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Global well-posedness and standing wave solutions for a class of nematic liquid crystal system

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Abstract. This paper is devoted to studying the Cauchy problem and a standing wave for a class of nematic liquid crystal system. This system appears in the recent studies of the propagation of a laser beam in a nematic liquid crystal. The above system couples the Schrödinger evolution equation to a nonlinear elliptic equation which describes the response of the director angle to the laser beam electric field. The global well-posedness will be established by using the Banach fixed point theorem and the continuity argument. Secondly, the existence of standing wave solution is established by using the constrained minimization approach.
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1. Introduction

In this paper, we consider the Cauchy problem of the following nematic liquid crystal system:

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u + u\sin 2\theta = \mu |u|^{p-2}u, & x \in \mathbb{R}^2, \\ -\nu\Delta\theta + q\sin 2\theta = 2|u|^2\cos 2\theta, & x \in \mathbb{R}^2, \end{cases} \tag{1}$$

where $2 < p < \infty$, $u(t,x,y)$ and $\theta(t,x,y)$ depend on the optical axis coordinate $t \in \mathbb{R}$ and the transverse coordinate $(x,y) \in \mathbb{R}^2$, $\Delta = \partial_x^2 + \partial_y^2$, is the Laplacian in the transverse directions, and μ, ν, q are positive constants. System (1) models the interaction between u and θ, where u represents the complex amplitude of the electric field of the polarized laser beam passing through the nematic liquid crystal sample, and θ represents the director angle of the macroscopic orientation of the liquid crystal molecule. In particular, system (1) can be described by the saturation effect in liquid crystal optics. When $\mu = 0$, system (1) can be simplified to the following nematic liquid crystal system with the saturation effect:

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u + u\sin 2\theta = 0, & x \in \mathbb{R}^2, \\ -\nu\Delta\theta + q\sin 2\theta = 2|u|^2\cos 2\theta, & x \in \mathbb{R}^2. \end{cases} \tag{2}$$

In [2], Borgna et al. considered the well-posed and solitary wave solutions of the above system (2), and proved the existence of global solutions via the Banach fixed

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point theorem, and presented that the pointing vector angle is bounded for any electric field, and additionally, normalized solutions were studied for the steady-state problems corresponding to system (2). For the arbitrary adviation angle case, in [3], Borgna et al. studied the well-posedness, decay and constrained minimizers of the Hamiltonian of the following coupled nonlinear Schrödinger system:

$$\begin{cases} \partial_z u = \frac{1}{2} i\Delta u + i\gamma(\sin^2(\psi+\theta_0) - \sin^2(\theta_0))u, \\ \nu\Delta\psi = \frac{1}{2}{E_0}^2 \sin(2\theta_0) - \frac{1}{2}({E_0}^2 + |u|^2)\sin(2(\psi+\theta_0)), \end{cases} \tag{3}$$

where E_0, ν, γ are positive constants, $\theta_0 \in (\frac{\pi}{4}, \frac{\pi}{2})$ is a constant. Model (3) arises in the study of optical beam propagation in nematic liquid crystals and models a set of experiments by Assanto and collaborators [5, 14, 15].

By using the small angle approximation $\sin\theta \approx \theta$, $\cos\theta \approx 1$, system (2) reduces to the following model:

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u + 2u\theta = 0, \ x \in \mathbb{R}^2, \\ -\nu\Delta\theta + 2q\theta = 2|u|^2, \quad x \in \mathbb{R}^2. \end{cases} \tag{4}$$

The directorial angle of system (4) has a unique solution $\theta = G * |u|^2$, where $G(x) = 2\nu^{-1} N_0(\sqrt{\frac{2q}{\nu}}|x|)$, and N_0 is the modified Bessel function. Replacing it in the first equation of system (4), it can be written as the Schrödinger equation with a Hartree-type nonlinearity

$$i\partial_t u + \frac{1}{2}\Delta u + 2\left(G * |u|^2\right) u = 0, \tag{5}$$

which describes the physical effect that a local electric field u can produce a deformed direction-vector angle θ over longer distances. As recognized by some authors [9, 16], this nonlocal effect regularizes the dynamics of the electric field and avoids the finite-time explosions that occur in two-dimensional cubic NLS (see [19]). In [11], Simon Louis et al. via the Newton conjugate gradient method calculated solitary waves of nematic liquid crystals. In 2004, Panayotars and Marchant ([12]) proved for the first time the existence and stability results of single spatial optical solitary waves of (5). In 2015, Wang and Li ([17]) discussed the propagation of two-color solitary waves with exponential-decay response by using a variational approach. In particular, Horikis and Frantzeskakis ([7]) were able to experimentally or numerically observe solitary waves pairs.

For some general forms of system (2), in [21], Zhang et al. have recently studied the existence of local and global solutions for the following Cauchy problems

$$\begin{cases} i\frac{\partial u}{\partial z} + \frac{d_1}{2}\triangle u + g_1 u\sin(2\theta) = 0, \quad x \in \mathbb{R}^2, z \in \mathbb{R}, \\ i\frac{\partial v}{\partial z} + \frac{d_2}{2}\triangle v + g_2 v\sin(2\theta) = 0, \quad x \in \mathbb{R}^2, z \in \mathbb{R}, \\ -\nu\triangle\theta + q\sin(2\theta) = 2(g_1|u|^2 + g_2|v|^2)\cos(2\theta), \quad x \in \mathbb{R}^2, z \in \mathbb{R}, \end{cases}$$

and proved the existence of positive radial ground state vector solitary wave solutions by using the symmetric decreasing rearrangement method and the minimization approach. Moreover, the experimental setup has been studied by physicists (see [7, 8]) who were able to experimentally or numerically observe solitary wave pairs which form bound state spinning about each other. Zhang et al. ([20]) studied a class of two-dimensional Hartree-type nonlinear Schrödinger systems with the Bessel potential kernel,

$$\begin{cases} i\frac{\partial u}{\partial z} + \frac{1}{2}D_u\triangle u + A_u u\sin(2\theta) = 0, \quad (x,y)\in\mathbb{R}^2, z\in\mathbb{R}, \\ i\frac{\partial v}{\partial z} + \frac{1}{2}D_v\triangle v + A_v v\sin(2\theta) = 0, \quad (x,y)\in\mathbb{R}^2, z\in\mathbb{R}, \\ -\nu\triangle\theta + q\sin(2\theta) = 2(A_u|u|^2 + A_v|v|^2)\cos(2\theta), \quad (x,y)\in\mathbb{R}^2, z\in\mathbb{R}, \end{cases}$$

established the global well-posedness and obtained the existence and orbital stability of ground state vector solitary waves by applying variational methods and the concentration-compactness lemma.

Compared with (2), there is a perturbation term $|u|^{p-2}u$ appearing in system (1); naturally, it is interesting to discuss the influence of the perturbation term on the existence of the global solution of system (1), even the ground state for the corresponding steady equations. To the best of our knowledge, there are few results about the existence of global solutions and a solitary wave for system (1) in the literature. The main purpose of this paper is to establish the local and global well-posedness of Cauchy problem (1) by using Banach fixed point theorem, the conservation law and the Gagliardo-Nirenberg inequality, we will prove the existence of the ground state solution for the steady equation of system (1) by using the constrained minimization approach.

The rest of this paper is organized as follows. In Section 2, we list some elementary results. Section 3 is devoted to proving the global existence of the solution of the initial problem (1). In Section 4, the existence of ground state solutions will be proved.

2. Preliminaries

Let $L^p(\mathbb{R}^2)$ $(p\geq 1)$ represent the usual Lebesgue space, for given $m\in\mathbb{N}$, and let $H^m(\mathbb{R}^2)$ represent the Sobolev space. For $I\subseteq\mathbb{R}$ and Banach space X, we have $L^p(I,X) = \{u : I\to X \mid \|u\|_X\in L^p(I)\}$.

Consider the following Schrödinger equation:

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u + f = 0, t\in\mathbb{R}, x\in\mathbb{R}^2, \\ u(0) = u_0, \end{cases} \tag{6}$$

where $u_0\in L^2(\mathbb{R}^2)$, the integral equation equivalent to (6) is

$$u(t) = W(t)u_0 + i\int_0^t W(t-s)f(s)ds,$$

where $W(t) = e^{i\frac{t}{2}\Delta}$, and $\{W(t) : t\in\mathbb{R}\}$ is the unitary group generated by $\frac{i}{2}\Delta$ in $L^2(\mathbb{R}^2)$. For any $s\in\mathbb{R}$, $W(t)$ is an isomorphism in $H^s(\mathbb{R}^2)$. Suppose $h(t) = W(t)u_0$, $g(t) = i\int_0^t W(t-s)f(s)ds$, then $u(t) = h(t)+g(t)$. The following Strichartz estimates hold.

Lemma 1 (see [4])**.** *Let $1 < r \leq 2 \leq p < \infty$, $q = \frac{2p}{p-2}$, $\gamma = \frac{2r}{3r-2}$; there exists $C_p, C_{p,r} > 0$ such that*

$$\begin{cases} \|h\|_{L^q(I,L^p)} \leq C_p\|u_0\|_{L^2}, \\ \|g\|_{L^q(I,L^p)} \leq C_{p,r}\|f\|_{L^\gamma(I,L^r)}, \end{cases} \tag{7}$$

for any interval $I \subset \mathbb{R}$.

