representation of a split connected reductive p-adic group, parabolically induced from a character of a parabolic subgroup. We show that the continuous dual of such a representation can be expressed in terms of the tensor product of modules. This explicit description can be useful when applying the Schneider-Teitelbaum duality theory for studying Banach space representations because the duality theory of Schneider and Teitelbaum relates p-adic Banach space representations to the corresponding Iwasawa modules [9].

Let F be a finite extension of $\mathbb{Q}_p$ and let o_F denote the ring of integers of F. For a profinite group H, there is a o_F-module

$$o_F[[H]] = \operatorname*{Proj\,lim}_{N \subset \mathcal{N}(H)} o_F[H/N].$$

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Here $\mathcal{N}(H)$ denotes the family of all open normal subgroups of H. Then we can define Iwasawa module $F[[H]] = F \otimes_{o_F} o_F[[H]]$, [9] with the finest locally convex topology such that the inclusion of $o_F[[H]]$ is continuous.

In this paper, we have considered sequence finite extensions of $\mathbb{Q}_p$ such that $K \supseteq L \supseteq \mathbb{Q}_p$. Let $\mathfrak{p}_L$ denote the unique maximal ideal of o_L. Let $\mathbf{G}$ be a split and connected reductive algebraic $\mathbb{Z}$-group, and G_0 the group of o_L-points. Fix a maximal split torus T_0 in G_0, and a minimal parabolic subgroup $P_0 = P_\emptyset$ containing T_0. Let W be the Weyl group of G_0 and Δ the set of simple roots. For $\Theta \subset \Delta$, P_Θ denotes the standard parabolic subgroup corresponding to the set Θ. Set $P_{\Theta,0} = P_\Theta(o_L)$. Let $K[[G_0]]$ be the completed group algebra defined in [8] (see section 2).

Let $\chi : M_{\Theta,0} \to o_K^\times$ be a continuous character, extended to $P_{\Theta,0}$ by making it trivial on $U_{\Theta,0}$. We consider the degenerate principal series representation

$$Ind_{P_{\Theta,0}}^{G_0}(\chi^{-1}) = \{f \in C(G_0, K) \mid f(gp) = \chi(p)f(g) \text{ for any } g \in G_0,\ p \in P_{\Theta,0}\}.$$

The character χ induces a $K[[P_{\Theta,0}]]$-module structure on K (proposition 1, corollary 1). We write $K^{(\chi)}$ for this $K[[P_{\Theta,0}]]$-module.

Our main result is the following theorem.

Theorem 1. *Let $\Theta \subseteq \Delta$. Let $P_\Theta = M_\Theta U_\Theta$ and χ is a continuous character of $M_{\Theta,0}$. Then the dual of the degenerate principal series representation $Ind_{P_{\Theta,0}}^{G_0}(\chi^{-1})$ is isomorphic to $K[[G_0]] \otimes_{K[[P_{\Theta,0}]]} K^{(\chi)}$.*

The statement of the theorem for principal series representations (case $\Theta = \emptyset$) can be found in [9] for $GL_2(\mathbb{Q}_p)$ and in [10]. A consequence of the theorem is that the degenerate principal series representation $Ind_{P_{\Theta,0}}^{G_0}(\chi^{-1})$ is an admissible Banach space representation (see [7], Proposition 2.4).

We briefly describe the content of the paper. In section 2, we introduce notations. In section 3, we present some results related to the dual of degenerate principal series representations. We construct a convenient and explicit decomposition for $G_0/P_{\Theta,0}$ in section 4. Finally, in section 5, we produce the proof of the main theorem.

2. Notations

Let L be a finite extension of $\mathbb{Q}_p$, o_L its ring of integers and $\mathfrak{p}_L$ a unique maximal ideal of o_L. Let K be a finite extension of L, and define o_K and $\mathfrak{p}_K$ analogously. Let $\mathbf{G}$ be a split and connected reductive algebraic $\mathbb{Z}$-group, and $G = \mathbf{G}(L)$. We write l for $o_L/\mathfrak{p}_L$. For any algebraic subgroup H of G, we write $\bar{H}$ for $H(l)$ and H_0 for $H(o_L)$.

We fix a split maximal torus $\mathbf{T} \subset \mathbf{G}$. Let Φ denote the set of roots of $\mathbf{T}$ in $\mathbf{G}$. Also, we fix a base Δ of Φ. The choice of Δ determines the corresponding Borel subgroup $\mathbf{B}$.

