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More Fibonacci-Bernoulli relations with and without balancing polynomials

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Abstract. We continue our study on relationships between Bernoulli polynomials and balancing (Lucas-balancing) polynomials. From these polynomial relations, we deduce new combinatorial identities with Fibonacci (Lucas) and Bernoulli numbers. Moreover, we prove a special identity involving Bernoulli polynomials and Fibonacci numbers in arithmetic progression. Special cases and some corollaries will highlight interesting aspects of our findings. Our results complement and generalize these of Frontczak (2019).

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1. Motivation and preliminaries

Let $B_n(x)$, $x \in \mathbb{C}$, be the n-th Bernoulli polynomial defined by

$$H(x,z) = \sum_{n=0}^{\infty} B_n(x)\frac{z^n}{n!} = \frac{ze^{xz}}{e^z-1} \qquad (|z| < 2\pi),$$

and let $B_n = B_n(0)$ be the n-th Bernoulli number [1].

Let further $B_n^*(x)$ be the n-th balancing polynomial [2], i.e., polynomials defined by the recurrence

$$B_n^*(x) = 6xB_{n-1}^*(x) - B_{n-2}^*(x), \qquad n \geq 2,$$

with the initial terms $B_0^*(x) = 0$ and $B_1^*(x) = 1$. Similarly, Lucas-balancing polynomials are defined by

$$C_n(x) = 6xC_{n-1}(x) - C_{n-2}(x), \qquad n \geq 2,$$

with the initial terms $C_0(x) = 1$ and $C_1(x) = 3x$. For more information about these polynomials see [2, 4, 5, 8, 9, 10]. The numbers $B_n^*(1) = B_n^*$ and $C_n(1) = C_n$

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are called balancing and Lucas-balancing numbers, respectively (see entries A001109 and A001541 in the On-Line Encyclopedia of Integer Sequences [11]).

Connections between Bernoulli polynomials $B_n(x)$ and balancing polynomials $B_n^*(x)$ are interesting, as they also give relations involving Bernoulli numbers and Fibonacci numbers (we refer to the papers [3, 6, 12]). The links are

$$B_n^*\Big(\frac{L_{2m}}{6}\Big) = \frac{F_{2mn}}{F_{2m}}, \qquad C_n\Big(\frac{L_{2m}}{6}\Big) = \frac{L_{2mn}}{2}, \tag{1}$$

$$B_n^*\Big(\frac{i}{6}L_{2m+1}\Big) = i^{n-1}\frac{F_{(2m+1)n}}{F_{2m+1}}, \qquad C_n\Big(\frac{i}{6}L_{2m+1}\Big) = i^n\frac{L_{(2m+1)n}}{2}, \tag{2}$$

where m is a nonnegative integer, $i=\sqrt{-1}$, and F_n and L_n denote Fibonacci and Lucas numbers, respectively. These sequences are defined by $F_0=0$, $F_1=1$, $L_0=2$, $L_1=1$ and $X_n=X_{n-1}+X_{n-2}$ for $n\ge 2$ (entries A000045 and A000032 in [11]).

Recently, Frontczak [3] showed, among other things, that

$$\sum_{\substack{k=0\\ n\equiv k \pmod 2}}^{n}\binom{n}{k}B_k^*(x)(2\sqrt{9x^2-1})^{n-k}B_{n-k} = nC_{n-1}(x). \tag{3}$$

Goubi [7] instantly “improved” this relation to

$$\sum_{k=0}^{n}\binom{n}{k}B_k^*(x)(2\sqrt{9x^2-1})^{n-k}B_{n-k} = n\big(C_{n-1}(x)-\sqrt{9x^2-1}B_{n-1}^*(x)\big). \tag{4}$$

We point out that since $B_{2n+1}=0$ for $n\ge 1$, the only non-zero contribution in Goubi's sum on the left comes from the index $k=n-1$, which obviously equals

$$\binom{n}{n-1}B_{n-1}^*(x)(2\sqrt{9x^2-1})\Big(-\frac{1}{2}\Big) = -n\sqrt{9x^2-1}B_{n-1}^*(x).$$

So, identities (3) and (4) are actually equivalent and the “improvement” is a simple reformulation. Nevertheless, to keep the notation simple, we will renounce the mod notation and work with the second formulation.