Lemma 2 (Gagliardo-Nirenberg inequalities [4])**.**

$$\|v\|_{L^q} \leq C_{p,q,r} \|\nabla v\|_{L^p}^{\alpha} \|v\|_{L^r}^{1-\alpha}, \tag{8}$$

where $\frac{1}{q} - \frac{1}{r} = \left(\frac{1}{p} - \frac{1}{2} - \frac{1}{r}\right)\alpha$, $0 \leq \alpha \leq 1, 1 \leq q, r \leq \infty$, $p > 2$.

For system (1), the Hamiltonian H is given by the following form:

$$H(u,\theta) = \frac{1}{4}\int_{\mathbb{R}^2} \left(|\nabla u|^2 + \nu|\nabla\theta|^2 - 2|u|^2\sin(2\theta) + q(1-\cos(2\theta))\right) dx - \frac{1}{p}\int_{\mathbb{R}^2} |u|^p dx.$$

Now we show that $\|u(x,t)\|_{L^2}$ and energy $H(u,\theta)$ are conserved.

Lemma 3. *Let u be the solution of system* (1)*; then the conservation law holds.*

$$\|u(x,t)\|_{L^2} = \|u_0\|_{L^2}, \quad H(u,\theta) = H(u_0,\theta). \tag{9}$$

Proof. Let u be the solution of system (1); then multiply both sides of the first equation of (1) by $\overline{u}$, and we have

$$i\partial_t u\overline{u} + \frac{1}{2}\Delta u\overline{u} + u\sin(2\theta)\overline{u} = |u|^{p-2}u\overline{u}.$$

Then integrating over $\mathbb{R}^2$,

$$\frac{i}{2}\frac{d}{dt}\int_{\mathbb{R}^2} |u|^2 dx - \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 dx + \int_{\mathbb{R}^2} |u|^2 \sin(2\theta) dx = \int_{\mathbb{R}^2} |u|^p dx,$$

we have $\frac{d}{dt}\int_{\mathbb{R}^2} |u|^2 = 0$. Therefore $\|u(x,t)\|_{L^2} = \|u_0\|_{L^2}$, and

$$\|u\|_{L^p}^p + \frac{1}{2}\|\nabla u\|_{L^2}^2 = \int_{\mathbb{R}^2} |u|^2 \sin(2\theta) \leq \|u\|_{L^2}^2 = \|u_0\|_{L^2}^2.$$

Since

$$\partial_u H = \frac{1}{2}\Delta u - u\sin(2\theta) - |u|^{p-2}u, \quad \partial_\theta H = -\frac{\nu}{2}\Delta\theta - |u|^2\cos(2\theta) + \frac{q}{2}\sin(2\theta),$$

by simple calculation, we have

$$\frac{\partial H}{\partial t}(u,\theta) = \langle \partial_u H, \partial_t u\rangle + \langle \partial_\theta H, \partial_t \theta\rangle = \langle i\partial_t u, \partial_t u\rangle = 0,$$

that is, $H(u,\theta) = H(u_0,\theta)$. Moreover, we see that

$$\|\nabla u\|_{L^2}^2 + \|u\|_{L^2}^2 \leq 4H(u,\theta) + 3\|u\|_{L^2}^2 + \frac{4}{p}\|u\|_{L^p}^p.$$

$\square$

Let $N(x,\theta):\mathbb{R}^2\times\mathbb{R}\to\mathbb{R}$ be a Carathéodory function, that is, for every $\theta\in\mathbb{R}$, $N(x,\theta)$ is measurable with respect to x, and for almost all $x\in\mathbb{R}^2$, $N(x,\theta)$ is continuous with respect to $\theta\in\mathbb{R}$. We write the second equation in system (1) as

$$-\Delta\theta=N(u,\theta),\quad N(u,\theta)=-\frac{q}{\nu}\sin(2\theta)+\frac{2}{\nu}|u|^2\cos(2\theta). \tag{10}$$

Obviously, $N(u,\theta)$ decreases in the interval $[0,\frac{\pi}{4}]$. For any $u\in\mathbb{C}$ and $\theta\in[0,\frac{\pi}{4}]$, the function $N(u(x),\theta)$, $x\in\mathbb{R}^2$, satisfies the Carathéodory condition and $|N(u,\theta)|\le C(|\theta|+|u|^2)$. Therefore $N(u(x),\theta):=N(x,\theta)$ is a Carathéodory function, and Nemytskii operator $\theta\to N(x,\theta)$ is a bounded continuous mapping on $L^2(\mathbb{R}^2)$.

In [2], the existence and uniqueness of equation (10) have been established, we list some results which will be used in the sequel.

Lemma 4. *Let $u\in L^4(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$; then $0\le\theta(x)\le\theta_{max}<\frac{\pi}{4}$ for all $x\in\mathbb{R}^2$, where $\theta_{max}=\frac{1}{2}\arctan\frac{2\|u\|_{L^\infty}^2}{q}$.*

Lemma 5. *Let $u\in L^4(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$; then equation (1) has a unique solution $\theta\in H^2(\mathbb{R}^2)$, and $0\le\theta(x)\le\frac{\pi}{4}$, $\forall x\in\mathbb{R}^2$. Moreover, $\|\theta\|_{H^2}\le C\|u\|_{L^4}^2$.*

3. Well-posedness of the problem

We now consider the initial value problem for system (1), written as

$$\begin{cases} u(t)=W(t)u_0+i\int_0^t W(t-s)(u(s)[\sin(2(\theta(s))-|u(s)|^{p-2})]ds, & t\in\mathbb{R},\\ -\Delta\theta=N(u,\theta), & x\in\mathbb{R}^2,\end{cases} \tag{11}$$

where $N(u,\theta)=-\frac{q}{\nu}\sin(2\theta)+\frac{2}{\nu}|u|^2\cos(2\theta)$, and W is the unitary group generated by $\frac{i}{2}\Delta$.

Given $\zeta>0$, define the set

$$X_\zeta=\{u\in C([0,\zeta],H^1(\mathbb{R}^2)):\nabla u\in L^{\frac{2p}{p-2}}([0,\zeta],L^p(\mathbb{R}^2))\},$$

endowed with the norm

$$\|u\|_{X_\zeta}=\|u\|_{C([0,\zeta],H^1(\mathbb{R}^2))}+\|\nabla u\|_{L^{\frac{2p}{p-2}}([0,\zeta],L^p(\mathbb{R}^2))}.$$

Clearly, X_ζ is a Banach space and if $u\in X_\zeta$, then $u\in L^{\frac{4(p-1)}{p-2}}([0,\zeta],L^\infty(\mathbb{R}^2))$. In fact, in Lemma 2, taking $q=\infty$, $r=2$, $\alpha=\frac{p}{2(p-1)}$,

$$\|u\|_{L^\infty}\le C\|\nabla u\|_{L^p}^{\frac{p}{2(p-1)}}\|u\|_{L^2}^{1-\frac{p}{2(p-1)}}=C\|\nabla u\|_{L^p}^{\frac{p}{2(p-1)}}\|u\|_{L^2}^{\frac{p-2}{2(p-1)}}.$$

Therefore, in view of $u\in C([0,\zeta],H^1(\mathbb{R}^2))$, we have

$$\begin{aligned}\|u\|_{L^\gamma([0,\zeta],L^\infty(\mathbb{R}^2))}^\gamma\le&C\int_0^\zeta\|\nabla u\|_{L^p}^{\frac{p}{2(p-1)}\gamma}\|u\|_{L^2}^{\frac{p-2}{2(p-1)}\gamma}\le C\int_0^\zeta\|\nabla u\|_{L^p}^{\frac{p}{2(p-1)}\gamma}\|u\|_{H^1}^{\frac{p-2}{2(p-1)}\gamma}\\ \le&C\|u\|_{L^\infty([0,\zeta],H^1(\mathbb{R}^2))}^{\frac{p-2}{2(p-1)}\gamma}\int_0^\zeta\|\nabla u\|_{L^p}^{\frac{p}{2(p-1)}\gamma}.\end{aligned}$$

Taking $\gamma = \frac{4(p-1)}{p-2}$ in the above inequality, we obtain that

$$\|u\|^{\frac{4(p-1)}{p-2}}_{L^{\frac{4(p-1)}{p-2}}([0,\zeta],L^\infty(\mathbb{R}^2))} \leq C\|u\|^2_{L^\infty([0,\zeta],H^1(\mathbb{R}^2))}\|\nabla u\|^{\frac{p-2}{2p}}_{L^{\frac{2p}{p-2}}([0,\zeta],L^p(\mathbb{R}^2))}.$$

Define the mapping $\eta : H^1(\mathbb{R}^2)\bigcap L^\infty(\mathbb{R}^2) \to H^2(\mathbb{R}^2)$, $\eta(u) = \theta$, where θ is the solution of equation (10). Lipschitz continuity of the mapping η was proved (see Proposition 4.1 in [2]). We recall it as follows.

Lemma 6.

$$\|\eta(u_1) - \eta(u_2)\|_{H^2} \leq C_{\nu,q}(\|u_1\|_{H^1} + \|u_2\|_{H^1})(1 + \|u_1\|^2_{L^\infty} + \|u_2\|^2_{L^\infty})\|u_1 - u_2\|_{H^1}.$$

From Strichartz estimates, i.e., Lemma 1, we can easily deduce the following estimates.

Lemma 7. *Let $f \in L^1([0,\zeta], H^1(\mathbb{R}^2))$, and let g be defined by*

$$g(t) = i\int_0^t W(t-s)f(s)ds.$$

Then $g \in X_\zeta$ and satisfies $\|g\|_{X_\zeta} \leq C\|f\|_{L^1([0,\zeta],H^1(\mathbb{R}^2))}$.

