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New regular two-graphs on 38 and 42 vertices

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Abstract. All regular two-graphs having up to 36 vertices are known, and the first open case is the enumeration of two-graphs on 38 vertices. It is known that there are at least 191 regular two-graphs on 38 vertices and at least 18 regular two-graphs on 42 vertices. The number of descendants of these two-graphs is 6760 and 120, respectively.
In this paper, we classify strongly regular graphs with parameters $(41, 20, 9, 10)$ having nontrivial automorphisms and show that there are exactly 7152 such graphs. We enumerate all regular two-graphs on 38 and 42 vertices with at least one descendant whose full automorphism group is nontrivial and establish that there are at least 194 regular two-graphs on 38 vertices and at least 752 regular two-graphs on 42 vertices. Furthermore, we construct descendants with a trivial automorphism group of the newly constructed two-graphs and increase the number of known strongly regular graphs with parameters $(37, 18, 8, 9)$ and $(41, 20, 9, 10)$ to 6802 and 18439m respectively. This significantly increases the number of known strongly regular graphs with parameters $(41, 20, 9, 10)$.

AMS subject classifications: 05E30, 20B25.

Keywords: regular two-graph, strongly regular graph, automorphism group, orbit matrix

1. Introduction

According to [16], the concept of regular two-graphs was introduced by G. Higman to study a doubly transitive representation of the third Conway's sporadic simple group Co_3. The connection between regular two-graphs and strongly regular graphs was established by Taylor in [16], and Bussemaker, Mathon and Seidel made the first step towards classifying regular two-graphs on at most 50 vertices and classified all regular two-graphs on $v < 30$ vertices (see [6, 16]). Their results were followed by Spence and McKay in [15] and [12]. To the best of our knowledge, the number of known regular two-graphs on up to 42 vertices is given in Table 1, where n and $N(n)$ denote the number of vertices and the number of known regular two-graphs, respectively, and the bar on the number indicates that the classification is completed.

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n	6	10	14	16	18	26	28	30	36	38	42
$N(n)$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{4}$	$\overline{1}$	$\overline{6}$	$\overline{227}$	191	18

Table 1: *Number of known regular two-graphs on up to 42 vertices*

Strongly regular graphs with at most 36 vertices are fully classified and for the parameters $(37, 18, 8, 9)$ all graphs with nontrivial automorphisms have been enumerated (see [4, 7]). The goal of this paper is to classify strongly regular graphs with parameters $(41, 20, 9, 10)$ admitting nontrivial automorphisms and to enumerate regular two-graphs on 38 and 42 vertices with at least one descendant with a nontrivial automorphism.

The paper is organized as follows: After a brief description of the terminology and some background results in Section 2, we establish the existence of three new regular two-graphs on 38 vertices in Section 3 and complete the classification of such two-graphs with at least one descendant with a nontrivial automorphism. We also construct 36 new strongly regular graphs with parameters $(37, 18, 8, 9)$ and a trivial automorphism group as descendants of the newly constructed two-graphs. In Section 4, we apply the method for constructing strongly regular graphs using orbit matrices to construct all strongly regular graphs with parameters $(41, 20, 9, 10)$ and nontrivial automorphism groups. Using this classification, in Section 5, we construct all regular two-graphs on 42 vertices with at least one descendant with a nontrivial automorphism group. Moreover, by constructing all descendants of the new two-graphs we obtain new strongly regular graphs with parameters $(41, 20, 9, 10)$ and a trivial automorphism group.
To eliminate isomorphic graphs and to determine order and the structure of their automorphism groups we use GAP [17].

2. Background and terminology

We assume that the reader is familiar with basic notions from the theory of finite groups. For basic definitions and properties of strongly regular graphs and two-graphs, we refer the reader to [5, 10, 12, 13, 18].

A graph is regular if all its vertices have the same valency. A simple regular graph $\Gamma = (\mathcal{V}, \mathcal{E})$ is strongly regular with parameters (v, k, λ, μ) if it has $|\mathcal{V}| = v$ vertices, valency k, and if any two adjacent vertices are together adjacent to λ vertices, while any two nonadjacent vertices are together adjacent to μ vertices. A strongly regular graph with parameters (v, k, λ, μ) is usually denoted by $\mathrm{SRG}(v, k, \lambda, \mu)$. A conference graph is a strongly regular graph with parameters $(v, k = (v-1)/2, \lambda = (v-5)/4, \mu = (v-1)/4)$.

