MATHEMATICAL COMMUNICATIONS
Math. Commun. **30**(2025), 39–59

Critical point approaches for doubly eigenvalue discrete boundary value problems driven by ϕ_c-Laplacian operator

Ahmad Ghobadi and Shapour Heidarkhani*

Department of Mathematics, Faculty of Sciences, Razi University, 67 149 Kermanshah, Iran

Received June 10, 2022; accepted April 18, 2024

Abstract. Under appropriate algebraic conditions on the nonlinearity, using variational methods and critical point theory we discuss the existence of one, two and three solutions for nonlinear discrete Dirichlet boundary value problems driven by ϕ_c-Laplacian operator involving two parameters λ and μ, without imposing the symmetry or oscillating behavior at infinity on the the nonlinearity, which has applications in the dynamic model of combustible gases, the capillarity problem in hydrodynamics, and the flux-limited diffusion phenomenon. Some applications and examples illustrate the obtained results.

AMS subject classifications: 39A23, 53A10, 49Q20

Keywords: multiple solutions, ϕ_c-Laplacian boundary value problem, critical point theory, variational methods, critical point theory

1. Introduction

In this paper, we study the following problem:

$$\begin{cases} -\Delta(\phi_c(\Delta u(k-1))) = \lambda f(k,u(k)) + \mu g(k,u(k)), \ k \in [1,N]_{\mathbb{Z}}, \\ u(0) = u(N+1) = 0, \end{cases} \tag{1}$$

where $\phi_c(s) = \frac{s}{\sqrt{1+s^2}}$, $[1,N]_{\mathbb{Z}} = \{1,2,...,N\}$ and $f,g : [1,N]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ are continuous functions. $\Delta u(k-1) = u(k) - u(k-1)$ and λ, μ are two positive parameters.

In the last decades, differential equations involving ϕ_c-Laplacian operator ($\phi_c(s) = \frac{s}{\sqrt{1+s^2}}$ is the mean curvature operator) regarded as a variant of the Liouville-Bratu-Gelfand problem, which is used to study the dynamic model of combustible gases [36, 37], the capillarity problem in hydrodynamics [22, 35], and the flux-limited diffusion phenomenon [29], have been studied by some researchers [3, 2, 4, 5, 15, 34].

On the other hand, difference equations describe evolution of certain phenomena over the route of time. For example, if a certain population has discrete generations, the size of the $k+1$th generation $u(k+1)$ is a function of the kth generation $u(k)$. Namely, since difference equations give a natural description of many discrete models in the real world, in various fields of science and technology such as statistics, computer science, electrical circuit analysis, biology, neural networks, optimal

*Corresponding author. *Email addresses:* ahmad673.1356@gmail.com (A. Ghobadi), sh.heidarkhani@razi.ac.ir (S. Heidarkhani)

https://www.mathos.unios.hr/mc

control, economics, finance, and so on, it is of practical importance to study the existence of solutions of difference equations and discrete boundary value problems. Lately, there have been many studies on discrete boundary value problems by different approaches, for example, see [11, 12, 17, 19, 20, 23, 24, 26, 27, 28, 30, 31, 32]. In [28], by using the three critical point theorems proposed by Bonanno [7], Jiang and Zhou have obtained sufficient conditions for the existence of at least three solutions of discrete p-Laplacian Dirichlet boundary value problems. In [12, 11], Bonanno and Candito have discussed the existence of multiple solutions for a class of discrete nonlinear boundary value problems under rather different assumptions via variational methods and critical point theory. In [17], by using critical point theory, the authors have studied the existence of at least three solutions for a perturbed nonlinear Dirichlet boundary value problem for difference equations depending on two positive parameters.

To the best of our knowledge, there are a few works in the literature studying the existence of solutions for boundary value problems involving ϕ_c-Laplacian, and we have just found [19, 38], in which the authors have discussed the existence of multiple solutions for problem (1) in the case $\mu = 0$, and the existence of infinitely many solutions for problem (1) in the case $\mu = 0$. We refer to [21], in which a partial discrete Dirichlet boundary value problem involving the mean curvature operator was studied, and under proper assumptions on the nonlinear term, some feasible conditions on the existence of multiple solutions by the method of critical point theory were obtained. Also, open intervals of the parameter to attain at least two positive solutions and an unbounded sequence of positive solutions with the help of the maximum principle were separately determined.

We refer to [25], where the author has studied the characterization of entire solutions of some system of Fermat type functional equations and also posed an open problem. The author has provided a nice discussion and presentation of the mathematical background to underline the relevance of the topic. The author has distinguished various situations and supported the finding with illustrative examples. We also refer to paper [6], where using critical point theory and variational methods, the existence of at least three solutions for a class of double eigenvalue discrete anisotropic Kirchhoff-type problems was discussed. Further, the effects of Kirchhoff weight on the principal operator were considered. We note the fact that establishing the Kirchhoff counterpart of existing models is a very actual topic of research. In [33], the authors have established the existence and multiplicity of non-zero homoclinic solutions to a nonlinear Laplacian difference equation without using Ambrosetti-Rabinowitz type-conditions. Similarly to our present paper, the main approach is based on the mountain pass theorem and the Palais-Smale compactness condition involving suitable functionals.

Inspired by the above results, in this the article, we investigate the existence of one, two and three solutions for problem (1). In these cases, we apply suitable conditions on the nonlinear terms and create openings for two parameters λ and μ in problem (1), without imposing the symmetry or oscillating behavior at infinity on the nonlinear terms f and g. We also give examples to show the use of the proven theorems. At the end, we discuss the existence of the solutions for the problem in the case when two parameters are the same. Precisely, we discuss the regularity properties of energy functionals associated to the main problem.

2. Preliminaries and basic notation

In this section, we introduce the tools that are necessary for our main results in the next section. Set

$$X = \{u : [0, N+1]_{\mathbb{Z}} \to \mathbb{R} | u(0) = u(N+1) = 0\}. \tag{2}$$

X is an N-dimensional Banach space with the following norm:

$$\|u\| := \Big(\sum_{k=1}^{N+1} |\Delta u(k-1)|^2 \Big)^{\frac{1}{2}}. \tag{3}$$

Let $\|u\|_\infty := \max\{u(k) : k \in [1, N]_{\mathbb{Z}}\}$. We see that $\|u\|_\infty$ is another norm in X. From Lemma 2.2 of [28], we have

Lemma 1. *For any $u \in X$, the following relation holds:*

$$\|u\|_\infty \leq \frac{\sqrt{N+1}}{2} \|u\|. \tag{4}$$

From (2.1) and (2.3) in [13], we have the following lemma.