Proof. Since $W(t)$ is a unitary operator, we have $\|g\|_{C([0,\zeta],H^1)} \leq \|f\|_{L^1([0,\zeta],H^1(\mathbb{R}^2))}$. By virtue of the fact that

$$\nabla g(t) = i\int_0^t W(t-s)\nabla f(s)ds,$$

and Lemma 1 (taking $r = 2, \gamma = 1$), we obtain

$$\|\nabla g\|_{L^{\frac{2p}{p-2}}([0,\zeta],L^p(\mathbb{R}^2))} \leq C_{1,2}\|\nabla f\|_{L^1([0,\zeta],L^2(\mathbb{R}^2))},$$

which leads to

$$\|g\|_{X_\zeta} = \|g\|_{C([0,\zeta],H^1(\mathbb{R}^2))} + \|\nabla g\|_{L^{\frac{2p}{p-2}}([0,\zeta],L^p(\mathbb{R}^2))} \leq C\|f\|_{L^1([0,\zeta],H^1(\mathbb{R}^2))}.$$

$\square$

Lemma 8. *Let $u_0 \in H^1(\mathbb{R}^2)$ and $h(t) = W(t)u_0$, therefore, $\|h\|_{X_\zeta} \leq C\|u_0\|_{H^1(\mathbb{R}^2)}$.*

Proof. By (7), we have $\|\nabla h\|_{L^{\frac{2p}{p-2}}([0,\zeta],L^p(\mathbb{R}^2))} \leq C\|u_0\|_{L^2}$. Using the property of W, we have

$$\|h(t)\|_{C([0,\zeta],H^1)} = \|W(t)u_0\|_{C([0,\zeta],H^1)} = \sup_{t\in[0,\zeta]}\|W(t)u_0\|_{H^1} = \sup_{t\in[0,\zeta]}\|u_0\|_{H^1} = \|u_0\|_{H^1},$$

which yields that

$$\|h\|_{X_\zeta} \leq C\|u_0\|_{H^1} + \|u_0\|_{H^1} \leq C\|u_0\|_{H^1}.$$

$\square$

Lemma 9. *Let $B(u) = u\sin(2\eta(u)) - |u|^{p-2}u$. Then the map $B : X_\zeta \to L^1([0,\zeta], H^1(\mathbb{R}^2))$ is bounded. Moreover, for any $R > 0$ there exists $C > 0$ such that $u \in X_\zeta$ and $\|u\|_{X_\zeta} \le R$ imply $\|B(u)\|_{L^1([0,\zeta],H^1(\mathbb{R}^2))} \le C\zeta \|u\|_{X_\zeta}$.*

Proof. Let $\theta = \eta(u)$; then $B(u) = u\sin(2\theta) - |u|^{p-2}u$. By the Sobolev embedding inequality, we have that $\|B(u)\|_{L^2} \le \|u\|_{L^2} + \|u\|_{H^1}^{p-1}$. Since $\nabla B(u) = \nabla u \sin(2\theta) + 2u\cos(2\theta)\nabla\theta - (p-1)|u|^{p-2} \cdot \nabla u$, by Lemma 4 and Hölder inequalities, we deduce that

$$\begin{aligned}
\|\nabla B(u)\|_{L^2} \le & \|\nabla u\|_{L^2} + 2\|u\|_{L^4}\|\nabla\theta\|_{L^4} + (p-1)\||u|^{p-2}\nabla u\|_{L^2} \\
\le & \|\nabla u\|_{L^2} + C_1\|u\|_{L^4}\|\theta\|_{H^2} + (p-1)\left(\int_{\mathbb{R}^2} |u|^{2(p-2)}|\nabla u|^2 dx\right)^{\frac{1}{2}} \\
\le & C_3(\|\nabla u\|_{L^2} + \|u\|_{H^1}^3) + C_2\left(\int_{\mathbb{R}^2} |\nabla u|^p\right)^{\frac{1}{p}}\left(\int_{\mathbb{R}^2} |u|^{2p}\right)^{\frac{p-2}{2p}} \\
\le & C(\|u\|_{H^1} + \|u\|_{H^1}^3) + C_3\|u\|_{H^1}^{(p-2)}\|\nabla u\|_{L^p}.
\end{aligned}$$

Integrating over $[0,\zeta]$, we get

$$\begin{aligned}
\int_0^\zeta \|B(u)\|_{H^1} dt \le & \int_0^\zeta (\|B(u)\|_{L^2} + \|\nabla B(u)\|_{L^2})\, ds \\
\le & \int_0^\zeta \left(\|u\|_{H^1} + \|u\|_{H^1}^3 + \|u\|_{H^1}^{p-1} + C_3\|u\|_{H^1}^{(p-2)}\|\nabla u\|_{L^p}\right) ds \\
\le & C\left(\|u\|_{L^\infty([0,\zeta],H^1(\mathbb{R}^2))} + \|u\|^3_{L^\infty([0,\zeta],H^1(\mathbb{R}^2))} + \|u\|^{p-1}_{L^\infty([0,\zeta],H^1(\mathbb{R}^2))}\right)\zeta \\
& + \|u\|^{p-2}_{L^\infty([0,\zeta],H^1)} \int_0^\zeta \|\nabla u\|_{L^p} ds \\
\le & C\left(\|u\|_{L^\infty([0,\zeta],H^1(\mathbb{R}^2))} + \|u\|^3_{L^\infty([0,\zeta],H^1(\mathbb{R}^2))} + \|u\|^{p-1}_{L^\infty([0,\zeta],H^1(\mathbb{R}^2))}\right)\zeta \\
& + \|u\|^{p-2}_{L^\infty([0,\zeta],H^1)} \left(\int_0^\zeta \|\nabla u\|_{L^p}^{\frac{2p}{p-2}}\right)^{\frac{p-2}{2p}} \zeta^{\frac{p+2}{2p}} \\
\le & C\left(\|u\|_{L^\infty([0,\zeta],H^1(\mathbb{R}^2))} + \|u\|^3_{L^\infty([0,\zeta],H^1(\mathbb{R}^2))} + \|u\|^{p-1}_{L^\infty([0,\zeta],H^1(\mathbb{R}^2))}\right)\zeta \\
& + \|u\|^{p-2}_{L^\infty([0,\zeta],H^1)}\|\nabla u\|_{L^{\frac{2p}{p-2}}([0,\zeta,L^p])} \zeta^{\frac{p+2}{2p}} \\
\le & \left(C\zeta(1 + R^2 + R^{p-2}) + \zeta^{\frac{p+2}{2p}} R^{p-2}\right)\|u\|_{L^\infty([0,\zeta],H^1(\mathbb{R}^2))}.
\end{aligned}$$

□

In view of the above estimates, we can obtain Lipschitz continuity of the mapping $B(u)$.

Lemma 10. *The mapping B defined in Lemma 9 is locally Lipschitz continuous, that is, for any $R > 0$, there exists $C(R) > 0$ such that when $u_1, u_2 \in X_\zeta$ and $\|u_1\|_{X_\zeta}$, $\|u_2\|_{X_\zeta} \le R$, we have*

$$\|B(u_1) - B(u_2)\|_{L^1([0,\zeta],H^1(\mathbb{R}^2)} \le C(R)(\zeta + \zeta^{\frac{p^2-2p+4}{p^2}} + \zeta^{\frac{p+2}{2p}})\|u_1 - u_2\|_{X_\zeta}.$$

Proof. Let $u_1, u_2 \in X_\zeta$ satisfy $\|u_1\|_{X_\zeta}, \|u_2\|_{X_\zeta} \leq R$. Since

$$\begin{aligned}||u_1|^{p-2}u_1 - |u_2|^{p-2}u_2| \leq& C_p|u_1-u_2|(|u_1|+|u_2|)^{p-2}\\ \leq& C_p|u_1-u_2|(|u_1|^{p-2}+|u_2|^{p-2}),\end{aligned}$$

we deduce that

$$\begin{aligned}|B(u_1)-B(u_2)| \leq& |u_1\sin(2\theta_1) - |u_1|^{p-2}u_1 - u_2\sin(2\theta_2) + |u_2|^{p-2}u_2|\\ \leq& |u_1-u_2| + 2|u_2||\theta_1-\theta_2| + ||u_2|^{p-2}u_2 - |u_1|^{p-2}u_1|\\ \leq& |u_1-u_2| + 2|u_2||\theta_1-\theta_2| + C_p|u_1-u_2|(|u_1|^{p-2}+|u_2|^{p-2}),\end{aligned}$$

where $\theta_j = \eta(u_j)$. By Hölder's inequality, we have

$$\begin{aligned}\|B(u_1)-B(u_2)\|_{L^2} \leq& \|u_1-u_2\|_{L^2} + 2\|u_2\|_{L^4}\|\theta_1-\theta_2\|_{L^4}\\ &+ C_3(\|u_1\|^{p-2}_{L^{2p}} + \|u_2\|^{p-2}_{L^{2p}})(\|u_1-u_2\|_{L^p})\\ \leq& C_4(\|u_1-u_2\|_{H^1} + \|u_2\|_{H^1}\|\theta_1-\theta_2\|_{H^1}\\ &+ C_5(\|u_1\|^{p-2}_{H^1} + \|u_2\|^{p-2}_{H^1})(\|u_1-u_2\|_{H^1}).\end{aligned} \tag{12}$$