In this paper, we prove more relations between Bernoulli polynomials and balancing polynomials. The proofs are based on our recent findings concerning exponential generating functions for these polynomials. From these polynomial relations, we deduce new combinatorial identities with Fibonacci (Lucas) and Bernoulli numbers. Moreover, we prove a special identity involving Bernoulli polynomials and Fibonacci numbers in arithmetic progression. Some consequences are stated as corollaries.

2. New Bernoulli-balancing relations

The next lemma [4] deals with exponential generating functions for balancing and Lucas-balancing polynomials. It will play a key role in the first part of the paper.

Lemma 1. *Let $b_1(x,z)$ and $b_2(x,z)$ be the exponential generating functions of odd and even indexed balancing polynomials, respectively. Then*

$$\begin{aligned} b_1(x,z) &= \sum_{n=0}^{\infty} B^*_{2n+1}(x)\frac{z^n}{n!} \\ &= \frac{e^{(18x^2-1)z}}{\sqrt{9x^2-1}}\big(3x\sinh(6x\sqrt{9x^2-1}z)+\sqrt{9x^2-1}\cosh(6x\sqrt{9x^2-1}z)\big) \end{aligned}$$

and

$$b_2(x,z) = \sum_{n=0}^{\infty} B^*_{2n}(x)\frac{z^n}{n!} = \frac{e^{(18x^2-1)z}}{\sqrt{9x^2-1}}\sinh(6x\sqrt{9x^2-1}z).$$

Similarly, for Lucas-balancing polynomials we have

$$\begin{aligned} c_1(x,z) &= \sum_{n=0}^{\infty} C_{2n+1}(x)\frac{z^n}{n!} \\ &= e^{(18x^2-1)z}\big(3x\cosh(6x\sqrt{9x^2-1}z)+\sqrt{9x^2-1}\sinh(6x\sqrt{9x^2-1}z)\big) \end{aligned}$$

and

$$c_2(x,z) = \sum_{n=0}^{\infty} C_{2n}(x)\frac{z^n}{n!} = e^{(18x^2-1)z}\cosh(6x\sqrt{9x^2-1}z).$$

We start with the following results involving even indexed balancing polynomials.

Theorem 1. *For each $n \geq 0$ and $x \in \mathbb{C}$, we have*

$$\begin{aligned} &\sum_{k=0}^{n}\binom{n}{k}(12x\sqrt{9x^2-1})^{n-k}B_{n-k}B^*_{2k}(x) \\ &\qquad = 6xn\big(C_{2n-2}(x)-\sqrt{9x^2-1}B^*_{2n-2}(x)\big). \end{aligned} \tag{5}$$

Proof. From

$$\frac{2}{e^{2x}-1} = \coth x - 1$$

we get

$$H(0,12x\sqrt{9x^2-1}z) = 6xz\sqrt{9x^2-1}\big(\coth(6x\sqrt{9x^2-1}z)-1\big).$$

This yields

$$\begin{aligned} &\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}\binom{n}{k}(12x\sqrt{9x^2-1})^{n-k}B_{n-k}B^*_{2k}(x)\Big)\frac{z^n}{n!} \\ &\quad = b_2(x,z)H(0,12x\sqrt{9x^2-1}z) \\ &\quad = 6xze^{(18x^2-1)z}\big(\cosh(6x\sqrt{9x^2-1}z)-\sinh(6x\sqrt{9x^2-1}z)\big) \\ &\quad = 6xzc_2(x,z)-6x\sqrt{9x^2-1}z\,b_2(x,z) \\ &\quad = 6x\sum_{n=0}^{\infty} n\big(C_{2n-2}(x)-\sqrt{9x^2-1}B^*_{2n-2}(x)\big)\frac{z^n}{n!}. \end{aligned}$$