Lemma 2. *For any $u \in X$, one has*

$$\frac{1}{\sqrt{N\lambda_N}} \|u\| \leq \|u\|_\infty \leq \frac{1}{\sqrt{\lambda_1}} \|u\|,$$

where $\lambda_1 = 4\sin^2 \frac{\pi}{2(N+1)}$ *and* $\lambda_N = 4\sin^2 \frac{N\pi}{2(N+1)}$.

We define the functionals $\Phi,\ \Psi : X \to \mathbb{R}$ in the following way:

$$\Phi(u) = \sum_{k=1}^{N+1} \Big(\sqrt{1 + (\Delta u(k-1))^2} - 1 \Big),\ \Psi(u) = \sum_{k=1}^{N} \Big(F(k, u(k)) + \frac{\mu}{\lambda} G(k, u(k)) \Big). \tag{5}$$

Corresponding to the functions f and g, we introduce the functions $F, G : [1, N]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$, respectively, as follows:

$$F(k, t) = \int_0^t f(k, \xi) d\xi, t \in \mathbb{R} \text{ and } G(k, t) = \int_0^t g(k, \xi) d\xi, t \in \mathbb{R}.$$

For $\gamma > 0$ and $\eta > 0$, set

$$G^\gamma := \sum_{k=1}^{N} \max_{|t| < \gamma} G(k, \xi) \ \text{ and } \ G_\eta := \sum_{k=1}^{N} \inf_{\xi \in [0, \eta]} G(k, \xi).$$

If g is sign-changing, then $G^\gamma \leq 0$ and $G_\eta \geq 0$. We say that a function $u \in X$ is a solution of problem (1) if

$$-\sum_{k=1}^{N} \Delta(\phi_c(\Delta u(k-1))) v(k) - \sum_{k=1}^{N} \Big(\lambda f(k, u(k)) v(k) + \mu g(k, u(k)) v(k) \Big) = 0$$

holds for all $v \in X$.

Definition 1. *Assume that X is a real reflexive Banach space. We say I satisfies the Palais-Smale condition (denoted as PS-condition for short) if any sequence $\{u_k\} \subset X$, for which $\{I(u_k)\}$ is bounded and $I'(u_k) \to 0$ as $k \to 0$, possesses a convergent subsequence.*

If $r_1, r_2 \in [-\infty, +\infty]$ such that $r_1 < r_2$ and $I = \Phi + \lambda\Psi$, where $\Phi, \Psi : X \to \mathbb{R}$ are two continuously Gâteaux differentiable functions, so that each sequence $\{u_n\}$ that has the following conditions:

(i_1) $\{I(u_n)\}$ is bounded;

(i_2) $\lim_{n\to\infty} \|I'(u_n)\|_{X^*} = 0$;

(i_3) $r_1 < \Phi(u_n) < r_2$,

possesses a convergent subsequence. At a time, the functional J applies to the Palais-Smale condition cut off lower at r_1 and upper at r_2 ($^{[r_1]}(PS)^{[r_2]}$-condition). Likewise we can make $(PS)^{[r_2]}$ in which $r_1 = -\infty$ and $r_2 \in \mathbb{R}$ and $^{[r_1]}(PS)$ in which $r_1 \in \mathbb{R}$ and $r_2 = \infty$. By definition (1) of the (PS)-condition, it is established that in $^{[r_1]}(PS)^{[r_2]}$-condition, $r_1 = -\infty$ and $r_2 = \infty$. Indeed, let Φ and Ψ be two continuously Gâteaux differentiable functionals defined on a real Banach space X; fix $r \in \mathbb{R}$. The functional $I = \Phi - \lambda\Psi$ is said to verify the Palais-Smale condition cut off upper at r (in short $(PS)^{[r]}$) if any sequence $\{u_n\}_{n\in\mathbb{N}}$ in X such that

(j_1) $\{I(u_n)\}$ is bounded;

(j_2) $\lim_{n\to\infty} \|I'(u_n)\|_{X^*} = 0$;

(j_3) $\Phi(u_n) < r$ for each $n \in \mathbb{N}$,

has a convergent subsequence.

The proofs of our theorems are based on the following four theorems.

Theorem 1 ([10, Theorem 2.3])**.** *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functions such that $\inf_{u\in X} \Phi(u) = \Phi(0) = \Psi(0) = 0$. Assume that there exist $r > 0$ and $\bar{u} \in X$, with $0 < \Phi(\bar{u}) < r$, such that:*

(k_1) $\frac{\sup_{\Phi(u)\le r} \Psi(u)}{r} < \frac{\Psi(\bar{u})}{\Phi(\bar{u})}$,

(k_2) *for each $\lambda \in \left(\frac{\Phi(\bar{u})}{\Psi(\bar{u})}, \frac{r}{\sup_{\Phi(u)\le r} \Psi(u)}\right)$, the functional $I_\lambda := \Phi - \lambda\Psi$ satisfies the $(PS)^{[r]}$-condition.*

Then, for each $\lambda \in \Lambda := \left(\frac{\Phi(\bar{u})}{\Psi(\bar{u})}, \frac{r}{\sup_{\Phi(u)\le r} \Psi(u)}\right)$, there exists $u_{0,\lambda} \in \Phi^{-1}(0, r)$ such that $I_\lambda(u_{0,\lambda}) \equiv \vartheta_{X^*}$ and $I_\lambda(u_{0,\lambda}) \le I_\lambda(u)$ for all $u \in \Phi^{-1}(0, r)$.

Theorem 2 ([10, Theorem 3.2])**.** *Let X be a real Banach space, $\Phi, \Psi : X \to \mathbb{R}$ two continuously Gâteaux differentiable functionals such that Φ is bounded from below and $\Phi(0) = \Psi(0) = 0$. Fix $r > 0$ and assume that for each*

$$\lambda \in \left(0, \frac{r}{\sup_{u\in\Phi^{-1}(-\infty,r)} \Psi(u)}\right),$$

the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies (PS)-condition and is unbounded from below. Then, for each

$$\lambda \in \Big(0, \frac{r}{\sup_{u\in\Phi^{-1}(-\infty,r)} \Psi(u)}\Big),$$

the functional I_λ admits two distinct critical points.