On the other hand, by a simple calculation, one has

$$\begin{aligned}|\nabla B(u_1)-\nabla B(u_2)| \leq& |\nabla(u_1-u_2)| + 2|\nabla u_2||\theta_1-\theta_2| + 2|u_1-u_2||\nabla\theta_1|\\ &+ 2|u_2||\nabla\theta_2-\nabla\theta_1| + 4|u_2||\theta_2-\theta_1||\nabla\theta_2|\\ &+ (p-1)|u_1|^{p-2}|\nabla(u_1-u_2)|\\ &+ (p-1)|\nabla u_2||u_1|^{p-2}-|u_2|^{p-2}|\\ =& I_1+I_2+I_3+I_4+I_5+I_6+I_7.\end{aligned}$$

By using the Sobolev embedding inequality and Hölder's inequalities, we have

$$\begin{aligned}\|I_1\|_{L^2} \leq& \|\nabla(u_1-u_2)\|_{L^2} \leq \|u_1-u_2\|_{H^1},\\ \|I_2\|_{L^2} \leq& C\|\nabla u_2\|_{L^2}\|\theta_1-\theta_2\|_{L^\infty} \leq C\|u_2\|_{H^1}\|\theta_1-\theta_2\|_{H^2},\\ \|I_3\|_{L^2} \leq& C\|\nabla\theta_1\|_{L^4}\|u_1-u_2\|_{L^4} \leq C\|\theta_1\|_{H^2}\|u_1-u_2\|_{H^1},\\ \|I_4\|_{L^2} \leq& C\|u_2\|_{L^4}\|\nabla\theta_2-\nabla\theta_1\|_{L^4} \leq C\|u_2\|_{H^1}\|\theta_1-\theta_2\|_{H^2},\\ \|I_5\|_{L^2} \leq& C\|u_2\|_{L^4}\|\nabla\theta_2\|_{L^4}\|\theta_1-\theta_2\|_{L^\infty} \leq C\|u_2\|_{H^1}\|\theta_2\|_{H^2}\|\theta_2-\theta_1\|_{H^2},\\ \|I_6\|^2_{L^2} \leq& C_p\left(\int_{\mathbb{R}^2}|\nabla(u_1-u_2)|^p dx\right)^{\frac{2}{p}}\left(\int_{\mathbb{R}^2}|u_1|^{2p}dx\right)^{\frac{p-2}{p}}\\ =& C_p\|\nabla(u_1-u_2)\|^2_{L^p}\|u_1\|^{2(p-2)}_{H^1},\end{aligned}$$

and

$$
\begin{aligned}
\|I_7\|_{L^2}^2 &= \int_{\mathbb{R}^2} (p-1)^2|\nabla u_2|^2 \left||u_1|^{p-2} - |u_2|^{p-2}\right|^2 dx \\
&\le \int_{\mathbb{R}^2} (p-1)^2|\nabla u_2|^2 \left||u_1|^{p-3}u_1 - |u_2|^{p-3}u_2\right|^2 dx \\
&= \int_{\mathbb{R}^2} (p-1)^2|\nabla u_2|^2 \left||u_1-u_2|\,||u_1|+|u_2||^{p-3}\right|^2 dx \\
&\le C_p \int_{\mathbb{R}^2} |\nabla u_2|^2|u_1-u_2|^2 \left||u_1|^{2(p-3)} + |u_2|^{2(p-3)}\right| dx \\
&= C_p \int_{\mathbb{R}^2} |\nabla u_2|^2|u_1-u_2|^2|u_1|^{2(p-3)}dx + C_p \int_{\mathbb{R}^2} |\nabla u_2|^2|u_1-u_2|^2|u_2|^{2(p-3)}dx \\
&\le C_p \left(\int_{\mathbb{R}^2} |\nabla u_2|^p dx\right)^{\frac{2}{p}} \left(\int_{\mathbb{R}^2} |u_1-u_2|^{2p}dx\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^2} |u_1|^{2p}dx\right)^{\frac{p-3}{p}} \\
&\quad +C_p \left(\int_{\mathbb{R}^2} |\nabla u_2|^p dx\right)^{\frac{2}{p}} \left(\int_{\mathbb{R}^2} |u_1-u_2|^{2p}dx\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^2} |u_2|^{2p}dx\right)^{\frac{p-3}{p}} \\
&= C_p\|\nabla u_2\|_{L^p}^2\|u_1-u_2\|_{L^{2p}}^2 \left(\|u_1\|_{L^{2p}}^{2(p-3)} + \|u_2\|_{L^{2p}}^{2(p-3)}\right) \\
&\le C\|\nabla u_2\|_{L^p}^2\|u_1-u_2\|_{H^1}^2 \left(\|u_1\|_{H^1}^{2(p-3)} + \|u_2\|_{H^1}^{2(p-3)}\right). \qquad (13)
\end{aligned}
$$

In view of Lemma 4, we know that $\|\theta_j\|_{H^2} \le C\|u_j\|_{H^1}^2$. By Lemma 2 and (8), we get that

$$
\|u\|_{L^\infty} \le C\|\nabla u\|_{L^p}^{\frac{2}{p}}\|u\|_{L^p}^{\frac{p-2}{p}} \le C\|\nabla u\|_{L^p}^{\frac{2}{p}}\|u\|_{H^1}^{\frac{p-2}{p}}.
$$

Therefore, according to (12), (13) and Lemma 6, we deduce that

$$
\begin{aligned}
&\|B(u_1)-B(u_2)\|_{H^1} \\
&\le C\|u_1-u_2\|_{H^1} + C\|u_2\|_{H^1}\|\theta_1-\theta_2\|_{H^1} + C(\|u_1\|_{H^1}^{p-2} \\
&\quad + \|u_2\|_{H^1}^{p-2})(\|u_1-u_2\|_{H^1}) + \|u_1-u_2\|_{H^1} + C\|u_2\|_{H^1}\|\theta_1-\theta_2\|_{H^2} \\
&\quad + C\|u_1-u_2\|_{H^1}\|\theta_1\|_{H^2} + C\|u_2\|_{H^1}\|\theta_1-\theta_2\|_{H^2} + C\|u_2\|_{H^1}\|\theta_1-\theta_2\|_{H^2}\|\theta_2\|_{H^2} \\
&\quad + C\|\nabla(u_1-u_2)\|_{L^p}\|u_1\|_{H^1}^{p-2} + C\|u_1-u_2\|_{H^1}\|\nabla u_2\|_{L^p}\left(\|u_1\|_{H^1}^{p-3} + \|u_2\|_{H^1}^{p-3}\right) \\
&\le (C + \|u_1\|_{H^1}^{p-2} + \|u_2\|_{H^1}^{p-2} + \|u_1\|_{H^1}^2 + 1 + C\|\nabla u_2\|_{L^p}(\|u_1\|_{H^1}^{p-3} + \|u_2\|_{H^1}^{p-3}) \\
&\quad + C(\|u_2\|_{H^1} + \|u_2\|_{H^1}^3)(\|u_1\|_{H^1} + \|u_2\|_{H^1})(1 + \|u_1\|_{L^\infty}^2 + \|u_2\|_{L^\infty}^2))\|u_1-u_2\|_{H^1} \\
&\quad + C\|\nabla(u_1-u_2)\|_{L^p}\|u_1\|_{H^1}^{p-2} \\
&\le (C + \|u_1\|_{H^1}^{p-2} + \|u_2\|_{H^1}^{p-2} + \|u_1\|_{H^1}^2 + 1 + C\|\nabla u_2\|_{L^p}(\|u_1\|_{H^1}^{p-3} + \|u_2\|_{H^1}^{p-3}) \\
&\quad + C(\|u_2\|_{H^1} + \|u_2\|_{H^1}^3)(\|u_1\|_{H^1} + \|u_2\|_{H^1})(1 + C\|\nabla u_1\|_{L^p}^{\frac{4}{p}}\|u_1\|_{H^1}^{\frac{2(p-2)}{p}} \\
&\quad + C\|\nabla u_2\|_{L^p}^{\frac{4}{p}}\|u_2\|_{H^1}^{\frac{2(p-2)}{p}})) \times \|u_1-u_2\|_{H^1} + C\|\nabla(u_1-u_2)\|_{L^p}\|u_1\|_{H^1}^{p-2}.
\end{aligned}
$$