An automorphism of a strongly regular graph Γ is a permutation of the vertices of Γ, such that two vertices are adjacent if and only if their images are adjacent. The full automorphism group of Γ, usually denoted by $Aut(\Gamma)$, is the group of all such permutations. Let $\Gamma_1 = (\mathcal{V}, \mathcal{E}_1)$ and $\Gamma_2 = (\mathcal{V}, \mathcal{E}_2)$ be strongly regular graphs and $G \leq Aut(\Gamma_1) \cap Aut(\Gamma_2)$. An isomorphism $\alpha : \Gamma_1 \to \Gamma_2$ is called a G-isomorphism if there exists an automorphism $\tau : G \to G$ such that for every $x, y \in \mathcal{V}$ and every

$g \in G$ the following holds:

$$(\tau g).(\alpha x) = \alpha y \Leftrightarrow g.x = y.$$

A two-graph is a pair $(\mathcal{V}, \Delta)$, where Δ is a collection of unordered triples chosen from a finite set of vertices $\mathcal{V}$, such that every 4-subset of $\mathcal{V}$ contains an even number of triples of Δ. The triples from Δ are called coherent. A regular two-graph has the property that every pair of vertices lies in the same number of triples of the two-graph. The complement of the two-graph $(\mathcal{V}, \Delta)$ is the two-graph $(\mathcal{V}, \overline{\Delta})$, where $\overline{\Delta}$ is the complement of Δ in the set of all 3-subsets of $\mathcal{V}$. The two-graphs $(\mathcal{V}, \Delta)$ and $(\mathcal{V}', \Delta')$ are isomorphic if there exists a bijection $\mathcal{V} \to \mathcal{V}'$ that induces a bijection $\Delta \to \Delta'$. A two-graph is called self-complementary if it is isomorphic to its complement. The automorphism group $Aut(\mathcal{V}, \Delta)$ of a two-graph $(\mathcal{V}, \Delta)$ is the group of permutations of $\mathcal{V}$ which preserves Δ.

From a two-graph $\Phi = (\mathcal{V}, \Delta)$ and any fixed $x \in \mathcal{V}$ we construct the graph Γ which has a vertex set $\mathcal{V}$ by setting the vertex x to be an isolated vertex and letting any two other vertices y, z be adjacent in Γ if $\{z, x, y\}$ is coherent in Φ. Deleting the isolated vertex x yields a graph on $|\mathcal{V}| - 1$ vertices, which is called the descendant of Φ. The two-graph $(\mathcal{V}, \Delta)$ is regular if and only if each descendant is strongly regular with parameters $(v-1, k, \lambda, \mu)$, where $\mu = k/2$. If the descendants are conference graphs, the corresponding two-graph is called a conference two-graph. The complement of any conference two-graph is again a conference two-graph.

In this paper, we classify SRGs$(41, 20, 9, 10)$ with nontrivial automorphisms. Note that these strongly regular graphs are conference graphs. We also enumerate regular two-graphs on 38 and 42 vertices with at least one descendant with nontrivial automorphisms, which are conference two-graphs. More details on conference graphs can be found in [5, 13] .

3. Regular two-graphs on 38 vertices

It is known that there are at least 191 regular two-graphs on 38 vertices (see [12]). These two-graphs are available at [14]. In total, they have 6760 nonisomorphic descendants, which are strongly regular graphs with parameters $(37, 18, 8, 9)$.

Recently, Crnković and the first author classified strongly regular graphs with parameters $(37, 18, 8, 9)$ admitting nontrivial automorphisms. They constructed six new strongly regular graphs with parameters $(37, 18, 8, 9)$ whose full automorphism group is of order three and showed that there are exactly forty SRGs$(37, 18, 8, 9)$ with nontrivial automorphisms (see [7, Theorem 4.3]). By analysing the new strongly regular graphs with parameters $(37, 18, 8, 9)$ constructed in [7], we obtain the following theorem.

Theorem 1. *Up to isomorphism, there exist at least* 194 *regular two-graphs on* 38 *vertices with* 6802 *nonisomorphic descendants. Among them, there are* 64 *self-complementary two-graphs. Exactly* 20 *regular two-graphs on* 38 *vertices have at least one descendant admitting nontrivial automorphisms, and there are no more two-graphs with this property.*

Proof. By constructing two-graphs corresponding to the six new SRGs$(37, 18, 8, 9)$ constructed in [7] and eliminating isomorphic copies, we obtain three new two-graphs Φ_i, $i \in \{1, 2, 3\}$. Further analysis of these two-graphs leads to the results presented in Table 2. The second column of the table contains the order of the corresponding full automorphism group G_{Φ_i}, and the third column gives the number of nonisomorphic descendants with a given full automorphism group, where E denotes a trivial group. In the last column, we indicate whether a two-graph is self-complementary or not.