Theorem 3 ([1, Theorem A])**.** *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^* and let $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that:*

(m_1) $\lim_{\|u\|\to\infty}(\Phi(u) + \lambda\Psi(u)) = \infty$, *for all $\lambda \in [0, \infty)$;*

(m_2) *there is $r \in \mathbb{R}$ such that $\inf_X \Phi < r$ and $\varphi_1(r) < \varphi_2(r)$, where*

$$\varphi_1(r) = \inf_{u\in\Phi^{-1}]-\infty,\bar{r}[} \frac{\Psi(u) - \inf_{\overline{\Phi^{-1}(-\infty,r)}^{\omega}} \Psi}{r - \Phi(u)},$$

$$\varphi_2(\bar{r}) = \inf_{u\in\Phi^{-1}(-\infty,r)} \sup_{v\in\Phi^{-1}[r,\infty)} \frac{\Psi(u) - \Psi(v)}{\Phi(v) - \Phi(u)},$$

and $\overline{\Phi^{-1}(-\infty, r)}^{\omega}$ is the closure of $\Phi^{-1}(-\infty, r)$ in the weak topology. Then, for each $\lambda \in (\frac{1}{\varphi_2(r)}, \frac{1}{\varphi_1(r)})$ the functional $\Phi + \lambda\Psi$ has at least three critical points in X.

Theorem 4 ([8, Theorem 1.1])**.** *Let X be a reflexive real Banach space, and let $\Phi,\ \Psi : X \to \mathbb{R}$ be two sequentially weakly lower semicontinuous and Gâteaux differentiable functions. Assume that Φ is (strongly) continuous and satisfies $\lim_{\|u\|\to\infty} \Phi(u) = \infty$. Also, suppose there exist two constants r_1 and r_2 such that*

(n_1) $\inf_X \Phi < r_1 < r_2$;

(n_2) $\varphi_1(r_1) < \varphi_2^*(r_1, r_2)$;

(n_3) $\varphi_1(r_2) < \varphi_2^*(r_1, r_2)$, *where φ_1 is defined as in Theorem 3 and*

$$\varphi_2^*(r_1, r_2) = \inf_{u\in\Phi^{-1}(-\infty,r)} \sup_{v\in\Phi^{-1}[r_1,r_2]} \frac{\Psi(v) - \Psi(u)}{\Phi(v) - \Phi(u)}.$$

Then, for each $\lambda \in (\frac{1}{\varphi_2^*(r_1,r_2)}, \min\{\frac{1}{\varphi_1(r_1)}, \frac{1}{\varphi_1(r_2)}\})$, the functional $\Phi + \lambda\Psi$ admits at least two critical points which lie in $\Phi^{-1}(-\infty, r_1]$ and $\Phi^{-1}[r_1, r_2)$, respectively.

For situations of successful employment of the results such as Theorems 1-4 in order to prove the existence of solutions for various boundary value problems, we refer the reader to [14, 16, 18].

3. Main results

In this section, we present the existence results.

Theorem 5. *Assume that there exist two positive constants* $\gamma > \frac{(N+1)\sqrt{N+1}}{2}$ *and* η *with the property*

$$2\sqrt{1+\eta^2}-1 < \frac{2}{\sqrt{N+1}}\gamma - N,$$

and assume that

(A_1) $\frac{\sum_{k=1}^{N}\sup_{|t|\leq\gamma}F(k,t)}{\frac{2}{\sqrt{N+1}}\gamma-N-1} < \frac{\sum_{k=1}^{N}F(k,\eta)}{2(\sqrt{1+\eta^2}-1)}$;

(A_2) $\limsup_{|\xi|\to\infty}\frac{\sup_{k\in[1,N]_{\mathbb{Z}}}F(k,\xi)}{|\xi|} < \infty.$

Then, for every

$$\lambda \in \Lambda := \Big(\frac{2(\sqrt{1+\eta^2}-1)}{\sum_{k=1}^{N}F(k,\eta)}, \frac{\frac{2}{\sqrt{N+1}}\gamma - N - 1}{\sum_{k=1}^{N}\sup_{|t|\leq\gamma}F(k,t)}\Big)$$

and for every continuous function $g : [1,N]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ *satisfying the following condition:*

$$\limsup_{|t|\to\infty}\frac{\sup_{k\in[1,N]_{\mathbb{Z}}}G(k,t)}{|t|} < \infty, \tag{6}$$

there exists δ_λ *given by*

$$\min\Big\{\frac{2(\sqrt{1+\eta^2}-1)-\lambda\sum_{k=1}^{N}F(k,\eta)}{G_\eta}, \frac{\frac{2}{\sqrt{N+1}}\gamma-N-1-\lambda\sum_{k=1}^{N}\sup_{|t|\leq\gamma}F(k,t)}{G^\gamma}\Big\} \tag{7}$$

such that for each $\mu \in [0,\delta_\lambda)$*, problem* (1) *admits at least one solution* u_λ *in* X *such that* $\max_{t\in[0,1]}|u_\lambda(t)| < \gamma$.

Proof. We want to apply Theorem 1 with regard to the space X with the norm defined in (3), and the functionals Φ and Ψ defined as in (5). According to the definition of Φ, we realize that Φ is Gâteaux differentiable and sequentially weakly lower semicontinuous, its Gâteaux derivative is the functional $\Phi'(u) \in X^*$, that is as follows:

$$\Phi'(u)(v) = -\sum_{k=1}^{N}\Delta(\phi_c(\Delta u(k-1)))v(k)$$

and

$$\Psi'(u)(v) = \sum_{k=1}^{N}\Big(f(k,u(k))v(k) + \frac{\mu}{\lambda}g(k,u(k))v(k)\Big),$$

for every $v \in X$. An easy computation ensures the regularity assumptions required on Φ. Moreover, the functional Ψ is in $C^1(X, \mathbb{R})$ and Ψ has a compact derivative. By the definition of Φ, one has

$$\begin{aligned}\Phi(u) &= \sum_{k=1}^{N+1} \sqrt{1 + (\Delta u(k-1))^2} - \sum_{k=1}^{N+1} 1 \\ &\geq \Big(\sum_{k=1}^{N+1} (1 + (\Delta u(k-1))^2) \Big)^{\frac{1}{2}} - \sum_{k=1}^{N+1} 1 \\ &\geq \|u\| - N - 1. \end{aligned} \tag{8}$$

Put $r = \frac{2}{\sqrt{N+1}}\gamma - N - 1$ and

$$\omega(k) = \begin{cases} \eta, & k \in [1, N]_{\mathbb{Z}}, \\ 0, & k = 0 = N+1. \end{cases}$$

Clearly, $w \in X$. Hence, we have

$$\Phi(\omega) = 2(\sqrt{1+\eta^2} - 1).$$

Thus, by assumption (12), we get $0 < \Phi(\omega) < r$. Moreover, by (4) and (8), we have

$$\begin{aligned}|u(t)| \leq \|u\|_\infty \leq \frac{\sqrt{N+1}}{2}\|u\| &\leq \frac{\sqrt{N+1}}{2}(\Phi(u) + N + 1) \\ &\leq \frac{\sqrt{N+1}}{2}(r + N + 1) = \gamma, \ \ \forall \ k \in [1, N]_{\mathbb{Z}}.\end{aligned}$$