We let W denote the Weyl group of G, which is naturally isomorphic to the quotient of the normalizer $N_{\mathbf{G}}(\mathbf{T})$ of $\mathbf{T}$ in $\mathbf{G}$. For $\Theta \subset \Delta$, we denote by Φ_Θ^+ (respectively, Φ_Θ^-) a set of positive (respectively, negative) roots in the linear span of Θ. We denote by W_Θ the subgroup of W generated by simple reflections for roots

$\alpha \in \Theta$. The set

$$[W/W_\Theta] = \{w \in W \mid w\Theta > 0\}$$

is a set of coset representatives of W/W_Θ as defined in [3]. Let $\mathbf{P}_\Theta = \mathbf{M}_\Theta \mathbf{U}_\Theta$ be a standard parabolic subgroup corresponding to Θ, with $\mathbf{M}_\Theta$ a standard Levi subgroup and $\mathbf{U}_\Theta$ its unipotent radical.

Let $\chi : M_{\Theta,0} \to o_K^\times$ be a continuous character. The character χ can be extended to $P_{\Theta,0}$ by making it trivial on $U_{\Theta,0}$. Define

$$Ind_{P_{\Theta,0}}^{G_0}(\chi^{-1}) = \{f \in C(G_0, K) \mid f(gp) = \chi(p)f(g) \ \ for \ \ any \ \ g \in G_0, \ p \in P_{\Theta,0}\}.$$

Via the left inverse translation action $g.f(x) = f(g^{-1}x)$, this is a K-Banach space representation of G_0. The representation $Ind_{P_{\Theta,0}}^{G_0}(\chi^{-1})$ is called the degenerate principal series representation.

Notice that G_0 is a compact p-adic group. Therefore we have the projective limit $o_K[[G_0]] := \operatorname{Proj} \lim_{H} o_K[G_0/H]$ taken over compact open normal subgroups H of G_0. We work with the completed group algebra $K[[G_0]] := K \otimes_{o_K} o_K[[G_0]]$. Let $C'(G_0, K)$ denote the vector space of continuous distributions on G_0 which, by definition, is the dual to the space $C(G_0, K)$ of continuous K-functions on G_0. Then $K[[G_0]]$ can be identified with $C'(G_0, K)$ by identifying $g \in G_0$ with the Dirac distribution δ_g.

Proposition 1. *Let $\chi : P_{\Theta,0} \to o_K^\times$ be a continuous character. Then χ induces an $o_K[[P_{\Theta,0}]]$-module structure on o_K.*

Proof. By continuity of χ at 1, for each integer $n \geq 0$ such that $\chi|_{P_{\Theta,n}} \neq 1$, there is a maximal integer l_n such that $\chi(P_{\Theta,n}) \subseteq 1 + \mathfrak{p}^{l_n}$. If $\chi|_{P_{\Theta,n}} = 1$, we take $l_n = \infty$. Thus, we can define a group homomorphism

$$\chi_n : P_{\Theta,0}/P_{\Theta,n} \longrightarrow (o_K/\mathfrak{p}^{l_n})^\times$$

in such a way that $\chi_n(pP_{\Theta,n}) = \chi(p)(1 + \mathfrak{p}^{l_n})$. By that, we have the corresponding ring homomorphism

$$\theta_n : o_K[P_{\Theta,0}/P_{\Theta,n}] \longrightarrow o_K/\mathfrak{p}^{l_n}$$

such that $\sum a_i \, p_i P_{\Theta,n} \longmapsto \sum \bar{a}_i \bar{\chi}(p_i)$, where $a_i \in o_K$ and $p_i \in P_{\Theta,0}$. It is clear from the construction that the maps $\{\theta_n\}_{n\geq 0}$ are compatible. Also, $\lim_{n\to\infty} l_n = \infty$ and hence $\operatorname{proj\,lim}_n o_K/\mathfrak{p}^{l_n} = o_K$. Then by the projective limit we obtain an o_K-linear continuous ring homomorphism, which we denote again by χ:

$$\chi : o_K[[P_{\Theta,0}]] \longrightarrow o_K.$$

The corresponding action $o_k[[P_{\Theta,0}]] \times o_K \to o_K$ is given by $(\mu, a) \mapsto \chi(\mu)a$. $\square$

Corollary 1. *Let $\chi : P_{\Theta,0} \to o_K^\times$ be a continuous character. Then χ induces a $K[[P_{\Theta,0}]]$-module structure on K.*

Proof. Tensoring with $\mathrm{id}_K \otimes_{o_K}$, we get a map $\chi : K[[P_{\Theta,0}]] \to K$. More specifically, $\mu \in K[[P_{\Theta,0}]]$ can be written as $\mu = a\mu_0$ for some $a \in K$ and $\mu_0 \in o_K[[P_{\Theta,0}]]$. Then $\chi(\mu) = a\chi(\mu_0)$. $\square$

We write $K^{(\chi)}$ for K equipped with the $K[[P_{\Theta,0}]]$-module structure induced by χ.