i	$\lvert G_{\Phi_i}\rvert$	Descendants of Φ_i	"S"
1	3	$[12 \times E, 2 \times Z_3]$	NO
2	3	$[12 \times E, 2 \times Z_3]$	NO
3	3	$[12 \times E, 2 \times Z_3]$	YES

Table 2: *Descendants of the new two-graphs on* 38 *vertices*

The two-graph Φ_3 is a self-complementary two-graph, and Φ_1 and Φ_2 are complements to each other. Each of the two-graphs Φ_i, $i \in \{1, 2, 3\}$, has 14 mutually nonisomorphic descendants. Among them, there are 12 strongly regular graphs with parameters $(37, 18, 8, 9)$ with a trivial automorphism group and two descendants with the full automorphism group of order three. We have thus obtained 36 descendants that are new SRGs$(37, 18, 8, 9)$ whose full automorphism group is trivial. Together with the previously known results, this includes all two-graphs that can have descendants with a nontrivial automorphism and gives the statement of the theorem. □

From the analysis in the proof of Theorem 1 and previously known results (see [7, Theorem 4.3]) we have the following statement.

Theorem 2. *Up to isomorphism, there exist at least* 6802 *strongly regular graphs with parameters* $(37, 18, 8, 9)$. *These are exactly forty SRGs*$(37, 18, 8, 9)$ *admitting nontrivial automorphisms, and at least* 6762 *SRGs*$(37, 18, 8, 9)$ *with the full automorphism group of order one.*

As a supplement to the two-graphs given in [6] (see also [14]), we give new two-graphs represented by the adjacency matrix of one of their descendants (Φ_2 can be obtained from Φ_1 as its complement). We denote by AM_{Φ_i} the adjacency matrix corresponding to the descendant of Φ_i.

4. Enumeration of SRGs$(41, 20, 9, 10)$ with nontrivial automorphisms

There are 120 mutually nonisomorphic SRGs$(41, 20, 9, 10)$ arising as descendants of 18 regular two-graphs Φ_i, $1 \leq i \leq 18$, on 42 vertices constructed by Bussemaker, Mathon and Seidel in [6]. These graphs are presented in Table 3, where we use the same notation as in Table 2. In Table 3, a pair of complementary two-graphs is represented by one of them.