Consequently,

$$\Phi^{-1}(-\infty, r] = \{u \in X; \Phi(u) \leq r\} \subseteq \{u \in X; |u(t)| \leq \gamma\}.$$

Therefore, one has

$$\sup_{u \in \Phi^{-1}(-\infty, r)} \sum_{k=1}^{N} F(k, u(k)) \leq \sup_{|t| \leq \gamma} \sum_{k=1}^{N} F(k, t),$$

and this in conjunction with the second inequality in (6) ensures

$$\begin{aligned}\sup_{u \in \Phi^{-1}(-\infty, r)} \sum_{k=1}^{N} \Big(F(k, u(k)) + \frac{\mu}{\lambda} G(k, u(k)) \Big) &\leq \sup_{u \in \Phi^{-1}(-\infty, r)} \sum_{k=1}^{N} F(k, u(k)) + \frac{\mu}{\lambda} G^\gamma \\ &\leq \sum_{k=1}^{N} \sup_{|t| \leq \gamma} F(k, t) + \frac{\mu}{\lambda} G^\gamma,\end{aligned}$$

for every $u \in X$ such that $\Phi(u) < r$. Thus,

$$\sup_{\Phi(u) \leq r} \Psi(u) \leq \sum_{k=1}^{N} \sup_{|t| \leq \gamma} F(k, t) + \frac{\mu}{\lambda} G^\gamma.$$

On the other hand, we have

$$\Psi(\omega) = \sum_{k=1}^{N} \Big(F(k,\omega(k)) + \frac{\mu}{\lambda} G(k,\omega(k)) \Big) \geq \sum_{k=1}^{N} F(k,\omega(k)) + \frac{\mu}{\lambda} G_\eta.$$

Therefore,

$$\begin{aligned} \frac{\sup_{u\in\Phi^{-1}(-\infty,r]} \Psi(u)}{r} &= \frac{\sup_{u\in\Phi^{-1}(-\infty,r]} \sum_{k=1}^{N} \Big(F(k,u(k)) + \frac{\mu}{\lambda} G(k,u(k)) \Big)}{r} \\ &\leq \frac{\sum_{k=1}^{N} \sup_{|t|\leq\gamma} F(k,t) + \frac{\mu}{\lambda} G^\gamma}{\frac{2}{\sqrt{N+1}}\gamma - N - 1}, \end{aligned} \tag{9}$$

and

$$\frac{\Psi(\omega)}{\Phi(\omega)} \geq \frac{\sum_{k=1}^{N} \Big(F(k,\eta) + \frac{\mu}{\lambda} G(k,\eta) \Big)}{2(\sqrt{1+\eta^2}-1)} \geq \frac{\sum_{k=1}^{N} F(k,\eta) + \frac{\mu}{\lambda} G_\eta}{2(\sqrt{1+\eta^2}-1)}.$$

Since

$$\mu < \frac{\frac{2}{\sqrt{N+1}}\gamma - N - 1 - \lambda \sum_{k=1}^{N} \sup_{|t|\leq\gamma} F(k,t)}{G^\gamma},$$

this means

$$\frac{\sum_{k=1}^{N} \sup_{|t|\leq\gamma} F(k,t) + \frac{\mu}{\lambda} G^\gamma}{\frac{2}{\sqrt{N+1}}\gamma - N - 1} < \frac{1}{\lambda}.$$

Furthermore,

$$\mu < \frac{2(\sqrt{1+\eta^2}-1) - \lambda \sum_{k=1}^{N} F(k,\eta)}{G_\eta},$$

this means

$$\frac{\sum_{k=1}^{N} F(k,\eta) + \frac{\mu}{\lambda} G_\eta}{2(\sqrt{1+\eta^2}-1)} > \frac{1}{\lambda}.$$

Then,

$$\frac{\sum_{k=1}^{N} \sup_{|t|\leq\gamma} F(k,t) + \frac{\mu}{\lambda} G^\gamma}{\frac{2}{\sqrt{N+1}}\gamma - N - 1} < \frac{1}{\lambda} < \frac{\sum_{k=1}^{N} F(k,\eta) + \frac{\mu}{\lambda} G_\eta}{2(\sqrt{1+\eta^2}-1)}. \tag{10}$$

Hence, from (9) to (10), the condition (k_1) of Theorem 1 is fulfilled. Finally, for $\lambda > 0$, we will show that the functional $I_\lambda = \Phi - \lambda\Psi$ is coercive. Since $\mu < \delta_k$ and by (6), we can fix $\alpha > 0$ such that $\alpha\mu < \frac{2}{N\sqrt{N+1}}$, and there exists $\rho \in \mathbb{R}$ such that $G(k,t) \leq \alpha|t| + \rho$, for every $(k,t) \in [1,N]_{\mathbb{Z}} \times \mathbb{R}$. Now, we fix $\varepsilon < \frac{2}{\lambda N\sqrt{N+1}} - \frac{\alpha\mu}{\lambda}$.

From the assumption (A_2) there exists $h \in \mathbb{R}$ such that $F(k,t) \le \varepsilon|t| + h$ for every $(k,t) \in [1,N]_{\mathbb{Z}} \times \mathbb{R}$. It follows that for each $u \in X$,

$$\begin{aligned}
\Phi(u) - \lambda\Psi(u) &= \sum_{k=1}^{N+1}\Big(\sqrt{1+(\Delta u(k-1))^2}-1\Big) - \lambda\sum_{k=1}^{N}\Big(F(k,u(k)) + \frac{\mu}{\lambda}G(k,u(k))\Big) \\
&\ge \Big(\sum_{k=1}^{N+1}[1+(\Delta u(k-1))^2]\Big)^{\frac{1}{2}} - \sum_{k=1}^{N+1} 1 - \lambda\varepsilon\sum_{k=1}^{N}|u(k)| - \lambda h \\
&\quad - \alpha\mu\sum_{k=1}^{N}|u(k)| - \mu\rho \\
&\ge \|u\| - \lambda\frac{\varepsilon N\sqrt{N+1}}{2}\|u\| - \mu\frac{\alpha N\sqrt{N+1}}{2}\|u\| - N(1+\lambda h+\mu\rho) - 1 \\
&\ge \Big(1 - \lambda\frac{\varepsilon N\sqrt{N+1}}{2} - \mu\frac{\alpha N\sqrt{N+1}}{2}\Big)\|u\| - N(1+\lambda h+\mu\rho) - 1,
\end{aligned}$$

so

$$\lim_{\|u\|_X\to\infty}(\Phi(u) - \lambda\Psi(u)) = \infty,$$

which means the functional $I_\lambda = \Phi(u) - \lambda\Psi(u)$ is coercive. Therefore, by [9, Proposition 2.1], the functional $I_\lambda = \Phi(u) - \lambda\Psi(u)$ verifies the $(PS)^{[r]}$-condition for each $r > 0$, so the condition (k_2) of Theorem 1 is fulfilled. From (9)-(10), one also has

$$\lambda \in \Big(\frac{\Phi(\omega)}{\Psi(\omega)}, \frac{r}{\sup_{\Phi(u)\le r}\Psi(u)}\Big).$$