3. The dual of a principal series representation

The representation $Ind_{P_{\Theta,0}}^{G_0}(\chi^{-1})$ is a closed subspace of the Banach space $C(G_0, K)$. Therefore, we can define a short exact sequence of left $K[[G_0]]$-modules

$$0 \longrightarrow Ind_{P_{\Theta,0}}^{G_0}(\chi^{-1}) \hookrightarrow C(G_0, K) \longrightarrow A \longrightarrow 0, \tag{1}$$

where $A = C(G_0, K)/Ind_{P_{\Theta,0}}^{G_0}(\chi^{-1})$. Because G_0 acts on $Ind_{P_{\Theta,0}}^{G_0}(\chi^{-1})$ by left inverse translation $g \cdot f(x) = f(g^{-1}x)$, $K[G_0]$ acts on $Ind_{P_{\Theta,0}}^{G_0}(\chi^{-1})$. Then by continuity there is a left $K[[G_0]]$-action on $Ind_{P_{\Theta,0}}^{G_0}(\chi^{-1})$.

Taking the continuous dual of (1), we obtain the following exact sequence:

$$0 \longrightarrow A' \longrightarrow C'(G_0, K) \longrightarrow Ind_{P_{\Theta,0}}^{G_0}(\chi^{-1})' \longrightarrow 0.$$

Here, $A' = \{\mu \in C'(G_0, K) \mid \mu(f) = 0 \ \ \forall f \in Ind_{P_{\Theta,0}}^{G_0}(\chi^{-1})\}$. Identifying $C'(G_0, K)$ with $K[[G_0]]$, we get the following isomorphism:

$$Ind_{P_{\Theta,0}}^{G_0}(\chi^{-1})' \cong K[[G_0]]/A'. \tag{2}$$

Henceforth, we use the set A' as defined above.

Lemma 1. *The set A' defined above is a left $K[[G_0]]$-module.*

Proof. Define a map $(g \cdot l)(a) := l(g^{-1}a)$ for any $g \in G_0$, $a \in A$ and $l \in A'$. It is not hard to show that this map is a left action. Hence, A' has a left G_0-action. This G_0-action on A' extends to a $K[G_0]$-action, and by linearity and continuity it extends to a $K[[G_0]]$-action. Then A' is a left $K[[G_0]]$-module. □

Furthermore, with set A' being a subgroup of $K[[G_0]]$ and having a $K[[P_{\Theta,0}]]$-action, we can introduce a left $K[[P_{\Theta,0}]]$-module structure for $K[[G_0]]/A'$, by section 10.2 in [4].

4. Coset representatives for $G_0/P_{\Theta,0}$

Let $w \in W$. If $B^- = TU^-$ is the Borel subgroup containing T opposite to B, by 28.1 in [5], we have T-stable subgroups of U

$$U'_w = U \cap wUw^{-1}, \quad U_w = U \cap wU^-w^{-1}.$$

Their respective sets of roots partition Φ^+, $\Phi_w^+ = \{\alpha > 0 \mid w(\alpha) > 0\}$ and $\Phi_w^- = \{\alpha > 0 \mid w(\alpha) < 0\}$. Proposition in 28.1 [5] shows that for each $w \in W$, $U = U_wU'_w = U'_wU_w$, but in general this direct span is not a semidirect product. This implies that the double coset BwB can also be written as U_wwB. The following propositon is the main result in this section.

Proposition 2. *Let Θ be a subset of the set of simple roots. Let P_Θ be a standard parabolic subgroup corresponding to Θ. Then there is a disjoint union decomposition*

$$G_0 = \bigsqcup_w wU^-_{\Theta,w,1/2}P_{\Theta,0},$$

where w ranges over the set $[W/W_\Theta]$ and $U^-_{\Theta,w,1/2} = \prod\limits_{\substack{\alpha<0,\\ w\alpha>0}} U_{\alpha,0} \times \prod\limits_{\substack{\alpha<0,\\ w\alpha<0,\\ \alpha\in\Phi^-\setminus\Phi^-_\Theta}} U_{\alpha,1}$.