$$AM_{\Phi_1} = \begin{pmatrix}
0&0&0&0&1&1&0&0&1&0&0&1&0&1&1&1&1&0&1&1&1&0&1&1&1&1&0&1&0&0&1&0&0&0&0&0&1\\
0&0&0&1&1&0&1&0&1&1&0&1&1&1&0&1&0&1&0&0&0&1&1&0&1&0&1&1&1&0&1&1&0&0&0&0&0\\
0&0&0&1&0&1&0&1&1&0&1&1&1&0&1&0&1&1&0&0&0&1&0&1&0&1&1&1&0&1&1&0&1&0&0&0&0\\
0&1&1&0&1&1&0&1&0&0&1&0&0&1&0&0&0&0&0&1&1&1&1&1&1&0&1&0&1&1&1&0&0&1&0&0&0\\
1&1&0&1&0&0&0&1&1&0&1&1&0&0&0&0&1&0&0&1&0&1&1&0&1&0&0&1&0&0&0&1&1&1&0&1&1\\
1&0&1&1&0&0&0&0&0&0&0&0&1&1&0&1&0&0&0&0&1&1&0&1&1&1&1&1&0&0&0&1&1&1&1&0&1\\
0&1&0&0&0&0&0&0&1&1&1&0&0&0&1&1&0&0&0&1&1&0&0&1&1&1&1&1&1&1&0&1&1&0&0&1&0\\
0&0&1&1&1&0&0&0&1&1&1&0&0&1&0&1&1&1&1&0&1&0&1&0&0&1&0&0&0&1&0&1&1&1&0&0&0\\
1&1&1&0&1&0&1&1&0&0&0&1&0&0&0&1&0&1&0&0&1&0&0&0&0&1&1&1&0&1&1&0&0&1&0&1&1\\
0&1&0&0&0&0&1&1&0&0&0&1&1&1&0&0&0&0&1&1&0&0&1&1&0&1&0&1&0&1&1&1&1&1&1&0&0\\
0&0&1&1&1&0&1&1&0&0&0&1&1&0&1&1&0&1&0&1&1&0&1&1&1&0&0&0&0&0&0&0&1&0&1&1&0\\
1&1&1&0&1&0&0&0&1&1&1&0&1&1&1&1&0&0&0&1&0&1&0&0&0&0&0&0&0&1&1&0&1&0&1&0&1\\
0&1&1&0&0&1&0&0&0&1&1&1&0&1&1&0&1&1&0&0&1&0&0&0&1&0&0&1&0&0&1&1&0&1&1&1&0\\
1&1&0&1&0&1&0&1&0&1&0&1&1&0&0&1&1&1&0&1&1&0&0&1&0&0&0&0&1&1&0&1&0&0&0&0&1\\
1&0&1&0&0&0&1&0&0&0&1&1&1&0&0&0&1&0&1&0&1&0&1&0&1&0&1&0&1&1&1&1&1&0&0&0&1\\
1&1&0&0&0&1&1&1&1&0&1&1&0&1&0&0&0&1&1&0&1&1&0&0&1&1&0&0&1&0&0&0&1&0&1&0&0\\
1&0&1&0&1&0&0&1&0&0&0&0&1&1&1&0&0&1&1&1&0&1&0&0&1&1&0&1&1&1&0&1&0&0&0&1&0\\
0&1&1&0&0&0&0&1&1&0&1&0&1&1&0&1&1&0&1&0&0&0&1&1&0&0&1&1&1&0&0&0&0&0&1&1&1\\
1&0&0&0&0&0&0&1&0&1&0&0&0&0&1&1&1&1&0&1&1&1&1&0&0&0&1&1&1&0&1&0&1&1&1&0&0\\
1&0&0&1&1&0&1&0&0&1&1&1&0&1&0&0&1&0&1&0&1&1&0&1&0&0&1&1&0&1&0&0&0&0&1&1&0\\
1&0&0&1&0&1&1&1&1&0&1&0&1&1&1&1&0&0&1&1&0&0&0&0&0&0&1&0&0&0&1&1&0&1&0&1&0\\
0&1&1&1&1&1&0&0&0&0&0&1&0&0&0&1&1&0&1&1&0&0&0&0&0&1&1&0&1&0&1&1&1&0&1&1&0\\
1&1&0&1&1&0&0&1&0&1&1&0&0&0&1&0&0&1&1&0&0&0&0&1&1&1&1&0&0&0&1&1&0&0&1&0&1\\
1&0&1&1&0&1&1&0&0&1&1&0&0&1&0&0&0&1&0&1&0&0&1&0&0&1&0&1&1&0&1&0&1&0&0&1&1\\
1&1&0&1&1&1&1&0&0&0&1&0&1&0&1&1&1&0&0&0&0&0&1&0&0&1&0&1&1&1&0&0&0&1&1&0&0\\
1&0&1&0&0&1&1&1&1&1&0&0&0&0&0&1&1&0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&0&0&1&1&0\\
0&1&1&1&0&1&1&0&1&0&0&0&0&0&1&0&0&1&1&1&1&1&1&0&0&0&0&1&0&1&0&1&0&0&1&0&1\\
1&1&1&0&1&1&1&0&1&1&0&0&1&0&0&0&1&1&1&1&0&0&0&1&1&0&1&0&0&0&0&0&1&1&0&0&0\\
0&1&0&1&0&0&1&0&0&0&0&0&0&1&1&1&1&1&1&0&0&1&0&1&1&0&0&0&0&1&1&0&1&1&0&1&1\\
0&0&1&1&0&0&1&1&1&1&0&1&0&1&1&0&1&0&0&1&0&0&0&0&1&1&1&0&1&0&0&0&0&1&1&0&1\\
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0&1&0&0&1&1&1&1&0&1&0&0&1&1&1&0&1&0&0&0&1&1&1&0&0&1&1&0&0&0&0&0&1&0&0&1&1\\
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0&0&0&1&1&1&0&1&1&1&0&0&1&0&0&0&0&0&1&0&1&0&0&0&1&0&0&1&1&1&1&0&1&0&1&1&1\\
0&0&0&0&0&1&0&0&0&1&1&1&1&0&0&1&0&1&1&1&0&1&1&0&1&1&1&0&0&1&0&0&0&1&0&1&1\\
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1&0&0&0&1&1&0&0&1&0&0&1&0&1&1&0&0&1&0&0&0&0&1&1&0&0&1&0&1&1&0&1&1&1&1&1&0
\end{pmatrix}$$