From $\|u_i\|_{X_\zeta} \le R$, we see that $\|u_i\|_{H^1} \le R (i = 1, 2)$, for $\forall t \in [0, \zeta]$. Thus, integrating over $[0,\zeta]$, and using Hölder's inequality, we get

$$\begin{aligned}
&\int_0^\zeta \|B(u_1)-B(u_2)\|_{H^1}ds\\
&\quad\le C(R)\int_0^\zeta \Big[(1+\|\nabla u_1\|_{L^p}^{\frac{4}{p}}+\|\nabla u_2\|_{L^p}^{\frac{4}{p}}+\|\nabla u_2\|_{L^p})\|u_1-u_2\|_{H^1}\\
&\qquad+\|\nabla(u_1-u_2)\|_{L^p}\Big]ds \qquad (14)\\
&\quad\le C(R)\Big[\sup_{[0,\zeta]}\|u_1-u_2\|_{H^1}\int_0^\zeta(1+\|\nabla u_1\|_{L^p}^{\frac{4}{p}}+\|\nabla u_2\|_{L^p}^{\frac{4}{p}}+\|\nabla u_2\|_{L^p})ds\\
&\qquad+\int_0^\zeta\|\nabla(u_1-u_2)\|_{L^p}ds\Big],
\end{aligned}$$

where $C(R)$ is a positive constant that only depends on R. By Hölder's inequality and $\|u_i\|_{X_\zeta}\le R$, we have

$$\begin{aligned}
\int_0^\zeta\|\nabla u_1\|_{L^p}^{\frac{4}{p}}ds &\le \Big(\int_0^\zeta\|\nabla u_1\|_{L^p}^{p\frac{2}{p-2}}ds\Big)^{\frac{2(p-2)}{p^2}}\Big(\int_0^\zeta 1^{\frac{p^2}{p^2-2p+4}}ds\Big)^{\frac{p^2-2p+4}{p^2}}\\
&\le R^{\frac{4}{p}}\zeta^{\frac{p^2-2p+4}{p^2}},
\end{aligned} \qquad (15)$$

$$\int_0^\zeta\|\nabla u_2\|_{L^p}ds\le(\int_0^\zeta\|\nabla u_2\|_{L^p}^{\frac{2p}{p-2}}ds)^{\frac{p-2}{2p}}(\int_0^\zeta 1^{\frac{2p}{p+2}}ds)^{\frac{p+2}{2p}}\le R\zeta^{\frac{p+2}{2p}}, \qquad (16)$$

and

$$\begin{aligned}
\int_0^\zeta\|\nabla(u_1-u_2)\|_{L^p}ds &\le\left(\int_0^\zeta\|\nabla(u_1-u_2)\|_{L^p}^{\frac{2p}{p-2}}ds\right)^{\frac{p-2}{2p}}\left(\int_0^\zeta 1ds\right)^{\frac{p+2}{2p}}\\
&\le\zeta^{\frac{p+2}{2p}}\|u_1-u_2\|_{X_\zeta}.
\end{aligned} \qquad (17)$$

Therefore, by (14), (15), (16), (17), we get

$$\|B(u_1)-B(u_2)\|_{L^1([0,\zeta],H^1)}\le C(R)(\zeta+\zeta^{\frac{p^2-2p+4}{p^2}}+\zeta^{\frac{p+2}{2p}})\|u_1-u_2\|_{X_\zeta},$$

where $C(R)$ is a positive constant that only depends on R. □

Theorem 1. *Let $u_0\in H^1(\mathbb{R}^2)$; then there exist $\zeta=\zeta(\|u_0\|_{H^1})>0$, problem* (11) *has a unique solution $(u,\theta)\in X_\zeta\times L^\infty([0,\zeta],H^2(\mathbb{R}^2))$ on the interval $[0,\zeta]$, and $\theta\in[0,\pi/4]$. Moreover, the mapping $u_0\mapsto u$ is continuous from $H^1(\mathbb{R}^2)$ to X_ζ.*

Proof. Define a mapping $\Gamma: X_\zeta\to X_\zeta$ given by the following form:

$$(\Gamma u)(t)=W(t)u_0+i\int_0^t W(t-s)B(u(s))ds,\quad t\in[0,\zeta].$$

In view of Lemma 4, we have $\theta=\eta(u)\in L^\infty([0,\zeta],H^2(\mathbb{R}^2))$ if $u\in X_\zeta$. From $u_0\in H^1(\mathbb{R}^2)$ and Lemma 8, we see that $h(t)=W(t)u_0\in X_\zeta$, $t\in[0,\zeta]$.

Now let us consider a closed ball $\overline{B_R(h)} \subset X_\zeta$, where $R > 0$. For any $u \in \overline{B_R(h)}$, by virtue of Lemma 7 and Lemma 9, taking ζ sufficiently small, we have

$$\|\Gamma(u) - h\|_{X_\zeta} = \|\int_0^t W(t-s)B(u(s))ds\|_{X_\zeta} \leq C\|B\|_{L^1([0,\zeta],H^1)} \leq C\zeta\|u\|_{X_\zeta} \leq R,$$

that is, $\Gamma : \overline{B_R(h)} \to \overline{B_R(h)}$ is well defined.

Next, we show that Γ is a contraction mapping on $\overline{B_R(h)}$. In fact, let $u_1, u_2 \in \overline{B_R(h)} \subset X_\zeta$; then

$$\Gamma(u_1)(t) - \Gamma(u_2)(t) = i\int_0^t W(t-s)(B(u_1(s)) - B(u_2(s)))ds.$$

Thus, by Lemma 7 and Lemma 10, we have

$$\begin{aligned} \|\Gamma(u_1) - \Gamma(u_2)\|_{X_\zeta} &\leq C\|B(u_1(s)) - B(u_2(s))\|_{L^1([0,\zeta],H^1)} \\ &\leq C(R)(\zeta + \zeta^{\frac{p^2-2p+4}{p^2}} + \zeta^{\frac{p+2}{2p}})\|u_1 - u_2\|_{X_\zeta}. \end{aligned} \tag{18}$$

Taking ζ sufficiently small such that $C(R)(\zeta + \zeta^{\frac{p^2-2p+4}{p^2}} + \zeta^{\frac{p+2}{2p}}) < \frac{1}{2}$ leads to

$$\|\Gamma(u_1) - \Gamma(u_2)\|_{X_\zeta} \leq \frac{1}{2}\|u_1 - u_2\|_{X_\zeta},$$

which means that Γ possesses the contraction property on $\overline{B_R(h)}$.

Applying the Banach contraction mapping theorem, there exists a unique $u \in \overline{B_R(h)}$, such that $\Gamma(u) = u$, that is;

$$u(t) = \Gamma(u)(t) = W(t)u_0 + i\int_0^t W(t-s)B(u(s))ds, \quad t \in [0,\zeta].$$

Thus the first equation of system (11) has a unique local solution $u(x,t)$ on the interval $[0,\zeta]$.

Finally, we prove the continuous dependence of the solution on the initial conditions. For this reason, let the solution corresponding to initial conditions $v_j \in H^1(\mathbb{R}^2)$ be u_j, $j = 1,2$. Obviously, $u_j = \Gamma_{v_j}(u_j)$, $j = 1,2$. Thus, by (18) and Lemma 8, we have

$$\begin{aligned} \|u_1 - u_2\|_{X_\zeta} &\leq \|\Gamma_{v_1}(u_1) - \Gamma_{v_2}(u_2)\|_{X_\zeta} \\ &= \|W(t)v_1 - W(t)v_2\|_{X_\zeta} + \|i\int_0^t W(t-s) \times (B(u_1(s)) - B(u_2(s)))\, ds\|_{X_\zeta} \\ &\leq \|W(t)v_1 - W(t)v_2\|_{X_\zeta} + C(R)(\zeta + \zeta^{\frac{p^2-2p+4}{p^2}} + \zeta^{\frac{p+2}{2p}})\|u_1 - u_2\|_{X_\zeta} \\ &= \|W(t)(v_1 - v_2)\|_{X_\zeta} + C(R)(\zeta + \zeta^{\frac{p^2-2p+4}{p^2}} + \zeta^{\frac{p+2}{2p}})\|u_1 - u_2\|_{X_\zeta} \\ &\leq C\|v_1 - v_2\|_{H^1} + C(R)(\zeta + \zeta^{\frac{p^2-2p+4}{p^2}} + \zeta^{\frac{p+2}{2p}})\|u_1 - u_2\|_{X_\zeta}. \end{aligned} \tag{19}$$

Let us take ζ sufficiently small such that $C(R)(\zeta+\zeta^{\frac{p^2-2p+4}{p^2}}+\zeta^{\frac{p+2}{2p}})<\frac{1}{2}$. Thus, (19) implies

$$\|u_1-u_2\|_{X_\zeta}\le C\|v_1-v_2\|_{H^1},$$

which implies the continuous dependence property of the solution on the initial data. □

Theorem 2 (Global existence)**.** *Let $u_0\in H^1(\mathbb{R}^2)$; then the problem has a unique solution $(u,\theta)\in C(\mathbb{R},H^1(\mathbb{R}^2))\times L^\infty(\mathbb{R},H^2(\mathbb{R}^2))$, such that $\theta\in[0,\pi/4]$, $\nabla u\in L^{\frac{2p}{p-2}}_{loc}(\mathbb{R},L^p(\mathbb{R}^2))$.*

Proof. Assume that (u,θ) is a solution of system (1). By Lemma 3, we get that

$$\|u(x,t)\|_{L^2(\mathbb{R}^2)}=\|u_0\|_{L^2(\mathbb{R}^2)},\quad H(u(x,t),\theta(x,t))=H(u_0,\theta_0),\ \theta_0=\eta(u_0).$$

Since

$$H(u,\theta)\ge\frac{1}{4}\|\nabla u\|^2_{L^2}-\frac{1}{2}\|u\|^2_{L^2}-\frac{1}{p}\|u\|^p_{L^p},$$

and

$$\|u\|^2_{H^1}\le 4H(u,\theta)+3\|u\|^2_{L^2}+C_p\|u\|^p_{L^p}\le C\|u_0\|^2_{H^1},\tag{20}$$

we deduce that $\|u\|_{H^1(\mathbb{R}^2)}$ is bounded, and its bounds are controlled by $\|u_0\|_{H^1}$.