Theorem 1 with $\overline{u} = \omega$ guarantees the existence of a local minimum point u_λ for the functional I_λ such that $0 < \Phi(u_\lambda) < r$ and so u_λ is a nontrivial weak solution of problem (1) such that $\max_{k\in[1,N]_{\mathbb{Z}}}|u_\lambda(k)| < \gamma$. □

Now we give an example to show the use of Theorem 5.

Example 1. *Consider the problem*

$$\begin{cases} -\Delta\Big(\frac{\Delta u(k-1)}{\sqrt{1+(\Delta u(k-1))^2}}\Big) = \lambda f(k,u(k)) + \mu g(k,u(k)), \ k \in [1,3]_{\mathbb{Z}}, \\ u(0) = u(N+1) = 0, \end{cases} \tag{11}$$

where $g(k,t) = 2k$ *for all* $(k,t) \in [1,3]_{\mathbb{Z}} \times \mathbb{R}$, *we have* $G(k,t) = 2kt + 1$ *for all* $(k,t) \in [1,3]_{\mathbb{Z}} \times \mathbb{R}$. *We see that* $\limsup_{|\xi|\to\infty}\frac{\sup_{k\in[1,3]_{\mathbb{Z}}}G(k,\xi)}{|\xi|} < \infty$, *namely,* (6) *holds. Letting*

$$f(k,t) = \begin{cases} 3k, & t \le 1, \\ \frac{3k}{t}, & t > 1, \end{cases}$$

for every $k \in [1,3]_{\mathbb{Z}}$, *we have*

$$F(k,t) = \begin{cases} 3kt, & t \le 1, \\ 3k(\ln(t)+1), & t > 1, \end{cases}$$

for every $k \in [1,3]_{\mathbb{Z}}$. *Hence,* $\limsup_{|\xi|\to\infty} \frac{\sup_{k\in[1,3]_{\mathbb{Z}}} F(k,\xi)}{|\xi|} < \infty$, *the condition* (A_2) *holds. Taking* $\gamma = 5$ *and* $\eta = 10^{-6}$, *then* $2\sqrt{1+10^{-12}} - 1 = 2\sqrt{1+\eta^2} - 1 < \frac{2}{\sqrt{N+1}}\gamma - N = 2$, *and also*

$$\frac{\sum_{k=1}^{N} \sup_{|t|\le\gamma} F(k,t)}{\frac{2}{\sqrt{N+1}}\gamma - N - 1} = 18 \times \ln 6 < \frac{18 \times 10^{-6}}{2(\sqrt{1+10^{-12}} - 1)} = \frac{\sum_{k=1}^{N} F(k,\eta)}{2(\sqrt{1+\eta^2} - 1)}.$$

Therefore, the condition (A_1) *holds. So, all conditions of Theorem* (5) *are satisfied. Consequently, it follows that for each* $\lambda \in (\frac{2(\sqrt{1+10^{-12}}-1)}{18\times 10^{-6}}, \frac{1}{18\ln 6})$ *and for every*

$$0 \le \mu < \min\Big\{\frac{2\times(\sqrt{1+10^{-12}}-1) - 18\times 10^{-6}\lambda}{12\times 10^{-6}+3}, \frac{1 - 18\ln 6 \times \lambda}{63}\Big\}$$

problem (11) *admits at least one weak solution in* X.

Now, we want to discuss the existence of at least two solutions for problem (1).

Theorem 6. *Assume that there exist two positive constants* $\gamma > \frac{(N+1)\sqrt{N+1}}{2}$ *and* η *with the property*

$$2\sqrt{1+\eta^2} - 1 < \frac{1}{\sqrt{N+1}}\gamma - N, \tag{12}$$

and assume that

(A_3) *there exist* $\nu > 1$ *and* $T > 0$ *such that*

$$0 < \nu F(k,\xi) < \xi f(k,\xi)$$

for all $|\xi| > T$ *and* $k \in [1,N]_{\mathbb{Z}}$.

Then, for each

$$\lambda \in \Big(0, \frac{\frac{2}{\sqrt{N+1}}\gamma - N - 1}{\sum_{k=1}^{N} \sup_{|t|\le\gamma} F(k,t)}\Big),$$

and for every continuous function $g : [1,N]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ *satisfying the following condition:*

(A_4) *there exist* $\nu \ge \nu' > 1$ *and* $T' > 0$ *such that*

$$0 < \nu' G(k,\xi) < \xi g(k,\xi)$$

for all $|\xi| > T'$ *and* $k \in [1,N]_{\mathbb{Z}}$,

and for each μ *in* (7), *problem* (1) *admits at least two solutions* u_1 *and* u_2 *in* X *such that* $\max_{k\in[1,N]_{\mathbb{Z}}} |u_1(k)| < \gamma$.

Proof. We want to apply Theorem 2. Recalling the functionals Φ and Ψ defined in Theorem 2, we first examine the (PS)-condition for the functional I_λ. To prove this, let $\{u_n\}$ be a sequence in X such that $\{I_\lambda(u_n)\}$ is bounded and $I'_\lambda(u_n) \to 0$ as $n \to +\infty$. Then, there exists a positive constant c_0 such that $|I_\lambda(u_n)| \le c_0$ and $|I'_\lambda(u_n)| \le c_0$ for all $n \in \mathbb{N}$. Therefore, by assumptions (A_3), (A_4), and the definition of I'_λ, we have

$$\begin{aligned} c_0 + c_1\|u_n\| &\ge \nu I_\lambda(u_n) - I'_\lambda(u_n)(u_n) \\ &\ge (\nu-1)\|u_n\| - N - 1 + \lambda\sum_{k=1}^{N}\Big(f(k,u_n(k))u_n(k) - \nu F(k,u_n(k))\Big) \\ &\quad + \mu\sum_{k=1}^{N}\Big(g(k,u_n(k))u_n(k) - \nu G(k,u_n(k))\Big) \\ &\ge (\nu-1)\|u_n\| - N - 1, \end{aligned}$$

for some $c_1 > 0$. Since $\nu \ge \nu' > 1$, this implies that $\{u_n\}$ is bounded. Next, we prove that there exists $u \in X$ such that $u_n \to u$ in X, as $n \to \infty$. Since X is a Banach space, there exist a subsequence, still denoted by $\{u_n\}$, and a function u in X such that $u_n \rightharpoonup u,\ in\ X$. We have

$$\sum_{k=1}^{N}\Big(\sqrt{1+(\Delta u_n(k-1))^2} - 1 - (\sqrt{1+(\Delta u(k-1))^2} - 1)\Big)dt \to 0,\ \ n \to \infty. \quad (13)$$