$$AM_{\Phi_3} = \begin{pmatrix}
0&0&0&0&1&1&0&0&1&0&0&1&0&1&1&1&1&0&1&1&1&0&1&1&1&1&0&1&0&0&1&0&0&0&0&0&1\\
0&0&0&1&1&0&1&0&1&1&0&1&1&1&0&1&0&1&0&0&0&1&1&0&1&0&1&1&1&0&1&1&0&0&0&0&0\\
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0&1&1&0&1&1&0&1&0&1&0&0&1&1&1&0&0&0&0&1&1&0&0&1&1&0&1&0&1&1&1&0&0&1&0&0&0\\
1&1&0&1&0&0&0&1&1&0&0&0&1&1&0&0&1&0&0&1&0&1&0&1&1&0&0&1&0&0&0&1&1&1&0&1&1\\
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1&1&0&1&1&0&0&1&0&0&1&1&1&0&0&1&0&0&1&1&0&0&1&0&0&1&1&0&1&1&0&0&1&0&0&0&1\\
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1&0&0&0&0&0&0&1&0&1&0&0&0&1&0&1&1&1&0&1&1&1&1&0&0&0&1&1&0&1&1&0&1&1&1&0&0\\
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1&0&0&1&0&1&1&1&1&0&1&0&0&0&1&1&0&0&1&1&0&0&0&0&0&0&1&1&1&0&1&1&0&1&0&1&0\\
0&1&1&0&1&0&1&0&1&0&0&1&1&0&1&1&1&0&1&1&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&0&0\\
1&1&0&0&0&0&1&0&0&1&1&0&1&1&1&0&0&1&1&0&0&0&0&1&0&1&0&1&0&1&1&1&0&1&0&0&1\\
1&0&1&1&1&0&1&1&1&1&1&0&0&0&1&0&0&1&0&1&0&0&1&0&1&0&0&0&0&0&1&0&1&0&1&0&1\\
1&1&0&1&1&1&1&0&0&1&0&0&0&0&0&1&1&0&0&0&0&0&0&1&0&1&0&1&1&1&1&0&1&0&1&1&0\\
1&0&1&0&0&1&1&1&1&1&1&1&1&1&0&1&1&0&0&0&0&0&1&0&1&0&0&0&0&1&0&0&0&1&0&1&0\\
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1&1&1&0&1&1&0&0&0&0&1&0&1&0&0&0&1&1&1&0&1&0&1&0&1&0&1&0&0&0&1&1&1&0&0&1&0\\
0&1&0&1&0&0&0&1&0&0&1&1&0&1&1&1&1&1&0&0&1&0&0&0&1&0&0&0&0&1&1&1&0&0&1&1&1\\
0&0&1&1&0&0&1&0&0&0&0&0&0&1&1&0&1&0&1&0&0&1&1&0&1&1&1&0&1&0&1&0&1&1&0&1&1\\
1&1&1&1&0&0&0&0&1&0&1&1&0&0&0&0&0&0&1&0&1&1&1&1&1&0&0&1&1&1&0&0&0&1&1&0&0\\
0&1&0&0&1&1&1&1&0&1&1&1&0&0&1&0&1&0&0&0&1&1&1&0&0&0&0&1&1&0&0&0&1&1&0&0&1\\
0&0&1&0&1&1&1&1&0&0&1&0&0&1&0&1&0&0&1&0&0&1&0&1&1&0&1&1&0&1&0&1&0&0&1&0&1\\
0&0&0&1&1&1&0&1&1&1&0&0&1&0&0&0&0&0&1&0&1&1&1&0&0&1&0&0&0&1&1&1&0&0&1&1&1\\
0&0&0&0&0&1&0&0&0&1&1&1&1&0&0&1&0&1&1&1&0&1&0&1&1&0&0&0&1&0&1&0&1&1&0&1&1\\
0&0&0&0&1&0&1&0&1&0&1&0&1&0&0&0&1&1&0&1&1&0&0&0&1&1&1&1&1&1&0&0&0&1&1&0&1\\
1&0&0&0&1&1&0&0&1&0&0&1&0&1&1&0&0&1&0&0&0&0&1&1&0&0&1&0&1&1&0&1&1&1&1&1&0
\end{pmatrix}$$

Further, there are 80 strongly regular graphs with parameters $(41, 20, 9, 10)$ having the full automorphism group isomorphic to the symmetric group S_3 (see [11, Theorem 5]). To the best of our knowledge, these 200 graphs are the only known SRGs$(41, 20, 9, 10)$. The aim of this section is to construct all SRGs$(41, 20, 9, 10)$ with nontrivial automorphisms.