For $t\in[0,\zeta]$, (1) has a unique solution on the interval $[0,\zeta]$, which satisfies equation (9). From (20), we have $\|u(x,\zeta)\|_{H^1}\le\|u_0\|_{H^1}$. Let $\widetilde{u}_0=u(x,\zeta)$, similarly to the proof of Theorem 1, a weak solution $\widetilde{u}(x,t+\zeta)$ of (1) exists on $[0,\zeta]$, and satisfies (9). Define

$$u(x,t)=\begin{cases}u(x,t), & 0\le t\le\zeta,\\ \widetilde{u}(x,t-\zeta), & 0\le\zeta\le t\le 2\zeta.\end{cases}\tag{21}$$

Obviously, $u(x,t)$ defined by (21) is the solution of system (1) on the interval $[0,2\zeta]$, and for $\zeta\le t\le 2\zeta$, we have

$$\|u(x,t)\|_{L^2}=\|\widetilde{u}(x,t-\zeta)\|_{L^2}=\|\widetilde{u}_0\|_{L^2}=\|u(x,\zeta)\|_{L^2}=\|u_0\|_{L^2},$$

and

$$H(u,\theta)=H(\widetilde{u}(x,t-\zeta),\theta)=H(\widetilde{u}_0,\theta)=H(u,\theta)=H(u_0,\theta_0).$$

Therefore, for all $t\in[0,2\zeta]$, $u(x,t)$ satisfies the conservation law (9). In addition, from (20), we have $\|u(x,2\zeta)\|_{H^1}\le C\|u_0\|_{H^1}$. Next, repeating the discussion above, constructing a solution of (1) on $[0,+\infty)$ for all $t>0$. In the same way, we can discuss the case of $t\le 0$ and obtain the global solution of (1) on $\mathbb{R}$, which satisfies (9). □

4. Existence of ground state solution

In this section, we will prove the existence of the ground state solution (u, θ) for a steady-state problem related to system (1).

Let $u(x,t) = e^{i\sigma t}v(x)$, where $\sigma \in \mathbb{R}^+$, $\theta(x,t) = \phi(x)$, system (1) becomes the following system:

$$\begin{cases} -\Delta v + 2\sigma v = 2v\sin(2\phi) + 2|v|^{p-2}v, & \text{in } \mathbb{R}^2, \\ -\nu\Delta\phi + q\sin(2\phi) = 2v^2\cos(2\phi), & \text{in } \mathbb{R}^2. \end{cases} \tag{22}$$

The energy function $J : H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \to \mathbb{R}$ corresponding to system (22) is defined as follows:

$$\begin{aligned} J(v,\phi) =& \frac{1}{4}\int_{\mathbb{R}^2} \left(|\nabla v|^2 + 2\sigma|v|^2 + \nu|\nabla\phi|^2 + q(1-\cos(2\phi))\right) dx \\ &- \frac{1}{2}\int_{\mathbb{R}^2} |v|^2\sin(2\phi)dx - \frac{1}{p}\int_{\mathbb{R}^2}|v|^p dx. \end{aligned}$$

From $\phi \in [0, \frac{\pi}{4}]$, we have

$$\int_{\mathbb{R}^2} |v|^2\sin(2\phi)dx \le \int_{\mathbb{R}^2}|v|^2dx, \quad \int_{\mathbb{R}^2}\phi^2dx \le \int_{\mathbb{R}^2}[1-\cos(2\phi)]dx \le 2\int_{\mathbb{R}^2}\phi^2dx.$$

Therefore, $J(v,\phi)$ is well defined on $H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$. By a standard argument, $J \in C^1(H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2), \mathbb{R})$, and $\forall (w,\psi) \in H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$,

$$\begin{aligned} \langle J'(v,\phi),(w,\psi)\rangle =& \frac{1}{2}\int_{\mathbb{R}^2}\Big(\nabla v\nabla w + 2\sigma vw - 2vw\sin(2\phi)\Big)dx - \int_{\mathbb{R}^2}|v|^{p-2}vwdx \\ &+ \frac{1}{2}\int_{\mathbb{R}^2}\Big(\nu\nabla\phi\nabla\psi - 2|v|^2\cos(2\phi)\psi + q\sin(2\phi)\psi\Big)dx. \end{aligned}$$

Obviously, if $(v,\phi) \in H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$ is a critical point of J, then (u,θ) is a weak solution of system (22). Let $V(x) = 2\sigma - 2\sin(2\phi)$, from $\phi \in [0, \frac{\pi}{4}]$ and assuming $\sigma > 1 + \alpha_0$, where $\alpha_0 > 0$, $V(x)$ is a bounded potential on $\mathbb{R}^2$ with positive and lower bounds. Using Lemma 1 in [13], we see that the solution v is a continuous function on $\mathbb{R}^2$, and $v \in L^\infty(\mathbb{R}^2)$, $\lim\limits_{|x|\to\infty} v(x) = 0$. By virtue of Lemma 4 in Section 2, we have $\phi \in [0, \theta_{max}]$, where $\theta_{max} = \frac{1}{2}\arctan\frac{2\|v\|^2_{L^\infty}}{q}$, $\theta_{\max} < \frac{\pi}{4}$. From Lemma 5 in Section 2, we know that $\|\phi\|_{H^2} \le C\|v\|^2_{L^4}$.

Let φ^* denote the symmetric decreasing rearrangement of $\varphi : \mathbb{R}^n \to \mathbb{R}_+$, which is the measurable function such that $|\{x \in \mathbb{R}^n : \varphi(x) > t\}| < \infty$, for all $t > 0$. From [10], we have the following lemma.

Lemma 11. *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a increasing continuous function such that $f(0) = 0$; then for all $\varphi : \mathbb{R}^n \to \mathbb{R}_+$, $(f \circ \varphi)^* = f \circ \varphi^*$.*

The symmetric properties of functional $J(v,\phi)$ can be obtained according to Lemma 11.

Proposition 1. *Let $(v,\phi) \in H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$, and $v \geq 0$, $0 \leq \phi \leq \frac{\pi}{4}$, i.e. in $\mathbb{R}^2$. Then $J(v^*,\phi^*) \leq J(v,\phi)$, where v^* and ϕ^* are the symmetric decreasing rearrangements of v and ϕ, respectively.*

Proof. By the Pólya-Szegö inequality, we have

$$\frac{1}{4}\|\nabla v^*\|_{L^2}^2 + \frac{\nu}{4}\|\nabla \phi^*\|_{L^2}^2 \leq \frac{1}{4}\|\nabla v\|_{L^2}^2 + \frac{\nu}{4}\|\nabla \phi\|_{L^2}^2. \tag{23}$$

The functions $1-\cos(2\phi)$, $\sin(2\phi)$ are increasing continuous on $[0,\pi/4]$ and vanishes at the origin, and Lemma 11 implies $(1-\cos(2\phi))^* = 1-\cos(2\phi^*)$, $(\sin(2\phi))^* = \sin(2\phi^*)$. Therefore,

$$\int_{\mathbb{R}^2} \frac{q}{4}(1-\cos(2\phi))dx = \int_{\mathbb{R}^2} \frac{q}{4}(1-\cos(2\phi))^* dx = \frac{q}{4}\int_{\mathbb{R}^2}(1-\cos(2\phi^*))dx. \tag{24}$$

By the Riesz inequality for the rearrangement inequality in [10], we have

$$\int_{\mathbb{R}^2} v^2 \sin(2\phi)dx \leq \int_{\mathbb{R}^2} v^{*2}\sin(2\phi)^* dx = \int_{\mathbb{R}^2} v^{*2}\sin(2\phi^*)dx, \tag{25}$$

and

$$\|v\|_{L^2}^2 = \|v^*\|_{L^2}^2, \quad \|v\|_{L^p}^p = \|v^*\|_{L^p}^p. \tag{26}$$

Therefore, from (23), (24), (25), (26), we have $J(v^*,\phi^*) \leq J(v,\phi)$. □

We introduce the Nehari manifold

$$\mathcal{N} = \{(v,\phi) \in H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2) : \ v \neq 0, \ \phi \neq 0, \ \langle J'(v,\phi),(v,\phi)\rangle = 0\}.$$

Lemma 12. *Let $p > 2$; then the Nehari manifold $\mathcal{N} \neq \emptyset$.*

Proof. Let $(v,\phi) \in H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$, and $v \neq 0$, $\phi \neq 0$. Let $v_t = tv(t^{-\frac{1}{2}}x)$, $\phi_t = \phi(t^{-\frac{1}{2}}x)$, where $t > 0$. By calculation, we have

$$\begin{aligned}\gamma(t) = \langle J'(v_t,\phi_t),(v_t,\phi_t)\rangle =& \frac{1}{2}t^2\int_{\mathbb{R}^2}|\nabla v|^2 dx + \sigma t^3\int_{\mathbb{R}^2}|v|^2 dx - t^3\int_{\mathbb{R}^2} v^2\sin(2\phi)dx \\ & - t^{2p+1}\int_{\mathbb{R}^2}|v|^p dx + \frac{\nu}{2}\int_{\mathbb{R}^2}|\nabla\phi|^2 dx \\ & - t^3\int_{\mathbb{R}^2} v^2\cos(2\phi)\cdot\phi dx + \frac{q}{2}t\int_{\mathbb{R}^2}\sin(2\phi)\cdot\phi dx.\end{aligned}$$

Clearly, $\gamma(t) \to \frac{\nu}{2}\int_{\mathbb{R}^2}|\nabla\phi|^2 dx > 0$ for $t \to 0^+$, and $\gamma(t) \to -\infty$ for $t \to +\infty$. Then there exists $t_1 > 0$, such that $\gamma(t_1) = 0$, that is, $\langle J'(v_{t_1},\phi_{t_1}),(v_{t_1},\phi_{t_1})\rangle = 0$. Obviously, $v_{t_1} \neq 0$, $\phi_{t_1}) \neq 0$. We conclude that $(v_{t_1},\phi_{t_1}) \in \mathcal{N}$, and $\mathcal{N} \neq \emptyset$. □