Proof. We have

$$\bar{G} = \bigsqcup_{w} \bar{B}w\bar{B}. \tag{3}$$

Identity (3) gives a canonical map $W \longrightarrow \bar{B}\backslash\bar{G}/\bar{B}$. We also know that $P_\Theta = \bigcup_{w\in W_\Theta} BwB$. Proposition 1.3.1 of [3] implies

$$\bar{G} = \bigsqcup_{w\in[W/W_\Theta]} \bar{B}w\bar{P}_\Theta. \tag{4}$$

In general, we can present BwP_Θ as $TUwP_\Theta$, which can be written as UwP_Θ since T normalizes U. Since $UwB = U_w wB$, we have $UwP_\Theta = U_w wP_\Theta$. It follows

$$\bar{G} = \bigsqcup_{w\in[W/W_\Theta]} \bar{U}_w w\bar{P}_\Theta.$$

Pulling back, we have $G_0 = \bigsqcup\limits_{w\in[W/W_\Theta]} (U_0 \cap wU^-w^{-1})wG_1P_{\Theta,0}$. Then

$$G_0 = \bigsqcup_{w\in[W/W_\Theta]} U_{w,0}wG_1P_{\Theta,0}. \tag{5}$$

Since $G_1 = U_1^-T_1U_1$, we can write

$$G_0 = \bigsqcup_{w\in[W/W_\Theta]} U_{w,0}wG_1P_{\Theta,0} = \bigsqcup_{w\in[W/W_\Theta]} wU^-_{w,1/2}P_{\Theta,0},$$

where $U^-_{w,1/2} = w^{-1}U_{w,0}wU_1^-$ by [2]. Again by [2], we have

$$U^-_{w,1/2} = \prod_{\substack{\alpha<0,\\ w\alpha>0}} U_{\alpha,0} \times \prod_{\substack{\alpha<0,\\ w\alpha<0}} U_{\alpha,1}.$$

From the above product we want to eliminate the subgroups U_α contained in P_Θ. Note that $U^-_{w,1/2}$ is a subgroup of U_0^-. We know that $U_0^- \cap P_{\Theta,0} = \prod\limits_{\alpha\in\Phi^-_\Theta} U_{\alpha,0}$. Since $w \in [W/W_\Theta] = \{w \in W \mid w\Theta > 0\}$, we have

$$\prod_{\alpha\in\Phi^-_\Theta} U_{\alpha,0} \cap \prod_{\substack{\alpha<0,\\ w\alpha>0}} U_{\alpha,1} = 1 \quad \text{and} \quad \prod_{\alpha\in\Phi^-_\Theta} U_{\alpha,0} \cap \prod_{\substack{\alpha<0,\\ w\alpha<0}} U_{\alpha,1} = \prod_{\alpha\in\Phi^-_\Theta} U_{\alpha,1}.$$

Therefore, we can write G_0 as a disjoint union

$$G_0 = \bigsqcup_{w\in[W/W_\Theta]} wU^-_{\Theta,w,1/2}P_{\Theta,0}, \tag{6}$$

where

$$U^-_{\Theta,w,1/2} = \prod_{\substack{\alpha<0,\\ w\alpha>0}} U_{\alpha,0} \times \prod_{\substack{\alpha<0,\\ w\alpha<0,\\ \alpha\in\Phi^-\setminus\Phi^-_\Theta}} U_{\alpha,1}.$$

□

The following technical result is analogous to Lemma 4.5 of [2].

Lemma 2. *Fix* $w_0 \in [W/W_\Theta]$ *and* $u_0 \in U^-_{\Theta,w_0,1/2}$. *Let* $n \geq 1$. *Then*

$$u_0^{-1}w_0^{-1}(\bigsqcup_w wU^-_{\Theta,w,1/2}) \cap G_nP_{\Theta,0} = U^-_{\Theta,n}$$

Proof. Consider the projection from G_0 to $\bar{G}$. The sets $wU^-_{\Theta,w,1/2}$ for $w \in [W/W_\Theta]$ all project into different elements of $\bar{B}\backslash\bar{G}/\bar{P}_\Theta$. Then,

$$w_0u_0G_1P_{\Theta,0} \cap wU^-_{\Theta,w,1/2} \neq \phi \Longrightarrow w = w_0.$$

Without loss of generality, let us assume $w = w_0$. Then

$$u_0^{-1}w_0^{-1}wU^-_{\Theta,w,1/2} = u_0^{-1}U^-_{\Theta,w,1/2} \subset U^-_{\Theta,0}.$$

Hence, it is enough to prove $U^-_{\Theta,0} \cap G_nP_{\Theta,0} \subset G_n$. We consider the projection to G_0/G_n. Since $U^-_{\Theta,0} \cap P_{\Theta,0} = \{1\}$, the only element of G_0/G_n which is in the image of both $P_{\Theta,0}$ and $U^-_{\Theta,0}$ is the identity. □

5. Main theorem

We begin by analysing $K[[G_0]] \otimes_{K[[P_{\Theta,0}]]} K^{(\chi)}$. We denote $K[[G_0]] \otimes_{K[[P_{\Theta,0}]]} K^{(\chi)}$ by $M^{(\chi)}$ and $o_K[[G_0]] \otimes_{o_K[[P_{\Theta,0}]]} o_K^{(\chi)}$ by $M_0^{(\chi)}$.