Let G be an automorphism group of the graph Γ with $|V| = v$ vertices, partitioning the set of vertices V into b orbits of sizes $n_1, \ldots, n_b$, respectively, where $\sum_{i=1}^{b} n_i = v$. It is known that n_i divides $|G|$, for $i = 1, \ldots, b$. Thus, to enumerate all strongly regular graphs with parameters $(41, 20, 9, 10)$ admitting nontrivial auto-

i	$\lvert\mathrm{Aut}(\Phi_i)\rvert$	Descendants of Φ_i	"S"
1	34440	$[1 \times Z_{41} : Z_{30}]$	YES
2	168	$[1 \times Z_4]$	YES
3	21	$[1 \times E]$	YES
4	14	$[2 \times E, 2 \times Z_2]$	YES
5	14	$[2 \times E, 2 \times Z_2]$	YES
6	14	$[2 \times E, 2 \times Z_2]$	NO
7	14	$[6 \times Z_2]$	YES
8	8	$[2 \times E, 6 \times Z_2, 2 \times D_8]$	YES
9	8	$[2 \times E, 6 \times Z_2, 2 \times D_8]$	YES
10	8	$[2 \times E, 6 \times Z_2, 2 \times D_8]$	NO
11	7	$[6 \times E]$	YES
12	7	$[6 \times E]$	YES
13	6	$[4 \times E, 6 \times Z_2]$	YES
14	6	$[4 \times E, 6 \times Z_2]$	YES
15	5	$[8 \times E, 2 \times Z_5]$	YES
16	4	$[10 \times E, 2 \times Z_4]$	YES

Table 3: *Descendants of the known two-graphs on* 42 *vertices*

morphisms, we consider automorphisms of prime order, following the construction method proposed by Behbahani and Lam in [3]. They introduced the concept of orbit matrices of strongly regular graphs and gave a method for constructing orbit matrices of strongly regular graphs with automorphisms of prime order and corresponding strongly regular graphs (see [2, 3]). In our construction, we will use the column orbit matrices introduced in [8].

Definition 1. *A $(b \times b)$-matrix $C = [c_{ij}]$ with entries satisfying conditions:*

$$\sum_{i=1}^{b} c_{ij} = \sum_{j=1}^{b} \frac{n_j}{n_i} c_{ij} = k, \tag{1}$$

$$\sum_{s=1}^{b} \frac{n_s}{n_j} c_{is} c_{js} = \delta_{ij}(k-\mu) + \mu n_i + (\lambda - \mu) c_{ij}, \tag{2}$$

where $0 \le c_{ij} \le n_i$, $0 \le c_{ii} \le n_i - 1$ and $\sum_{i=1}^{b} n_i = v$, is called a **column orbit matrix** *for a strongly regular graph with parameters (v, k, λ, μ) and the orbit lengths distribution $(n_1, \ldots, n_b)$.*

There is exactly one $\mathrm{SRG}(41, 20, 9, 10)$ admitting an automorphism group isomorphic to Z_{41}, namely the Paley graph with 41 vertices having the full automorphism group isomorphic to $Z_{41} : Z_{30}$. In the sequel, we will consider automorphisms of prime order p, where $2 \le p \le 37$. Such automorphism acts in orbits of at most two different lengths. If the group G acts with d_1 orbits of length 1 and d_p orbits of length p, we denote the corresponding orbit lengths distribution by $(d_1 \times 1, d_p \times p)$.

Lemma 1. *If an automorphism of prime order p, $2 \le p \le 37$, acts on a strongly regular graph with parameters $(41, 20, 9, 10)$, then $p \in \{2, 3, 5\}$.*

Proof. The first step in constructing strongly regular graphs with parameters $(41, 20, 9, 10)$ admitting an automorphism of prime order p is to determine all permissible distributions $(d_1 \times 1, d_p \times p)$, $2 \leq p \leq 37$. A nontrivial automorphism acting on $\mathrm{SRG}(v, k, \lambda, \mu)$ with eigenvalues $s < r < k$ fixes at most $\frac{\max(\lambda,\mu)}{k-r} v$ vertices ([2, Theorem 3.7]). Therefore, $d_1 \leq 23$, $d_1 + d_p \cdot p = 41$, and the number of possible orbit lenghts distributions is as given in the second row of Table 4.

p	2	3	5	7	11	13	17	19	23	29	31	37
No. of distributions	12	8	5	3	2	2	1	2	1	1	1	1
Distributions with prototypes	10	2	2	0	0	0	0	0	0	0	0	0

Table 4: *The number of orbit lenghts distributions and the existence of prototypes*

In [2], the concept of a prototype for a row of a column orbit matrix $C = [c_{ij}]$ of a strongly regular graph with a presumed automorphism group of prime order p was introduced.

A prototype of a fixed row (a row corresponding to an orbit of length 1) for the distribution $(d_1 \times 1, d_p \times p)$ is a nonnegative integer solution of x_0, x_1, y_0 and y_1 satisfying the following set of linear equations:

$$\begin{aligned} x_0 + x_1 &= d_1 \\ y_0 + y_1 &= d_p \\ x_1 + py_1 &= k, \end{aligned} \tag{3}$$

where x_0 and x_1 are the number of zeros and ones, respectively, in the fixed columns of a fixed row, and y_0 and y_1 are the number of zeros and ones, respectively, in the nonfixed columns of a fixed row.