The proof is complete. □

Corollary 1. *For each $n \geq 0$, the following relation holds:*

$$\sum_{k=0}^{n} \binom{n}{k} (24\sqrt{2})^{n-k} B_{2k}^{*} B_{n-k} = 6n\big(C_{2n-2} - 2\sqrt{2} B_{2n-2}^{*}\big).$$

Proof. Set $x = 1$ in (5). $\square$

Corollary 2. *For each $n \geq 0$ and $j \geq 1$,*

$$\sum_{k=0}^{n} \binom{n}{k} (\sqrt{5} F_{2j})^{n-k} F_{2kj} B_{n-k} = \frac{n}{2} F_{2j}\big(L_{2j(n-1)} - \sqrt{5} F_{2j(n-1)}\big). \tag{6}$$

Proof. Evaluate (5) at the points $x = \frac{i}{6} L_{2m+1}$ and $x = \frac{1}{6} L_{2m}$, respectively, and use the links from (1) and (2). To simplify the square root recall that $L_n^2 = 5F_n^2 + (-1)^n 4$. $\square$

The special case

$$\sum_{k=0}^{n} \binom{n}{k} (\sqrt{5})^{n-k} F_{2k} B_{n-k} = \frac{n}{2}\big(L_{2n-2} - \sqrt{5} F_{2n-2}\big)$$

appears as equation (22) in [3]. We will derive an extension of this result in the sequel.

Remark 1. *By reindexing, we can write* (5) *as follows:*

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \big(144x^2(9x^2-1)\big)^k B_{2k} B_{2(n-2k)}^{*}(x) = 6nx C_{2(n-1)}(x). \tag{7}$$

Another interesting identity involving even indexed balancing polynomials is our next theorem.

Theorem 2. *For each $n \geq 0$ and $x \in \mathbb{C}$, we have the relation*

$$\begin{aligned} &\sum_{k=0}^{n} \binom{n}{k} (12x\sqrt{9x^2-1})^{n-k} B_{2k}^{*}(x) B_{n-k}(x) \\ &\qquad = 6nx\big(18x^2 - 1 + 6x(2x-1)\sqrt{9x^2-1}\big)^{n-1}. \end{aligned} \tag{8}$$

Proof. Since

$$H(x,z) = \frac{z}{2} \frac{e^{(x-1/2)z}}{\sinh \frac{z}{2}},$$

it follows that

$$\begin{aligned} &\sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} \binom{n}{k} (12x\sqrt{9x^2-1})^{n-k} B_{2k}^{*}(x) B_{n-k}(x) \Big) \frac{z^n}{n!} \\ &\quad = b_2(x,z) H(x, 12x\sqrt{9x^2-1}z) \\ &\quad = 6xze^{(18x^2-1)z + 6x(2x-1)\sqrt{9x^2-1}z} \\ &\quad = 6x \sum_{n=0}^{\infty} n\big(18x^2 - 1 + 6x(2x-1)\sqrt{9x^2-1}\big)^{n-1} \frac{z^n}{n!}. \end{aligned}$$

The proof is complete. □

Corollary 3. *For each $n \geq 0$,*

$$\sum_{k=0}^{n} \binom{n}{k} (3\sqrt{5})^{n-k} \big(2^{1-(n-k)} - 1\big) F_{4k} B_{n-k} = 3n \Big(\frac{7}{2}\Big)^{n-1}. \tag{9}$$

Proof. Set $x = \frac{1}{2}$ in (8) and use $B^*_{2n}(\frac{1}{2}) = F_{4n}$ and $B_n(\frac{1}{2}) = (2^{1-n} - 1)B_n$ [1, Corollary 9.1.5]. □