Lemma 3. *Let* $\Theta \subset \Delta$. *Let* $P_\Theta = M_\Theta U_\Theta$ *and let* χ *be a continuous character of* $M_{\Theta,0}$. *Then*

$$\begin{aligned} M_0^{(\chi)} &= (\operatorname{Proj}\lim_n o_K[G_0/G_n]) \otimes_{o_K[[P_{\Theta,0}]]} o_K^{(\chi)} \\ &\cong \operatorname{Proj}\lim_n \ (o_K[G_0/G_n] \otimes_{o_K[P_{\Theta,0}]} o_K^{(\chi)}). \end{aligned}$$

Proof. The case $\Theta = \emptyset$ is proposition 3.2 of [1]. With minor changes, the proof applies to an arbitrary Θ. □

Lemma 4. *Let* $\Theta \subset \Delta$. *Let* $P_\Theta = M_\Theta U_\Theta$ *and* χ *let be a continuous character of* $M_{\Theta,0}$. *Then*

$$o_K[G_0/G_n] \otimes_{o_K[P_{\Theta,0}]} o_K^{(\chi)} \cong \bigoplus_{w\in[W/W_\Theta]} w\ o_K[U^-_{\Theta,w,1/2}/U^-_{\Theta,n}]$$

as o_K*-modules.*

Proof. From proposition 2, $G_0 = \bigsqcup_{w\in[W/W_\Theta]} wU^-_{\Theta,w,1/2}P_{\Theta,0}$. Hence, if $g \in G_0$, we can write this in a unique way as $g = wup$, where $u \in U^-_{\Theta,w,1/2}$ and $p \in P_{\Theta,0}$. Any $x \in o_K[G_0/G_n]$ can be written as a finite sum:

$$x = \sum_{i=1}^{m} a_i w_i u_i p_i G_n,$$

where $a_i \in o_K$, $w_i \in [W/W_\Theta]$, $u_i \in U^-_{\Theta,w_i,1/2}$, and $p_i \in P_{\Theta,0}$.

Let $agG_n \otimes 1 \in o_K[G_0/G_n] \otimes_{o_K[P_{\Theta,0}]} o_K^{(\chi)}$. Writing $g = wup$ as above, we get $agG_n \otimes 1 = a\chi(p)wuG_n \otimes 1$. Let $S = \{u_1, \cdots, u_t\}$ be a set of representatives of $U^-_{\Theta,w,1/2}/U^-_{\Theta,n}$. Then $agG_n \otimes 1$ can be written as $a\chi(p)wu_\ell G_n \otimes 1$ for some $u_\ell \in S$. We want to show that this expression is unique. First we show that w is unique.

Let us denote by I the Iwahori subgroup of G_0. Then $I = G_1P_0$. So $G_n \subseteq I$ for all $n \geq 1$. Fix $w_0 \in [W/W_\Theta]$. The coset $w_0U^-_{\Theta,w,1/2}P_{\Theta,0}$ is a disjoint union of several cosets IwB_0 because $P_\Theta = \bigcup_{w\in W_\Theta} BwB$. However, w_0 has least length in w_0W_Θ, by Lemma 1.1.2 in [3]. Hence, if $wU^-_{\Theta,w_0,1/2}P_{\Theta,0} \cap w_0U^-_{\Theta,w,1/2}P_{\Theta,0} \neq \phi$, for some $w \in [W/W_\Theta]$, it follows $w = w_0$. In particular, for $g \in G_0$, there is a unique $w \in [W/W_\Theta]$ such that $gG_n = wupG_n$, for some $u \in U^-_{\Theta,w,1/2}$ and $p \in P_{\Theta,0}$.

To show that $u_\ell \in S$ is unique, assume that $a\chi(p)wu_\ell G_n = a\chi(p)wu_jG_n$ for some $u_j \in S$. Then $u_j^{-1}u_\ell \in G_n \cap U^-_{\Theta,w,1/2} = U^-_{\Theta,n}$, so $u_j = u_\ell$.

Using the uniqueness of the above expression of $agG_n \otimes 1$, we can define

$$\Psi_n(agG_n \otimes 1) = a\chi(p)wu_\ell U^-_{\Theta,n}.$$

This is well-defined and does not depend on the choice of the set of representatives S. We extend Ψ_n o_K-linearly to a map

$$\Psi_n : o_K[G_0/G_n] \otimes_{o_K[P_{\Theta,0}]} o_K^{(\chi)} \longrightarrow \bigoplus_{w\in[W/W_\Theta]} w\, o_K[U^-_{\Theta,w,1/2}/U^-_{\Theta,n}].$$

Then Ψ_n is clearly surjective and injective. □

The above results yield the following lemma.