A prototype of a nonfixed row (a row corresponding to an orbit of length p) for the distribution $(d_1 \times 1, d_p \times p)$ is a nonnegative integer solution of $x_0, x_p, y_0, \ldots, y_p$, satisfying this set of linear equations:

$$\begin{aligned} x_0 + x_p &= d_1 \\ y_0 + y_1 + \cdots + y_p &= d_p \\ x_p + y_1 + 2y_2 \cdots + py_p &= k \\ px_p + y_1 + 4y_2 \cdots + p^2 y_p &= \frac{(k-\mu)p + \mu p^2 + (\lambda - \mu)c_{rr}p}{p} \end{aligned} \tag{4}$$

for any nonfixed row r, where x_0 and x_p are the number of zeros and p's, respectively, in the fixed columns of the row r, and $y_i, i = 0, 1, \ldots, p$, is the number of i's on the nonfixed columns of the row r. So, for different c_{rr} we get different equations. Since p is a prime, the number c_{rr} must be even ([2, Lemma 3.2]).

Solving the systems of equations (3) and (4) for all possible orbit lengths distributions from Table 4, we obtain that prototypes exist only when $p \in \{2, 3, 5\}$, as presented in the third row of Table 4. (The orbit lengths distributions for which row prototypes exist are given in the first three rows of Table 5.) $\square$

After eliminating orbit lengths distributions for which there are no row prototypes, we must consider the orbit lengths distributions $(d_1 \times 1, d_p \times p)$ shown in Table 5. Using the prototypes, we construct the orbit matrices row by row, eliminating mutually G-isomorphic orbit matrices. For eliminating orbit matrices yielding G-isomorphic strongly regular graphs we use the same method as for eliminating orbit matrices of G-isomorphic designs (see [9, 11]). The construction was performed using our programs written in GAP [17]. The number of constructed orbit matrices is listed in Table 5.

p	2	2	2	2	2	2	2	2	2	2	3	3	5	5
d_1	1	3	5	7	9	11	13	15	17	19	5	8	1	6
d_p	20	19	18	17	16	15	14	13	12	11	12	11	8	7
$\#OM$	6	0	2872	0	6	0	232	0	0	0	18	0	3	2
$\#SRG$	12	0	2362	0	64	0	4544	0	0	0	264	0	3	0

Table 5: *The number of constructed orbit matrices and SRGs(*$41, 20, 9, 10$*)*

Orbit matrices exist for the orbit lenghts distributions $(1\times1, 20\times2)$, $(5\times1, 18\times2)$, $(9 \times 1, 16 \times 2)$, $(13 \times 1, 14 \times 2)$, $(5 \times 1, 12 \times 3)$, $(1 \times 1, 8 \times 5)$ and $(6 \times 1, 7 \times 5)$, and in the final step of the construction, we consider these cases to construct adjacency matrices of strongly regular graphs with parameters $(41, 20, 9, 10)$. The number of nonisomorphic graphs we obtain is given in the fifth row of Table 5. Among the constructed strongly regular graphs, there are 7152 mutually nonisomorphic graphs, and 7089 of them are new. An analysis of the full automorphism groups of these 7152 graphs gives us the following theorem.

Theorem 3. *Up to isomorphism, there exist exactly* 7152 *strongly regular graphs with parameters* $(41, 21, 9, 10)$ *having nontrivial automorphisms, with the full automorphism groups as presented in Table 6.*

$\mathrm{Aut}(\Gamma)$	$Z_{41} : Z_{30}$	D_8	S_3	Z_5	Z_4	Z_3	Z_2
$\|\mathrm{Aut}(\Gamma)\|$	820	8	6	5	4	3	2
$\#SRGs$	1	8	80	2	3	184	6874

Table 6: *SRGs*$(41, 20, 9, 10)$ *with nontrivial automorphisms*

The list of adjacency matrices of all SRGs$(41, 20, 9, 10)$ with nontrivial automorphisms is available online at:

`http://www.math.uniri.hr/~mmaksimovic/nontrivial_41.txt` .

5. Regular two-graphs on 42 vertices

In [6], Bussemaker, Mathon and Seidel constructed 18 regular two-graphs on 42 vertices (see Table 3). To the best of our knowledge, these are the only known regular two-graphs on 42 vertices.