From the continuity of f and g we get

$$\lambda\sum_{k=1}^{N}(f(k,u_n(k)) - f(k,u(k)))(u_n(k) - u(k)) \to 0,\ \ n \to \infty, \quad (14)$$

and

$$\mu\sum_{k=1}^{N}(g(k,u_n(k)) - g(k,u(k)))(u_n(k) - u(k)) \to 0,\ \ n \to \infty. \quad (15)$$

On the other hand, we have

$$\begin{aligned} < I'_\lambda(u_n) - I'_\lambda(u), u_n - u > &= -\sum_{k=1}^{N+1}\Delta(\phi_c(\Delta u_n(k-1)))(u_n(k) - u(k)) \\ &\quad - \lambda\sum_{k=1}^{N}\Big(f(k,u_n(k))(u_n(k) - u(k))\Big) \\ &\quad - \mu\sum_{k=1}^{N}\Big(g(k,u_n(k))(u_n(k) - u(k))\Big) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=1}^{N+1}\Delta(\phi_c(\Delta u(k-1)))(u_n(k)-u(k))\\
&+\lambda\sum_{k=1}^{N}\Big(\lambda f(k,u(k))(u_n(k)-u(k))\Big)\\
&+\mu\sum_{k=1}^{N}\Big(g(k,u(k))(u_n(k)-u(k))\Big)\\
\geq\ &\|u_n-u\|\\
&-\sum_{k=1}^{N}\Big(\lambda(f(k,u_n(k))-f(k,u(k)))(u_n(k)-u(k))\\
&+\mu(g(k,u_n(k))-g(k,u(k)))(u_n(k)-u(k))\Big). \qquad (16)
\end{aligned}
$$

It is easy to see that

$$< I'_\lambda(u_n)-I'_\lambda(u), u_n-u > \to 0.$$

From (13)-(16), we have

$$\|u_n-u\|\to 0.$$

Consequently, the sequence u_n converges strongly to u in X. Therefore, I_λ satisfies the (PS)-condition. From the conditions A_3 and A_4, there exist constants $a_1, a_2, b_1, b_2 > 0$ such that

$$F(k,t)\geq a_1|t|^\nu - a_2 \qquad (17)$$

for all $k\in[1,N]_{\mathbb{Z}}$ and $t\in\mathbb{R}$.

$$G(k,t)\geq b_1|t|^{\nu'} - b_2 \qquad (18)$$

for all $k\in[1,N]_{\mathbb{Z}}$ and $t\in\mathbb{R}$. Now, choosing any $u\in X\backslash\{0\}$, using (17) and (18) for each $\tau>0$, one has

$$
\begin{aligned}
I_\lambda(\tau u) &\leq \|\tau u\| - N - 1 - \lambda\sum_{k=1}^{N}F(k,\tau u(k)) - \mu\sum_{k=1}^{N}G(k,\tau u(k))\\
&\leq \tau\|u\| - \lambda\tau^\nu\sum_{k=1}^{N}a_1|u(k)|^\nu + \lambda a_2 - \mu\tau^{\nu'}\sum_{k=1}^{N}b_1|u(k)|^{\nu'} + \mu b_2 - N - 1.
\end{aligned}
$$

Since $\nu>2$, this indicates that the functional I_λ is unbounded from below. Thus, all hypotheses of Theorem 2 are verified. Therefore, for each $\lambda\in\left(0, \frac{\frac{2}{\sqrt{N+1}}\gamma-N-1}{\sum_{k=1}^{N}\sup_{|t|\leq\gamma}F(k,t)}\right)$ the functional I_λ admits two critical points that are weak solutions of problem (1). □

Remark 1. *In Theorem 1 we observe that if $f(k,0) \neq 0$, then Theorem 7 ensures the existence of two nontrivial solutions for problem* (1)*. If the condition $f(k,0) \neq 0$ for all $k \in [1,N]_{\mathbb{Z}}$ does not hold, the second solution u_2 of problem* (1) *may be trivial, but the problem has at least a nontrivial solution.*

Now, we want to present an example to illustrate the application of Theorem 6.

Example 2. *Let $N = 20$. Put $g(k,t) = 3t^2 + t + k$ for all $(k,t) \in [1,20]_{\mathbb{Z}} \times \mathbb{R}$, so $G(k,t) = t^3 + t^2 + kt$ for all $(k,t) \in [1,20]_{\mathbb{Z}} \times \mathbb{R}$, hence $\lim_{\xi\to+\infty} \frac{\xi g(k,\xi)}{G(k,\xi)} = 3$, by choosing $\nu' = 3$ and $T' = 1$, condition (A_4) holds. Letting*

$$f(k,t) = \begin{cases} t^3 + \cos \pi t + 4, & t \leq 1, \\ 5t^4 - 1, & t > 1, \end{cases}$$

for all $k \in [1,20]_{\mathbb{Z}}$, we have

$$F(k,t) = \begin{cases} \frac{1}{4}t^4 + \frac{1}{\pi}\sin \pi t + 4t, & t \leq 1, \\ t^5 + \frac{13}{4}, & t > 1. \end{cases}$$

Hence, $\lim_{\xi\to+\infty} \frac{\xi f(k,\xi)}{F(k,\xi)} = 5 < \infty$ and $\lim_{\xi\to-\infty} \frac{\xi f(k,\xi)}{F(k,\xi)} = 4 < \infty$, thus by choosing $\nu = 5 > 2$ and $T = 1$, condition (A_3) is satisfied, also $\nu > \nu'$. Taking $\gamma = 100$ and $\eta = 1$ such that $2\sqrt{1+\eta^2} - 1 = 2\sqrt{2} - 1 < \frac{2}{\sqrt{21}} \times 100 - 21 = \frac{2}{\sqrt{N+1}}\gamma - N - 1$, we clearly see that all assumptions of Theorem 6 are fulfilled. Therefore, it follows that for each $\lambda \in \left(0, \frac{\frac{2}{\sqrt{21}} \times 100 - 21}{21(100^5 + \frac{13}{4})}\right)$ the following problem:

$$\begin{cases} -\Delta\left(\frac{\Delta u(k-1)}{\sqrt{1+(\Delta u(k-1))^2}}\right) = \lambda f(k,u(k)) + \mu(3t^2 + t + k), \ k \in [1,20]_{\mathbb{Z}} \\ u(0) = u(21) = 0, \end{cases}$$

has at least two nontrivial solutions.