Lemma 5. *$M^{(\chi)}$ maps isomorphically onto $\bigoplus_{w\in[W/W_\Theta]} wK[[U^-_{\Theta,w,1/2}]]$ as a topological K-module.*

Proof. By Lemma 4, we have a collection of maps

$$\Psi_n : o_K[G_0/G_n] \otimes_{o_K[P_{\Theta,0}]} o_K^{(\chi)} \longrightarrow \bigoplus_{w\in[W/W_\Theta]} w\, o_K[U^-_{\Theta,w,1/2}/U^-_{\Theta,n}].$$

First we show that these maps are compatible. Let $m > n$. Then the following diagram commutes.

$$\begin{tikzcd} o_K[G_0/G_m] \otimes_{o_K[P_{\Theta,0}]} o_K^{(\chi)} \arrow[r, "\Psi_m"] \arrow[d, "\varphi_{mn}"'] & \bigoplus_{w\in[W/W_\Theta]} w\, o_K[U^-_{\Theta,w,1/2}/U^-_{\Theta,m}] \arrow[d, "\varphi'_{mn}"] \\ o_K[G_0/G_n] \otimes_{o_K[P_{\Theta,0}]} o_K^{(\chi)} \arrow[r, "\Psi_n"] & \bigoplus_{w\in[W/W_\Theta]} w\, o_K[U^-_{\Theta,w,1/2}/U^-_{\Theta,n}] \end{tikzcd}$$

Let $\{g_1, \cdots, g_r\}$ and $\{u_1, \cdots, u_t\}$ be sets of representatives of G_0/G_m and $U^-_{\Theta,w,1/2}/U^-_{\Theta,m}$, respectively. Let $(\mu \otimes a) \in o_K[G_0/G_m] \otimes_{o_K[P_{\Theta,0}]} o_K^{(\chi)}$, where $\mu = \sum_{i=1}^{r} a_i g_i G_m$. Then,

$$\Psi_n \circ \varphi_{mn}(\mu \otimes a) = \Psi_n(\sum_{i=1}^{r} a_i g_i G_n \otimes a) = \sum_{i=1}^{t} a_i \chi(p_i) w_i u_i U^-_{\Theta,n}.$$

Similarly,

$$\varphi'_{mn} \circ \Psi_m(\mu \otimes a) = \varphi'_{mn}(\sum_{i=1}^{t} a_i \chi(p_i) w_i u_i U^-_{\Theta,m}) = \sum_{i=1}^{t} a_i \chi(p_i) w_i u_i U^-_{\Theta,n}.$$

Hence we have,

$$\Psi = \operatorname{Proj}_n \lim \Psi_n : \operatorname{Proj}_n \lim (o_K[G_0/G_n] \otimes_{o_K[P_{\Theta,0}]} o_K^{(\chi)}) \longrightarrow \operatorname{Proj}_n \lim (\bigoplus_{w \in [W/W_\Theta]} w o_K[U^-_{\Theta,w,1/2}/U^-_{\Theta,n}]).$$

The components Ψ_n are injective, and by general properties of projective limits Ψ is injective as well. Surjectivity follows from Lemma 1.1.5 in [6] because Ψ is a map of inverse systems of compact topological groups. In conclusion, Ψ is an isomorphism from $M_0^{(\chi)}$ to $\bigoplus_{w \in [W/W_\Theta]} w o_K[[U^-_{\Theta,w,1/2}]]$. This implies that $M^{(\chi)}$ is isomorphic to $\bigoplus_{w \in [W/W_\Theta]} wK[[U^-_{\Theta,w,1/2}]]$. $\square$

Every element of $M^{(\chi)} = K[[G_0]] \otimes_{K[[P_0]]} K^{(\chi)}$ can be written as $\mu \otimes 1$, for some $\mu \in K[[G_0]]$. The map

$$K[[G_0]] \longrightarrow M^{(\chi)}$$
$$\mu \longmapsto \mu \otimes 1$$

realizes $M^{(\chi)}$ as a quotient of $K[[G_0]]$. For $\mu \in K[[G_0]]$, set $[\mu] := \mu \otimes 1 \in M^{(\chi)}$. The embedding

$$\bigoplus_{w \in [W/W_\Theta]} wK[[U^-_{\Theta,w,1/2}]] \hookrightarrow K[[G_0]],$$

together with Lemma 5 gives us the following corollary:

Corollary 2. $\{ [\mu] \mid \mu \in K[[G_0]] \} = \{ [\mu] \mid \mu \in \bigoplus_{w \in [W/W_\Theta]} wK[[U^-_{\Theta,w,1/2}]] \}$.