Theorem 4. *Up to isomorphism, there exist at least* 752 *regular two-graphs on* 42 *vertices with* 18439 *nonisomorphic descendants. Among them, there are* 64 *self-complementary two-graphs. Exactly* 749 *regular two-graphs on* 42 *vertices have at least one descendant with a nontrivial automorphism, and there are no more two-graphs with this property.*

Proof. Using the classification of SRGs(41, 20, 9, 10) with nontrivial automorphisms given in Section 4, we constructed all corresponding two-graphs on 42 vertices and their descendants. (Note that if there are more two-graphs on 42 vertices, they can only have descendants with a trivial automorphism group.) Our results are summarized in Table 7.

i	$\lvert \mathrm{Aut}(\Phi_i) \rvert$	Descendants of Φ_i	"S"	New
1	34440	$[1 \times Z_{41} : Z_{30}]$	YES	NO
2	168	$[1 \times Z_4]$	YES	NO
3-4	14	$[2 \times E, 2 \times Z_2]$	YES	NO
5	14	$[2 \times E, 2 \times Z_2]$	NO	NO
6	14	$[6 \times Z_2]$	YES	NO
7	9	$[4 \times E, 2 \times Z_3]$	YES	YES
8-9	8	$[2 \times E, 6 \times Z_2, 2 \times D_8]$	YES	NO
10	8	$[2 \times E, 6 \times Z_2, 2 \times D_8]$	NO	NO
11-12	6	$[4 \times E, 6 \times Z_2]$	YES	NO
13-32	6	$[4 \times E, 4 \times Z_2, 2 \times Z_3, 2 \times S_3]$	NO	YES
33	5	$[8 \times E, 2 \times Z_5]$	YES	NO
34	4	$[10 \times E, 2 \times Z_4]$	YES	NO
35	3	$[12 \times E, 6 \times Z_3]$	YES	YES
36-43	3	$[12 \times E, 6 \times Z_3]$	NO	YES
44-63	2	$[18 \times E, 6 \times Z_2]$	YES	YES
64-227	2	$[18 \times E, 6 \times Z_2]$	NO	YES
228-253	2	$[14 \times E, 14 \times Z_2]$	YES	YES
254-402	2	$[14 \times E, 14 \times Z_2]$	NO	YES
403-404	2	$[16 \times E, 10 \times Z_2]$	YES	YES
405	2	$[16 \times E, 10 \times Z_2]$	NO	YES

Table 7: *Descendants of the constructed two-graphs on* 42 *vertices*

Among the 749 constructed two-graphs, there are 61 self-complementary two-graphs and 344 pairs of complementary two-graphs. In total, these 749 two-graphs have 18426 descendants. In Table 7, each pair of complementary two-graphs is represented by one of them.

The only known two-graphs on 42 vertices not included in Table 7 are those which have only descendants with a trivial automorphism group; there are three such two-graphs known (see [6]), and together they have 13 nonisomorphic descendants with a trivial automorphism group (see Table 3). So, the statement of the theorem holds. □

For each two-graph from Table 7 the adjacency matrix of one of its descendants with the smallest nontrivial automorphism group is available online at:

`http://www.math.uniri.hr/~mmaksimovic/descendants_41.txt` .

Up to isomorphism, all two-graphs on 42 vertices with at least one descendant with nontrivial automorphism group can be reconstructed from this list, as can their 18426 descendants.

As can be seen from Table 3, the number of SRGs$(41, 20, 9, 10)$ with a trivial full automorphism group, which have been known so far is 55. Newly constructed two-graphs on 42 vertices have together 11274 nonisomorphic descendants whose full automorphism group is trivial and among them, there are 11232 new SRGs$(41, 20, 9, 10)$. Therefore, the following theorem holds.

Theorem 5. *Up to isomorphism, there exist at least* 18439 *strongly regular graphs with parameters* $(41, 20, 9, 10)$. *There are exactly* 7152 *strongly regular graphs with parameters* $(41, 20, 9, 10)$ *having nontrivial automorphisms, and at least* 11287 *strongly regular graphs with parameters* $(41, 20, 9, 10)$ *having the full automorphism group of order one.*

Remark 1. *Strongly regular configurations with parameters* $(41_5, 9, 10)$ *are the smallest strongly regular configurations for which (non)existence is not known (see [1]). Such a configuration could arise from a strongly regular graph having parameters* $(41, 20, 9, 10)$. *However, we have checked all* 18439 *strongly regular graphs from Theorem 5 and none of them yields a strongly regular configuration. Thus, if a strongly regular configuration* $(41_5, 9, 10)$ *could be constructed from a strongly regular graph with parameters* $(41, 20, 9, 10)$*, such a graph has no nontrivial automorphisms.*

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