Now, we consider $\lambda = \mu$, and in this case we want to discuss the existence of at least two and three solutions for problem (1).

Theorem 7. *Assume that there exist two positive constants $\bar{\gamma} > \frac{(N+1)\sqrt{N+1}}{2}$ and $\bar{\eta}$ with the property*

$$\frac{\sqrt{N+1}}{2}(2\sqrt{1+\bar{\eta}^2} + N - 1) > \bar{\gamma}, \tag{19}$$

and let the assumption (A_2) in Theorem 5 hold. Moreover, assume that

(A_5) $\sum_{k=1}^{N} F(k,\eta) \geq 0$;

(A_6)

$$\frac{\sum_{k=1}^{N} \sup_{|t|\leq\bar{\gamma}} F(k,t)}{\frac{2}{\sqrt{N+1}}\bar{\gamma} - N - 1} < \frac{\sum_{k=1}^{N} F(k,\eta) - \sum_{k=1}^{N} \sup_{|t|\leq\bar{\gamma}} F(k,t)}{2(\sqrt{1+\bar{\eta}^2} - 1)}.$$

Then, for every

$$\lambda \in \Big(\frac{2(\sqrt{1+\bar{\eta}^2}-1)}{\sum_{k=1}^{N} F(k,\eta) - \sum_{k=1}^{N} \sup_{|t|\le\bar{\gamma}} F(k,t)}, \frac{\frac{2}{\sqrt{N+1}}\bar{\gamma} - N - 1}{\sum_{k=1}^{N} \sup_{|t|\le\bar{\gamma}} F(k,t)} \Big)$$

and for every sign-changing function $g : [1,N]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ *satisfying in* (6) *problem* (1)*, in the case* $\lambda = \mu$*, admits at least three solutions in* X*.*

Proof. Put $I_\lambda = \Phi(u) + \lambda\Psi(u)$, where

$$\Phi(u) = \sum_{k=1}^{N+1}\Big(\sqrt{1+(\Delta u(k-1))^2} - 1\Big) \text{ and } \Psi(u) = -\sum_{k=1}^{N}\Big(F(k,u(k)) + G(k,u(k))\Big) \quad (20)$$

for all $u \in X$. Standard arguments show that Φ and Ψ are Gâteaux differentiable functionals whose Gâteaux derivatives at the point $u \in X$ are given by

$$\Phi'(u)(v) = -\sum_{k=1}^{N} \Delta(\phi_c(\Delta u(k-1)))v(k)$$

and

$$\Psi'(u)v = -\sum_{k=1}^{N} \Big(f(k,u(k))v(k) + g(k,u(k))v(k) \Big)$$

for all $u, v \in X$, respectively. We know that a critical point for function $\Phi(u)+\lambda\Psi(u)$ represents a solution of problem (1) in the case $\lambda = \mu$. Our objective is to implement Theorem 3 for Φ and Ψ. By sequentially weakly lower semicontinuity of the norm, the functional Φ is sequentially weakly lower semicontinuous. Moreover, Φ is continuously Gâteaux differentiable and its Gâteaux derivative admits a continuous inverse on X^*. The functional $\Psi : X \to \mathbb{R}$ is well-defined and continuously Gâteaux differentiable and its Gâteaux derivative is compact. Thus it is enough to show that Φ and Ψ satisfy (m_1) and (m_2) in Theorem 3. By (6), we can fix $\alpha > 0$ such that $\alpha\lambda < \frac{2}{N\sqrt{N+1}}$, and there exists $\rho \in \mathbb{R}$ such that $G(k,t) \le \alpha|t| + \rho$. Now, we fix $\varepsilon < \frac{2}{\lambda N\sqrt{N+1}} - \alpha$. From the assumption (A_2) there is $h \in \mathbb{R}$ such that $F(k,t) \le \varepsilon|t| + h$ for every $(k,t) \in [1,N]_{\mathbb{Z}} \times \mathbb{R}$. It follows that for each $u \in X$,

$$\begin{aligned}
\Phi(u) + \lambda\Psi(u) &= \sum_{k=1}^{N+1} \Big(\sqrt{1+(\Delta u(k-1))^2} - 1\Big) - \lambda \sum_{k=1}^{N} \Big(F(k,u(k)) + G(k,u(k))\Big) \\
&\ge \Big(\sum_{k=1}^{N+1} [1+(\Delta u(k-1))^2]\Big)^{\frac{1}{2}} - \sum_{k=1}^{N+1} 1 - \lambda\varepsilon \sum_{1}^{N} |u(k)| - \lambda h \\
&\quad - \alpha\lambda \sum_{1}^{N} |u(k)| - \lambda\rho
\end{aligned}$$

$$\geq \|u\| - N - 1 - \lambda \frac{\varepsilon N\sqrt{N+1}}{2}\|u\| - \lambda N h - \lambda \frac{\alpha N\sqrt{N+1}}{2}\|u\| - \lambda N\rho$$
$$\geq \Big(1 - \lambda \frac{\varepsilon N\sqrt{N+1}}{2} - \lambda \frac{\alpha N\sqrt{N+1}}{2}\Big)\|u\| - \lambda N h - \lambda N\rho - N - 1,$$

thus $\lim_{\|u\|_X\to\infty}(\Phi(u)+\lambda\Psi(u)) = \infty$, which means the functional $I_\lambda = \Phi(u)+\lambda\Psi(u)$ is coercive. Now it remains to show that (m_2) of Theorem 3 is satisfied. Put $\bar{r} = \frac{2}{\sqrt{N+1}}\bar{\gamma} - N - 1$ and

$$\omega(k) = \begin{cases} \bar{\eta}, & k \in [1,N]_{\mathbb{Z}}, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $w \in X$ and

$$\Phi(\omega) = 2(\sqrt{1+\bar{\eta}^2} - 1). \tag{21}$$

Thus, by (19) we see that $\Phi(\omega) > \bar{r}$. Moreover, since g is sign-changing and using the condition (A_5), we have

$$\Psi(\omega) = -\sum_{k=1}^{N}\Big(F(k,w(k)) + G(k,w(k))\Big) \leq -\sum_{k=1}^{N} F(k,\eta).$$