The last identity could be compared with

$$\sum_{k=0}^{n} \binom{n}{k} (\sqrt{5})^{n-k} \big(2^{1-(n-k)} - 1\big) F_{2k} B_{n-k} = n \Big(\frac{3}{2}\Big)^{n-1}, \tag{10}$$

which is equation (30) in [3]. It is maybe worth remarking that the value $x = -\frac{1}{2}$ in conjunction with $B^*_n(-x) = (-1)^{n+1} B^*_n(x)$ [2] and the difference equation for Bernoulli polynomials $B_n(x+1) - B_n(x) = nx^{n-1}$ [1, Proposition 9.1.3] gives

$$\begin{aligned}\sum_{k=0}^{n} \binom{n}{k} (-3\sqrt{5})^{n-k} F_{4k} \big((2^{1-(n-k)} - 1)B_{n-k} + (n-k)(-1)^{n-k} 2^{1-(n-k)}\big) \\ = 3n \Big(\frac{7+6\sqrt{5}}{2}\Big)^{n-1}.\end{aligned}$$

So, by Corollary 3, we end with

$$\sum_{k=0}^{n} \binom{n}{k} (n-k)(3\sqrt{5})^{n-k} 2^{1-(n-k)} F_{4k} = 3n \bigg(\Big(\frac{7+6\sqrt{5}}{2}\Big)^{n-1} - \Big(\frac{7}{2}\Big)^{n-1}\bigg)$$

or, equivalently,

$$\sum_{k=0}^{n} \binom{n}{k} k \Big(\frac{3\sqrt{5}}{2}\Big)^{k} F_{4(n-k)} = \frac{3n}{2^n} \big((7+6\sqrt{5})^{n-1} - 7^{n-1}\big).$$

Theorem 3. *For each $n \geq 0$ and $x \in \mathbb{C}$, we have the relation*

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} (4^k - 1) \big(144x^2(9x^2 - 1)\big)^k B_{2k} C_{2(n-2k)}(x) = 6nx(9x^2-1) B^*_{2(n-1)}(x). \tag{11}$$

Proof. Combine $c_1(x,z)$ with $c_2(x,z)$. □

Remark 2. *Combining $b_2(x,z)$ with $c_1(x,z)$ yields*

$$\begin{aligned}\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \big(144x^2(9x^2 - 1)\big)^k B_{2k} B^*_{2(n-2k)}(x) \\ = 2n\big(C_{2n-1}(x) - (9x^2 - 1) B^*_{2(n-1)}(x)\big).\end{aligned}$$

Since $C_n(x) = 3xC_{n-1}(x) + (9x^2 - 1)B^*_{n-1}(x)$ [2, Proposition 2.3], the right-hand side equals $2n\big(C_{2n-1}(x) - (9x^2-1)B^*_{2(n-1)}(x)\big) = 6nxC_{2(n-1)}(x)$, so we again have (7)*. Similarly, relating $b_1(x,z)$ to $c_2(x,z)$ gives*

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}(4^k-1)\big(144x^2(9x^2-1)\big)^k B_{2k}C_{2(n-2k)}(x) \\ = 2n(9x^2-1)\left(B^*_{2n-1}(x) - C_{2(n-1)}(x)\right),$$

but since $C_n(x) = B^*_{n+1}(x) - 3xB^*_n(x)$ [2, Proposition 2.3], the right-hand side equals $2n(9x^2-1)\big(B^*_{2n-1}(x) - C_{2(n-1)}(x)\big) = 6nx(9x^2-1)B^*_{2(n-1)}(x)$, and we again end with (11).

The next identity is the counterpart of Corollary 2.