Finally, after combining the above result with (2), it remains to prove

$$\bigoplus_{w \in [W/W_\Theta]} wK[[U^-_{\Theta,w,1/2}]] \cong K[[G_0]]/A'.$$

Proposition 3. *Define $\Psi : K[[G_0]] \times K^{(\chi)} \longrightarrow K[[G_0]]/A'$ by $\Psi = \pi \circ \varphi$, where π is the projection from $K[[G_0]]$ to $K[[G_0]]/A'$ and $\varphi : K[[G_0]] \times K^{(\chi)} \to K[[G_0]]$ is given by $\varphi(\mu, a) = a\mu$. Then the map Ψ is $K[[P_{\Theta,0}]]$-balanced and $K[[G_0]]$-linear in the first coordinate.*

$$
\begin{tikzcd}
K[[G_0]] \times K^{(\chi)} \arrow[rr, "\varphi"] \arrow[dr, "\Psi"'] & & K[[G_0]] \arrow[dl, "\pi"] \\
& K[[G_0]]/A' &
\end{tikzcd}
$$

Proof. First, we observe that φ is $K[[P_{\Theta,0}]]$-bilinear and that π is $K[[P_{\Theta,0}]]$-linear, so Ψ is $K[[P_{\Theta,0}]]$-bilinear.

Let $\eta \in K[[P_{\Theta,0}]]$. Then we have to prove

$$\Psi(\mu, \eta a) = \Psi(\mu\eta, a). \tag{7}$$

Let us first prove this for Dirac distributions. Take any δ_g and δ_p with any $g \in G_0$ and $p \in P_{\Theta,0}$ respectively. Then the expression in (7) can be written as

$$\Psi(\delta_g, \delta_p a) = \Psi(\delta_g\delta_p, a). \tag{8}$$

According to the way we have defined Ψ, we can show that $\Psi(\delta_g, \delta_p a) = \chi(p)a\delta_g + A'$ and $\Psi(\delta_g\delta_p, a) = a\delta_{gp} + A'$. Then we can see that $[\ \chi(p)a\delta_g - a\delta_{gp}] \in A'$ because $\chi(p)a\delta_g(f) - a\delta_{gp}(f) = 0$ for any $f \in Ind_{P_{\Theta,0}}^{G_0}(\chi^{-1})$. Thus expression (8) is true for $K[P_{\Theta,0}]$; then by continuity it is true for $K[[P_{\Theta,0}]]$.

For the second part of the proposition, let $\eta \in K[[G_0]]$. Then $\Psi(\eta\mu, a) = \pi(\varphi(\eta\mu, a)) = a\eta\mu + A'$, which is explicitly the same as $\eta\Psi(\mu, a)$. Thus Ψ is $K[[G_0]]$-linear. □

With these required technical results, now we present the main result.

Theorem 2. *Let $\Theta \subset \Delta$. Let $P_\Theta = M_\Theta U_\Theta$ and χ let be a continuous character of $M_{\Theta,0}$. Then the dual of the principal series representation $Ind_{P_{\Theta,0}}^{G_0}(\chi^{-1})$ is isomorphic to $K[[G_0]] \otimes_{K[[P_{\Theta,0}]]} K^{(\chi)}$.*

Proof. By proposition 3, there is a $K[[P_{\Theta,0}]]$-balanced map

$$\Psi : K[[G_0]] \times K^{(\chi)} \longrightarrow K[[G_0]]/A',$$

$K[[G_0]]$-linear in the first coordinate. Therefore, there exists the corresponding $K[[G_0]]$-linear map

$$\Phi : K[[G_0]] \otimes_{K[[P_{\Theta,0}]]} K^{(\chi)} \longrightarrow K[[G_0]]/A'. \tag{9}$$

We have to prove that Φ is an isomorphism. It is clearly surjective because Ψ is surjective.

For injectivity, we take a non-zero element $[\eta]$ of $K[[G_0]] \otimes K^{(\chi)}$, and construct a representative $\eta = \sum_{w \in [W/W_\Theta]} w\eta_w$ for it as in Lemma 5. Here $\eta_w \in K[[U^-_{\Theta,w,1/2}]]$ for each $w \in [W/W_\Theta]$.

By scaling, we may assume that $\eta_w = (\eta_{w,l})_{l=0}^{\infty} \in o_K[[U^-_{\Theta,w,1/2}]]$ for each w, and that there exists $n \geq 1$, $w_0 \in [W/W_\Theta]$ and $\bar{u_0} \in U^-_{\Theta,w,1/2}/U^-_{\Theta,n}$, such that the coefficient c_0 of $\bar{u_0}$ of $\eta_{w,n}$ is a unit.