From Lemma 12, we can consider the following minimum problem:

$$\inf_{(v,\phi)\in\mathcal{N}} J(v,\phi) := m. \tag{27}$$

Lemma 13. *It holds that $m > 0$.*

Proof. By $(v,\phi)\in\mathcal{N}$, we deduce that

$$\begin{aligned}
J(v,\phi) &= J(v,\phi)-\frac{1}{2}\langle J'(v,\phi),(v,\phi)\rangle\\
&=\frac{1}{4}\int_{\mathbb{R}^2}[|\nabla v|^2+2\sigma v^2+\nu|\nabla\phi|^2-2v^2\sin(2\phi)+q(1-\cos(2\phi))]dx\\
&\quad-\frac{1}{p}\int_{\mathbb{R}^2}|v|^p dx-\frac{1}{4}\int_{\mathbb{R}^2}|\nabla v|^2dx-\frac{\sigma}{2}\int_{\mathbb{R}^2}v^2dx+\frac{1}{2}\int_{\mathbb{R}^2}v^2\sin(2\phi)dx\\
&\quad+\frac{1}{2}\int_{\mathbb{R}^2}|v|^pdx-\frac{\nu}{4}\int_{\mathbb{R}^2}|\nabla\phi|^2dx+\frac{1}{2}\int_{\mathbb{R}^2}v^2\cos(2\phi)\cdot\phi dx\\
&\quad-\frac{q}{4}\int_{\mathbb{R}^2}\sin(2\phi)\cdot\phi dx\\
&=(\frac{1}{2}-\frac{1}{p})\int_{\mathbb{R}^2}|v|^pdx+\frac{q}{4}\int_{\mathbb{R}^2}[(1-\cos(2\phi))-\sin(2\phi)\phi]dx\\
&\quad+\frac{1}{2}\int_{\mathbb{R}^2}v^2\cos(2\phi)\phi dx\\
&\geq(\frac{1}{2}-\frac{1}{p})\int_{\mathbb{R}^2}|v|^pdx+\frac{q}{4}\int_{\mathbb{R}^2}[1-\cos(2\phi)-\sin(2\phi)\phi]dx>0,
\end{aligned}$$

where we have used the following fact: $1-\cos(2x)-x\sin(2x)\geq 0$, for any $x\in[0,\frac{\pi}{4}]$. The proof is completed. $\square$

Next, we use the minimization method to prove that the minimum energy m can be achieved.

Lemma 14. *Let $\sigma>1+\alpha_0$, $p>2$, where $\alpha_0>0$ is a constant; then there exist $(v,\phi)\in H_r^1(\mathbb{R}^2)\times H_r^1(\mathbb{R}^2)$, such that $J(v,\phi)=m$.*

Proof. Set

$$\mathcal{N}_r=\{(v,\phi)\in H_r^1(\mathbb{R}^2)\times H_r^1(\mathbb{R}^2):\ v\neq 0,\ \phi\neq 0,\ \langle J'(v,\phi),(v,\phi)\rangle=0\},$$

and then

$$m=\inf_{(v,\phi)\in\mathcal{N}}J(v,\phi)=\inf_{(v,\phi)\in\mathcal{N}_r}J(v,\phi):=m_1.$$

In fact, since $J(v,\phi)=J(|v|,\phi)$, we may assume that v is non-negative. By Proposition 1, we see that $J(v^*,\phi^*)\leq J(v,\phi)$, which implies that $m_1\geq m$. On the other hand, let $(v^*,\phi^*)\in\mathcal{N}_r$. Then $(v^*,\phi^*)\in\mathcal{N}$, which leads to $m_1\leq m$. Then $m=m_1$. Let $(v_n,\phi_n)\in H_r^1(\mathbb{R}^2)\times H_r^1(\mathbb{R}^2)$ be a minimizing sequence such that $J(v_n,\phi_n)\to m$ as $n\to\infty$.

By calculation, we can get

$$
\begin{aligned}
J(v_n,\phi_n) &- \frac{1}{p}\langle J'(v_n,\phi_n),(v_n,\phi_n)\rangle \\
=&(\frac{1}{4}-\frac{1}{2p})\|\nabla v_n\|_{L^2}^2+(\frac{\sigma}{2}-\frac{\sigma}{p})\|v_n\|_{L^2}^2+(\frac{\nu}{4}-\frac{\nu}{2p})\|\nabla\phi_n\|_{L^2}^2 \\
&-(\frac{1}{2}-\frac{1}{p})\int_{\mathbb{R}^2}v_n^2\sin(2\phi_n)dx+\frac{q}{4}\int_{\mathbb{R}^2}[1-\cos(2\phi_n)]dx \\
&+\frac{1}{p}\int_{\mathbb{R}^2}v_n^2\cos(2\phi_n)\cdot\phi_n dx-\frac{q}{2p}\int_{\mathbb{R}^2}\sin(2\phi_n)\cdot\phi_n dx \\
\geq&(\frac{1}{4}-\frac{1}{2p})\|\nabla v_n\|_{L^2}^2+(\frac{\sigma}{2}-\frac{\sigma}{p})\|v_n\|_{L^2}^2+(\frac{\nu}{4}-\frac{\nu}{2p})\|\nabla\phi_n\|_{L^2}^2 \\
&-(\frac{1}{2}-\frac{1}{p})\int_{\mathbb{R}^2}v_n^2\sin(2\phi_n)dx+\frac{q}{4}\int_{\mathbb{R}^2}[1-\cos(2\phi_n)-\sin(2\phi_n)\cdot\phi_n]dx \\
&+\frac{1}{p}\int_{\mathbb{R}^2}v_n^2\cos(2\phi_n)\cdot\phi_n dx \\
\geq&(\frac{1}{4}-\frac{1}{2p})\|\nabla v_n\|_{L^2}^2+(\frac{\sigma}{2}-\frac{\sigma}{p})\|v_n\|_{L^2}^2+(\frac{\nu}{4}-\frac{\nu}{2p})\|\nabla\phi_n\|_{L^2}^2-(\frac{1}{2}-\frac{1}{p})\|v\|_{L^2}^2 \\
\geq&\min\{\frac{1}{4}-\frac{1}{2p},(\sigma-1)((\frac{1}{2}-\frac{1}{p})\}\|v_n\|_{H^1}^2+(\frac{\nu}{4}-\frac{\nu}{2p})\|\nabla\phi_n\|_{L^2}^2.
\end{aligned}
$$

Since $\sigma>1$ and $p>2$, using Lemma 5, we obtain that $\{(v_n,\phi_n)\}$ is bounded in $H_r^1(\mathbb{R}^2)\times H_r^1(\mathbb{R}^2)$ up to a subsequence still denoted by $\{(v_n,\phi_n)\}$, and we can assume that there exist $v,\phi\in H_r^1(\mathbb{R}^2)$ such that

$$
\begin{cases}
v_n\rightharpoonup v,\phi_n\rightharpoonup\phi, & \text{in } H_r^1(\mathbb{R}^2),\\
v_n\to v,\phi_n\to\phi, & \text{in } L^s(\mathbb{R}^2),(2<s<\infty),\\
v_n(x)\to v(x),\phi_n(x)\to\phi(x), & \text{a.e. in } \mathbb{R}^2,
\end{cases}
\tag{28}
$$

and $v,\phi\geq 0$, i.e. in $\mathbb{R}^2$. We now prove that $(v,\phi)\in\mathcal{N}$, and $J(v,\phi)=m$. By (28) and weak lower semi-continuity of the norm function, we have

$$
\frac{1}{2}\|\nabla v\|_{L^2}^2+\frac{\nu}{2}\|\nabla\phi\|_{L^2}^2\leq\liminf_{n\to\infty}\left(\frac{1}{2}\|\nabla v_n\|_{L^2}^2+\frac{\nu}{2}\|\nabla\phi_n\|_{L^2}^2\right). \tag{29}
$$

By (29), Hölder's inequality and Lebesgue's dominated convergence theorem, we can obtain

$$
\begin{aligned}
\int_{\mathbb{R}^2}&\left|v_n^2\sin(2\phi_n)-v^2\sin(2\phi)\right|dx \\
&=\int_{\mathbb{R}^2}|(v_n^2-v^2)\sin(2\phi_n)+v(\sin(2\phi_n)-\sin(2\phi))|dx \\
&\leq\int_{\mathbb{R}^2}|v_n^2-v^2||2\phi_n|dx+\int_{\mathbb{R}^2}v^2|\sin(2\phi_n)-\sin(2\phi)|dx \\
&\leq(\int_{\mathbb{R}^2}|v_n^2-v^2|^2dx)^{\frac{1}{2}}(\int_{\mathbb{R}^2}|\phi_n|^2dx)^{\frac{1}{2}}+\int_{\mathbb{R}^2}v^2|\sin(2\phi_n)-\sin(2\phi)|dx\to 0.
\end{aligned}
\tag{30}
$$

By Fatou's lemma, we have

$$q\int_{\mathbb{R}^2}\sin(2\phi)\cdot\phi dx \le q\liminf_{n\to\infty}\int_{\mathbb{R}^2}\sin(2\phi_n)\cdot\phi_n dx. \tag{31}$$