Corollary 4. *For each $n \ge 0$ and $j \ge 1$,*

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}(20^k-5^k)F_{2j}^{2k}B_{2k}L_{2j(n-2k)} = \frac{5n}{2}F_{2j}F_{2j(n-1)}.$$

Proof. Insert $x = \frac{i}{6}L_{2m+1}$ and $x = \frac{1}{6}L_{2m}$, respectively, in (11) to get

$$\sum_{k=1}^{\lfloor n/2\rfloor}\binom{n}{2k}(4^k-1)B_{2k}\big(L_{2m}^4 - 4L_{2m}^2\big)^k L_{4m(n-2k)} = \frac{n}{2}\frac{L_{2m}}{F_{2m}}(L_{2m}^2-4)F_{4m(n-1)},$$

$$\sum_{k=1}^{\lfloor n/2\rfloor}\binom{n}{2k}(4^k-1)B_{2k}\big(L_{2m}^4 + 4L_{2m}^2\big)^k L_{2(2m+1)(n-2k)} \\ = \frac{n}{2}\frac{L_{2m+1}}{F_{2m+1}}(L_{2m+1}^2+4)F_{2(2m+1)(n-1)}.$$

Simplify using $L_n^2 = 5F_n^2 + (-1)^n 4$ and $L_nF_n = F_{2n}$. □

When $j = 1$, then

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}(4^k-1)5^kB_{2k}L_{2(n-2k)} = \frac{5n}{2}F_{2(n-1)},$$

which is equation (23) in [3]. When $j = 2$, then

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}(4^k-1)45^kB_{2k}L_{4(n-2k)} = \frac{15n}{2}F_{4(n-1)}.$$

We conclude the analysis with the following result.

Theorem 4. *For each $n \geq 0$ and $x \in \mathbb{C}$, we have the relations*

$$(6x)^{n-1} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} (36x^2 - 4)^k B_{2k} B^*_{n-2k}(x)$$
$$= n \left(\sum_{k=0}^{n-1} \binom{n-1}{k} B^*_{2k+1}(x) - \frac{(6x)^n}{2} B^*_{n-1}(x) \right) \tag{12}$$

and

$$(6x)^{n-1} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} (4^k - 1)(36x^2 - 4)^k B_{2k} C_{n-2k}(x)$$
$$= n \left(\sum_{k=0}^{n-1} \binom{n-1}{k} C_{2k+1}(x) - \frac{(6x)^n}{2} C_{n-1}(x) \right). \tag{13}$$

Proof. For the first identity, combine $b_1(x,z)$ with $b(x,z)$, where $b(x,z)$ is the exponential generating function for $B^*_n(x)$ [2],

$$b(x,z) = \sum_{n=0}^{\infty} B^*_n(x) \frac{z^n}{n!} = \frac{e^{3xz}}{\sqrt{9x^2-1}} \sinh(\sqrt{9x^2-1}z).$$

The second identity follows from relating $c_1(x,z)$ to $c(x,z)$ with

$$c(x,z) = \sum_{n=0}^{\infty} C_n(x) \frac{z^n}{n!} = e^{3xz} \cosh(\sqrt{9x^2-1}z).$$

□

Corollary 5. *For each $n \geq 0$ and $j \geq 1$,*

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} (5F_j^2)^k F_{j(n-2k)} B_{2k}$$
$$= (-1)^{nj} \frac{n}{L_j^{n-1}} \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^{kj} F_{j(2k+1)} - \frac{n}{2} F_{j(n-1)} L_j,$$
$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} (4^k - 1)(5F_j^2)^k L_{j(n-2k)} B_{2k}$$
$$= (-1)^{nj} \frac{n}{L_j^{n-1}} \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^{kj} L_{j(2k+1)} - \frac{n}{2} L_{j(n-1)} L_j.$$

Proof. Set $x = \frac{i}{6} L_{2m+1}$ and $x = \frac{1}{6} L_{2m}$ in (12) and (13), respectively, and simplify as before. □

3. A special polynomial identity

Equations (6), (9) and (10) give rise to the question if there is a connection between them. The answer to that question is positive, as will be shown in the next theorem. The theorem generalizes Theorem 9 in [3], which has been generalized in a different way in [6]. The proof of the extension presented here does not require the notion of balancing polynomials.