Let us now choose $u_0 \in U^-_{\Theta,w,1/2}$, which projects to $\bar{u_0}$, and let $\mu = u_0^{-1}w_0^{-1}\eta$. Since we can write μ as an element of the projective limit $\mu = (\mu_l)_{l=0}^{\infty}$, then

$$\mu_n = c_0 + \sum_{\bar{g}\in G_0/G_n, \bar{g}\neq 1} c_{\bar{g}}\bar{g}, \qquad c_0 \in o_K^{\times},\, c_{\bar{g}} \in o_K. \tag{10}$$

We can write $\mu = \mu' + \mu''$, where $\mu' \in o_K[[G_n]]$ and $\operatorname{supp}(\mu'') \subset G_0\backslash G_n$. Also note that the support of μ lies in $u_0^{-1}w_0^{-1}(\bigsqcup_w wU^-_{\Theta,w,1/2})$ and by Lemma 2 $\operatorname{supp}(\mu) \cap G_nP_{\Theta,0}$ is in G_n. Thus the support of μ'' is actually disjoint from $G_nP_{\Theta,0}$.

Moreover, we have the image of μ' under the augmentation map is precisely c_0, which is the coefficient of the identity coset of μ' in equation (10). Since c_0 is a unit, we know from proposition 7.1 in [2] that μ' is an invertible element of $o_K[[G_n]]$. Multiplying by its inverse,

$$(\mu')^{-1}\mu = 1 + (\mu')^{-1}\mu''.$$

Let us denote a new form of the element as $\eta_0 = 1 + \nu$, where the support of ν is disjoint from $G_nP_{\Theta,0}$. We remark that $1 \in o_K[[G_0]]$ is the Dirac distribution δ_1. We show here that $[\eta_0] \notin ker\ \Phi$.

Recall that $G_0 = \bigsqcup_{w\in[W/W_\Theta]} wU^-_{\Theta,w,1/2}P_{\Theta,0}$, where

$$U^-_{\Theta,w,1/2} = \prod_{\substack{\alpha<0,\\ w\alpha>0}} U_{\alpha,0} \times \prod_{\substack{\alpha<0,\\ w\alpha<0,\\ \alpha\in\Phi^-\backslash\Phi^-_\Theta}} U_{\alpha,1}.$$

Since $U^-_{\Theta,1,1/2} = \prod_{\alpha\in\Phi^-\backslash\Phi^-_\Theta} U_{\alpha,1} = U^-_{\Theta,1}$, we have $G_n \cap U^-_{\Theta,1,1/2} = G_n \cap U^-_{\Theta,1}$. Furthermore, as $G_n \supset U^-_{\Theta,n}$, $G_n \cap U^-_{\Theta,1} = U^-_{\Theta,n}$. We may define

$$f(g) = \begin{cases} \chi(p), & \text{if } g = xp,\ x \in U^-_{\Theta,n}, p \in P_{\Theta,0}, \\ 0, & \text{otherwise.} \end{cases}$$

The function f is in the induced space with support in $G_nP_{\Theta,0}$. Then we have $\eta_0(f) \neq 0$ because $(1+\nu)(f) = \delta_1(f) + \nu(f) = f(1) = \chi(1) \neq 0$. That gives $[\eta_0] \notin \ker\Phi$, which implies $[(\mu')^{-1}\mu]$ is not an element of the kernel of Φ. Hence, $\Phi([(\mu')^{-1}\mu]) \notin A'$.

As Φ is a $K[[G_0]]$-linear map, $\Phi([(\mu')^{-1}\mu]) = (\mu')^{-1} \cdot \Phi([\mu])$. Furthermore, since A' is a subset of $K[[G_0]]$ and a left $K[[G_0]]$-module, it is a left $K[[G_0]]$-ideal. Therefore, $\Phi([\mu]) \notin A'$. Similarly, $\Phi([\eta]) \notin A'$.

This means Φ is an injective map. This completes the proof. $\square$

Corollary 3. *$Ind_{P_{\Theta,0}}^{G_0}(\chi^{-1})$ is an admissible G_0-representation.*

We know that $K[[G_0]] \otimes_{K[[P_{\Theta,0}]]} K^{(\chi)}$ is finitely generated $K[[G_0]]$-module. It is generated by the element $1 \otimes 1$. As the dual of the principal series representation $Ind_{P_{\Theta,0}}^{G_0}(\chi^{-1})$ is isomorphic to $K[[G_0]] \otimes_{K[[P_{\Theta,0}]]} K^{(\chi)}$, by Lemma 3.4 of [9], $Ind_{P_{\Theta,0}}^{G_0}(\chi^{-1})$ is an admissible G_0-representation.

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