Since

$$\begin{aligned}
&\int_{\mathbb{R}^2}|\cos(2\phi_n)\cdot\phi_n-\cos(2\phi)\cdot\phi|^3dx\\
&\quad\le\int_{\mathbb{R}^2}\Big||\phi_n-\phi|\cos(2\phi_n)+\phi|\cos(2\phi_n)-\cos(2\phi)|\Big|^3dx\\
&\quad\le 4\int_{\mathbb{R}^2}|\phi_n-\phi|^3\cos^3(2\phi_n)dx+\int_{\mathbb{R}^2}\phi^3|\cos(2\phi_n)-\cos(2\phi)|^3dx\\
&\quad\le 4\int_{\mathbb{R}^2}|\phi_n-\phi|^3\cos^3(2\phi_n)dx+\int_{\mathbb{R}^2}\phi^3|\phi+\phi_n|^3|\phi-\phi_n|^3dx\\
&\quad\to 0,
\end{aligned} \tag{32}$$

then by (32), we have

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^2}v_n^2\cos(2\phi_n)\cdot\phi_n-v^2\cos(2\phi)\cdot\phi dx\Big|\\
&\quad=\Big|\int_{\mathbb{R}^2}(v_n^2-v^2)\cos(2\phi_n)\cdot\phi_n+v^2\left(\cos(2\phi_n)\cdot\phi_n-\cos(2\phi)\cdot\phi\right)\Big|\\
&\quad\le\left(\int_{\mathbb{R}^2}|v_n^2-v^2|^{\frac{3}{2}}dx\right)^{\frac{2}{3}}\left(\int_{\mathbb{R}^2}|\cos(2\phi_n)\cdot\phi_n|^3dx\right)^{\frac{1}{3}}+\left(\int_{\mathbb{R}^2}|v|^3dx\right)^{\frac{2}{3}}\\
&\quad\quad\times\left(\int_{\mathbb{R}^2}\Big|\cos(2\phi_n)\cdot\phi_n-\cos(2\phi)\cdot\phi\Big|^3dx\right)^{\frac{1}{3}}\to 0.
\end{aligned} \tag{33}$$

Therefore, according to (29), (30), (31), and (33), we can obtain

$$\begin{aligned}
&\frac{1}{2}\|\nabla v\|_{L^2}^2+\sigma\|v\|_{L^2}^2+\frac{\nu}{2}\|\nabla\phi\|_{L^2}^2+\frac{q}{2}\int_{\mathbb{R}^2}\sin(2\phi)\cdot\phi dx\\
&\quad\le\int_{\mathbb{R}^2}v^2\sin(2\phi)dx+\|v\|_{L^p}^p+\int_{\mathbb{R}^2}v^2\cos(2\phi)\cdot\phi dx.
\end{aligned} \tag{34}$$

If the equality in (34) holds, then $(v,\phi)\in\mathcal{N}_r$. So, arguing by contradiction, we assume that

$$\begin{aligned}
&\frac{1}{2}\ \|\nabla v\|_{L^2}^2\ +\sigma\|v\|_{L^2}^2+\frac{\nu}{2}\|\nabla\phi\|_{L^2}^2+\frac{q}{2}\int_{\mathbb{R}^2}\sin(2\phi)\cdot\phi\\
&\quad<\int_{\mathbb{R}^2}v^2\sin(2\phi)+\|v\|_{L^p}^p+\int_{\mathbb{R}^2}v^2\cos(2\phi)\cdot\phi.
\end{aligned}$$

Let $\gamma(t)=\langle J'(v_t,\phi_t),(v_t,\phi_t)\rangle$, where $v_t=tv(t^{-\frac{1}{2}}x)$, $\phi_t=\phi(t^{-\frac{1}{2}}x)$. Clearly, $\gamma(t)\to\frac{\nu}{2}\|\nabla\phi\|_{L^2}^2\ge 0$ for $t\to 0^+$, and $\gamma(1)<0$. Therefore, there exists $t\in(0,1)$, such that $\gamma(t)=0$, i.e., $(v_t,\phi_t)\in\mathcal{N}_r$. Thus, by (29), (30), and Fatou's lemma, we deduce that

$$\begin{aligned}
m \le & J(v_t,\phi_t)=J(v_t,\phi_t)-\frac{1}{2}\langle J'(v_t,\phi_t),(v_t,\phi_t)\rangle\\
=&\frac{qt}{4}\int_{\mathbb{R}^2}(1-\cos(2\phi)-\phi\sin(2\phi))\,dx+(\frac{1}{2}-\frac{1}{p})t^{2p+1}\int_{\mathbb{R}^2}|v|^p dx\\
&+\frac{t^3}{2}\int_{\mathbb{R}^2}|v|^2\cos(2\phi)\phi dx\\
<&\frac{q}{4}\int_{\mathbb{R}^2}(1-\cos(2\phi)-\phi\sin(2\phi))\,dx+(\frac{1}{2}-\frac{1}{p})\int_{\mathbb{R}^2}|v|^p dx+\frac{1}{2}\int_{\mathbb{R}^2}|v|^2\cos(2\phi)\phi dx\\
\le&\liminf_{n\to\infty}\frac{q}{4}\int_{\mathbb{R}^2}(1-\cos(2\phi_n)-\phi_n\sin(2\phi_n))\,dx+\liminf_{n\to\infty}(\frac{1}{2}-\frac{1}{p})\int_{\mathbb{R}^2}|v_n|^p dx\\
&+\liminf_{n\to\infty}\frac{1}{2}\int_{\mathbb{R}^2}v_n^2\cos(2\phi_n)\phi_n dx\\
\le&\liminf_{n\to\infty}\Big[\frac{q}{4}\int_{\mathbb{R}^2}(1-\cos(2\phi_n)-\phi_n\sin(2\phi_n))\,dx+(\frac{1}{2}-\frac{1}{p})\int_{\mathbb{R}^2}|v_n|^p dx\\
&+\frac{1}{2}\int_{\mathbb{R}^2}v_n^2\cos(2\phi_n)\phi_n dx\Big]\\
=&\liminf_{n\to\infty}\Big(J(v_n,\phi_n)-\frac{1}{2}\langle J'(v_n,\phi_n),(v_n,\phi_n)\rangle\Big)=\lim_{n\to\infty}J(v_n,\phi_n)=m,
\end{aligned}$$

which leads to a contradiction. Therefore, the equality in (34) holds and then $(v,\phi)\in\mathcal{N}_r$. By weak lower semi-continuity of the norm function, (29), and (30), we get that

$$J(v,\phi)\le\liminf_{n\to\infty}J(v_n,\phi_n)=m.$$

On the other hand, owing to $(v,\phi)\in\mathcal{N}_r$, we have that $J(v,\phi)\ge m$, and so $J(v,\phi)=m$. This completes the proof. $\square$

Lemma 15. *Let $(v,\phi)\in H_r^1(\mathbb{R}^2)\times H_r^1(\mathbb{R}^2)$ be the minimizer of the minimization problem (27); then there exists $q_0>0$ such that for any $q\ge q_0$, there holds $J'(v,\phi)=0$.*

Proof. Let $G(v,\phi)=\langle J'(v,\phi),(v,\phi)\rangle=0$. Applying the Lagrange multiplier theorem, there exists $\mu\in\mathbb{R}$, such that $J'(v,\phi)=\mu G'(v,\phi)$. Next, we prove $\mu=0$. Indeed, by simple computation, we get

$$\begin{aligned}
\langle G'(v,\phi),(v,\phi)\rangle =& \int_{\mathbb{R}^2}|\nabla v|^2+2\sigma|v|^2dx-2\int_{\mathbb{R}^2}(v^2\sin(2\phi)+2v^2\cos(2\phi)\phi\\
&-v^2\sin(2\phi)\phi^2+\frac{1}{2}v^2\cos(2\phi)\phi)dx+q\int_{\mathbb{R}^2}(\cos(2\phi)\phi^2\\
&+\frac{1}{2}\sin(2\phi)\phi)dx-p\int_{\mathbb{R}^2}|v|^p dx+\nu\int_{\mathbb{R}^2}|\nabla\phi|^2dx\\
=& (2-p)\|v\|_{L^p}^p+\int_{\mathbb{R}^2}(2v^2\sin(2\phi)\phi^2-3v^2\cos(2\phi)\phi)dx\\
&+q\int_{\mathbb{R}^2}(\cos(2\phi)\phi^2-\frac{1}{2}\sin(2\phi)\phi)dx. \qquad (35)
\end{aligned}$$

Observing that $2x^2\cos(2x)-x\sin(2x)\le 0$, for any $x\in[0,\frac{\pi}{4}]$. Thus, there exists $q_0>0$, such that $2\int_{\mathbb{R}^2}(v^2\sin(2\phi)\phi^2-\frac{3}{2}v^2\cos(2\phi)\phi)dx+q\int_{\mathbb{R}^2}(\cos(2\phi)\phi^2-\frac{1}{2}\sin(2\phi)\phi)dx\le 0$, for any $q\ge q_0$. Therefore, by (35), we deduce that

$$\langle G'(v,\phi),(v,\phi)\rangle<0.$$

This implies that $\mu=0$. Thus, we conclude that $J'(v,\phi)=0$. $\square$

Theorem 3. *Let $\sigma>1+\sigma_0$, $p>2$, and there exists $q_0>0$, satisfying $q\ge q_0$, where $\sigma_0>0$ is a constant. Then system* (22) *has a normal ground state solution $(v,\phi)\in H_r^1(\mathbb{R}^2)\times H_r^1(\mathbb{R}^2)$.*

Proof. From Lemma 14, there exists $(v,\phi)\in\mathcal{N}_r$, such that $J(v,\phi)=m$. By Lemma 15, we have $J'(v,\phi)=0$, i.e., (v,ϕ) is the ground solution of system (22), and the proof is completed. $\square$

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