Theorem 5. *Let α be the golden ratio, $\alpha = (1+\sqrt{5})/2$, and $\beta = (1-\sqrt{5})/2 = -1/\alpha$. Then, for each $n \ge 0$, $j \ge 1$, and $x \in \mathbb{C}$, we have the relations*

$$\sum_{k=0}^{n} \binom{n}{k} F_{jk} (\sqrt{5}F_j)^{n-k} B_{n-k}(x) = nF_j\big((\sqrt{5}x+\beta)F_j + F_{j-1}\big)^{n-1} \tag{14}$$

and

$$\sum_{k=0}^{n} \binom{n}{k} F_{jk} (-\sqrt{5}F_j)^{n-k} B_{n-k}(x) = nF_j\big((\alpha-\sqrt{5}x)F_j + F_{j-1}\big)^{n-1}. \tag{15}$$

Proof. Let $F(z)$ be the exponential generating function for $(F_{jn})_{n\ge 0}, j \ge 1$. Evidently, the Binet formula for F_{jn} gives

$$F(z) = \sum_{n=0}^{\infty} F_{jn}\frac{z^n}{n!} = \frac{1}{\sqrt{5}}\big(e^{\alpha^j z} - e^{\beta^j z}\big).$$

Now we use the relations $\alpha^j = \alpha F_j + F_{j-1}$ and $\beta^j = \beta F_j + F_{j-1}$ to write

$$F(z) = \frac{2}{\sqrt{5}} e^{(1/2F_j + F_{j-1})z} \sinh\Big(\frac{\sqrt{5}F_j}{2}z\Big).$$

Hence, it follows that

$$\begin{aligned}\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}\binom{n}{k}F_{jk}(\sqrt{5}F_j)^{n-k}B_{n-k}(x)\Big)\frac{z^n}{n!} &= F(z)H(x,\sqrt{5}F_j z)\\ &= F_j z\, e^{((x-1/2)\sqrt{5}F_j + 1/2F_j + F_{j-1})z}\\ &= F_j z\, e^{((\sqrt{5}x+\beta)F_j + F_{j-1})z}.\end{aligned}$$

This proves the first equation. The second follows upon replacing x by $1-x$ and using $B_n(1-x) = (-1)^n B_n(x)$ [1, Proposition 9.1.3] and $\alpha - \beta = \sqrt{5}$. $\square$

When $x = 0$, then

$$\sum_{k=0}^{n}\binom{n}{k}(\sqrt{5}F_j)^{n-k}F_{jk}B_{n-k} = nF_j\beta^{j(n-1)} = \frac{n}{2}F_j\big(L_{j(n-1)} - \sqrt{5}F_{j(n-1)}\big),$$

which generalizes (6). Similarly,

$$\sum_{k=0}^{n}\binom{n}{k}(-\sqrt{5}F_j)^{n-k}F_{jk}B_{n-k} = nF_j\alpha^{j(n-1)} = \frac{n}{2}F_j\big(L_{j(n-1)} + \sqrt{5}F_{j(n-1)}\big).$$

A combination of both yields

$$\sum_{k=0}^{n}\binom{n}{k}(\sqrt{5}F_j)^{n-k}\big(1+(-1)^{n-k}\big)F_{jk}B_{n-k}=nF_jL_{j(n-1)}.$$

Corollary 6. *For each $n\ge 0$ and $j\ge 1$,*

$$\sum_{k=0}^{n}\binom{n}{k}(\sqrt{5}F_j)^{n-k}\big(2^{1-(n-k)}-1\big)F_{jk}B_{n-k}=n2^{1-n}F_jL_j^{n-1}. \tag{16}$$

Proof. Set $x=\frac{1}{2}$ in (14) or (15) and use once more $B_n(\frac{1}{2})=(2^{1-n}-1)B_n$. When simplifying, keep in mind the relation $F_j+2F_{j-1}=L_j$. $\square$

When $j=2$ and $j=4$, we get (9) and (10), respectively. For $j=3$, the identity becomes

$$\sum_{k=0}^{n}\binom{n}{k}(2\sqrt{5})^{n-k}\big(2^{1-(n-k)}-1\big)F_{3k}B_{n-k}=n2^{n}.$$

Corollary 7. *Let n, j and q be integers with $n,j\ge 1$ and $q\ge 2$. Then it holds that*

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}(\sqrt{5}F_j)^{n-k}\big(q^{1-(n-k)}-1\big)F_{jk}B_{n-k}&\\ =nF_jq^{1-n}\sum_{r=1}^{q-1}\big(r\alpha^j+(q-r)\beta^j\big)^{n-1}.&\end{aligned} \tag{17}$$

Proof. The multiplication theorem [1, Proposition 9.1.3]

$$\frac{1}{q}\sum_{r=0}^{q-1}B_n\Big(x+\frac{r}{q}\Big)=\frac{B_n(qx)}{q^n}$$

gives

$$\big(q^{1-(n-k)}-1\big)B_{n-k}=\sum_{r=1}^{q-1}B_n\Big(\frac{r}{q}\Big).$$

Therefore, we can write

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}F_{jk}(\sqrt{5}F_j)^{n-k}\big(q^{1-(n-k)}-1\big)B_{n-k}&\\ =nF_j\sum_{r=1}^{q-1}\Big(\Big(\sqrt{5}\,\frac{r}{q}+\beta\Big)F_j+F_{j-1}\Big)^{n-1}&\\ =nF_jq^{1-n}\sum_{r=1}^{q-1}\big(\sqrt{5}rF_j+q(\beta F_j+F_{j-1})\big)^{n-1}&\\ =nF_jq^{1-n}\sum_{r=1}^{q-1}\big(r\alpha^j+(q-r)\beta^j\big)^{n-1}.&\end{aligned}$$

$\square$

When $q=2$, then (17) gives (16). When $q=3$, then we obtain

$$\sum_{k=0}^{n}\binom{n}{k}F_{jk}(\sqrt{5}F_j)^{n-k}\big(3^{1-(n-k)}-1\big)B_{n-k} = nF_j 3^{1-n}\big((L_j+\beta^j)^{n-1}+(L_j+\alpha^j)^{n-1}\big).$$

Corollary 8. *Let n, j and m be integers with $n, j\geq 1$ and $0\leq m\leq n-1$. Then*

$$\sum_{k=0}^{n-m}\binom{n}{k}F_{jk}(\sqrt{5}F_j)^{n-k}(n-k)_m B_{n-m-k}(x) = (n)_{m+1}F_j(\sqrt{5}F_j)^m\big((\sqrt{5}x+\beta)F_j+F_{j-1}\big)^{n-1-m},$$

and

$$\sum_{k=0}^{n-m}\binom{n}{k}F_{jk}(-\sqrt{5}F_j)^{n-k}(n-k)_m B_{n-m-k}(x) = (n)_{m+1}F_j\big(-\sqrt{5}F_j\big)^m\big((\alpha-\sqrt{5}x)F_j+F_{j-1}\big)^{n-1-m},$$

where $(y)_n=y(y-1)\cdots(y-n+1)$, $(y)_0=1$, denotes the falling factorial.

Proof. Differentiate the identities in Theorem 5 m times and use the fact $B_n'(x)=nB_{n-1}(x)$ [1, Proposition 9.1.2]. When $m\geq n$, then both sides of the identities become zero. $\square$

Corollary 9. *For nonnegative integers n, N and $j\geq 1$, we have the identities*

$$\sum_{s=0}^{N}\Big(\big(\alpha^j+\sqrt{5}F_j s\big)^n-\big(\beta^j+\sqrt{5}F_j s\big)^n\Big)=\big(\sqrt{5}F_jN+\alpha^j\big)^n-\beta^{jn},$$

$$\sum_{s=0}^{N}\Big(\big(\beta^j-\sqrt{5}F_j s\big)^n-\big(\alpha^j-\sqrt{5}F_j s\big)^n\Big)=\big(-\sqrt{5}F_jN+\beta^j\big)^n-\alpha^{jn}.$$

Proof. We only prove the first identity. Integrate both sides of (14) from 0 to $N+1$ and use the formula

$$\sum_{s=0}^{N}s^n=\int_0^{N+1}B_n(x)dx.$$

The last integral identity actually reads:

$$\sum_{s=0}^{N}s^n=\int_0^{N+1}B_n(x)dx=\frac{1}{N+1}\big(B_{n+1}(N+1)-B_{n+1}\big)$$

(Faulhaber's formula) and holds for all $n\geq 2$, so justification is needed. As we will work with the integral part only, with the convention that $0^0=1$, the cases $n=0$

and $n = 1$ can be checked explicitly. Hence, for the LHS of (14) we obtain

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}F_{jk}\big(\sqrt{5}F_j\big)^{n-k}\int_0^{N+1}B_{n-k}(x)dx\\ &= \sum_{s=0}^{N}\sum_{k=0}^{n}\binom{n}{k}F_{jk}\big(\sqrt{5}F_j s\big)^{n-k}\\ &= \frac{1}{\sqrt{5}}\sum_{s=0}^{N}\Big(\big(\alpha^j+\sqrt{5}F_j s\big)^n-\big(\beta^j+\sqrt{5}F_j s\big)^n\Big).\end{aligned}$$

The integral on the RHS of (14) is easily evaluated as

$$nF_j\int_0^{N+1}\big((\sqrt{5}x+\beta)F_j+F_{j-1}\big)^{n-1}dx = \frac{1}{\sqrt{5}}\big(\sqrt{5}F_j(N+1)+\beta^j\big)^n-\beta^{jn}.$$

The proof of the second formula is similar. □

4. Conclusion

In this paper, we have discovered new identities relating Bernoulli numbers (polynomials) to balancing and Lucas-balancing polynomials. We have also derived a general identity involving Bernoulli polynomials and Fibonacci numbers in arithmetic progression. In our future papers, we will discuss the analogue results for Euler polynomials and Lucas-balancing polynomials as well as identities connecting Bernoulli polynomials with Fibonacci and Chebyshev polynomials.

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References

[1] H. Cohen, *Number Theory. Vol. II: Analytic and Modern Tools*, Springer, New York, 2007.

[2] R. Frontczak, *On balancing polynomials*, Appl. Math. Sci. **13**(2019), 57–66.

[3] R. Frontczak, *Relating Fibonacci numbers to Bernoulli numbers via balancing polynomials*, J. Integer Seq. **22**(2019), Article 19.5.3.

[4] R. Frontczak, T. Goy, *Additional close links between balancing and Lucas-balancing polynomials*, Adv. Stud. Contemp. Math. **31**(2021), to appear, 14 pp.

[5] R. Frontczak, T. Goy, *Combinatorial sums associated with balancing and Lucas-balancing polynomials*, Ann. Math. Inform. **52**(2020), 97–105.

[6] R. Frontczak, Z. Tomovski, *Generalized Euler-Genocchi polynomials and Lucas numbers*, Integers **20**(2020), #A52.

[7] M. Goubi, *Relating balancing polynomials to Lucas-balancing polynomials via Bernoulli numbers*, J. Integer Seq. **23**(2020), Article 20.4.2.

[8] D. S. KIM, T. KIM, *On sums of finite products of balancing polynomials*, J. Comput. Appl. Math. **377**(2020), Article 112913.

[9] Y. MENG, *A new identity involving balancing polynomials and balancing numbers*, Symmetry **11**(2019), 1141.

[10] P. K. RAY, *Balancing polynomials and their derivatives*, Ukrainian Math. J. **69**(2017), 646–663.

[11] N. J. A. SLOANE, *The On-Line Encyclopedia of Integer Sequences*, available at `https://oeis.org`.

[12] T. ZHANG, Y. MA, *On generalized Fibonacci polynomials and Bernoulli numbers*, J. Integer Seq. **8**(2005), Article 05.5.3.