Taking (4) into account, for every $u \in X$ such that $\Phi(u) < \bar{r}$, we get

$$\sup_{k\in[1,N]_{\mathbb{Z}}} |u(k)| \leq \bar{\gamma}.$$

Thus

$$\begin{aligned} \sup_{u\in\Phi^{-1}(-\infty,\bar{r})} \sum_{k=1}^{N}\Big(F(k,u(k)) + G(k,u(k))\Big) &\leq \sup_{u\in\Phi^{-1}(-\infty,\bar{r})} \sum_{k=1}^{N} F(k,u(k)) \\ &\leq \sum_{k=1}^{N} \sup_{|t|\leq\bar{\gamma}} F(k,t), \end{aligned} \tag{22}$$

for every $u \in X$ with $\Phi(u) < \bar{r}$. So

$$\sup_{\Phi(u)\leq\bar{r}} \Psi(u) \leq \sum_{k=1}^{N} \sup_{|t|\leq\bar{\gamma}} F(k,t).$$

By simple calculations and from the definition of $\varphi_1(\bar{r})$, since $\Phi(0) = \Psi(0) = 0$ and $\overline{\Phi^{-1}(-\infty,\bar{r})}^{\omega} = \Phi^{-1}(-\infty,a\bar{r})$, we have

$$\begin{aligned} \varphi_1(\bar{r}) &= \inf_{u\in\Phi^{-1}]-\infty,\bar{r}[} \frac{\Psi(u) - \inf_{\overline{\Phi^{-1}(-\infty,\bar{r})}^{\omega}} \Psi(u)}{\bar{r} - \Phi(u)} \\ &\leq \frac{-\inf_{\overline{\Phi^{-1}(-\infty,\bar{r})}^{\omega}} \Psi(u)}{\bar{r}} \\ &\leq \frac{\sum_{k=1}^{N} \sup_{|t|\leq\bar{\gamma}} F(k,t)}{\frac{2}{\sqrt{N+1}}\bar{\gamma} - N - 1}. \end{aligned}$$

On the other hand, by (22), one has

$$\begin{aligned}
\varphi_2(\bar{r}) &= \inf_{u\in\Phi^{-1}(-\infty,\bar{r})} \sup_{v\in\Phi^{-1}[\bar{r},\infty)} \frac{\Psi(u)-\Psi(v)}{\Phi(v)-\Phi(u)} \\
&\geq \inf_{u\in\Phi^{-1}(-\infty,\bar{r})} \frac{\Psi(u)-\Psi(\omega)}{\Phi(\omega)-\Phi(u)} \\
&\geq \frac{\inf_{u\in\Phi^{-1}(-\infty,\bar{r})} \Psi(u)-\Psi(\omega)}{\Phi(\omega)-\Phi(u)} \\
&\geq \frac{-\sum_{k=1}^{N}\sup_{|k|\leq\bar{\gamma}} F(k,t)+\sum_{k=1}^{N} F(k,\eta)}{\Phi(\omega)-\Phi(u)} \\
&\geq \frac{\sum_{k=1}^{N} F(k,\eta)-\sum_{k=1}^{N}\sup_{|t|\leq\bar{\gamma}} F(k,t)}{2(\sqrt{1+\bar{\eta}^2}-1)}.
\end{aligned}$$

From (A_6) one has

$$\varphi_1(\bar{r}) < \varphi_2(\bar{r}).$$

Therefore, from Theorem 3, taking into account that

$$\frac{1}{\varphi_2(\bar{r})} \leq \frac{2(\sqrt{1+\bar{\eta}^2}-1)}{\sum_{k=1}^{N} F(k,\eta)-\sum_{k=1}^{N}\sup_{|t|\leq\bar{\gamma}} F(k,t)}$$

and

$$\frac{1}{\varphi_1(\bar{r})} \geq \frac{\frac{2}{\sqrt{N+1}}\bar{\gamma}-N-1}{\sum_{k=1}^{N}\sup_{|t|\leq\bar{\gamma}} F(k,t)},$$

we obtain the desired result. □

Remark 2. *In Theorem 7, if we replace the condition*

(A_7) $\frac{\sum_{k=1}^{N}\sup_{|t|\leq\bar{\gamma}} F(k,t)}{\frac{2}{\sqrt{N+1}}\bar{\gamma}-N-1} < \frac{\sum_{k=1}^{N} F(k,\eta)}{2(\sqrt{1+\bar{\eta}^2}-1)}$

with the condition (A_6)*, the assumptions* (19), (A_5) *and* (A_7) *hold. Then, for each*

$$\lambda \in \Big(\frac{2(\sqrt{1+\bar{\eta}^2}-1)}{\sum_{k=1}^{N} F(k,\eta)}, \frac{\frac{2}{\sqrt{N+1}}\bar{\gamma}-N-1}{\sum_{k=1}^{N}\sup_{|t|\leq\bar{\gamma}} F(k,t)}\Big),$$

and for every sign-changing function $g : [1,N]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ *satisfying condition* (6)*, in the case* $\lambda = \mu$ *problem* (1) *admits at least three solutions.*

We now want to show the existence and multiplicity of solutions for problem (1) by using Theorem 4 in the case $\lambda = \mu$.

Theorem 8. *Assume that there exist three positive constants* $\bar{\gamma}_1 > \frac{(N+1)\sqrt{N+1}}{2}$, $\bar{\gamma}_2 > \frac{(N+1)\sqrt{N+1}}{2}$ *and* $\bar{\eta}$ *with the property*

$$\frac{2}{\sqrt{N+1}}\bar{\gamma}_1 - N < 2\sqrt{1+\bar{\eta}^2} - 1 < \frac{2}{\sqrt{N+1}}\bar{\gamma}_2 - N \tag{23}$$

such that the assumption (A_5) *in Theorem 3 holds, and*

(A_8)

$$\max\Big\{\frac{\sum_{k=1}^{N}\sup_{|t|\leq\bar{\gamma}_1} F(k,t)}{\frac{2}{\sqrt{N+1}}\bar{\gamma}_1 - N - 1}, \frac{\sum_{k=1}^{N}\sup_{|t|\leq\bar{\gamma}_2} F(k,t)}{\frac{2}{\sqrt{N+1}}\bar{\gamma}_2 - N - 1}\Big\} < \frac{\sum_{k=1}^{N} F(k,\eta)}{2(\sqrt{1+\bar{\eta}^2}-1)}.$$

Then, for each

$$\lambda \in \Lambda = \Big(\frac{2(\sqrt{1+\bar{\eta}^2}-1)}{\sum_{k=1}^{N} F(k,\eta)}, \min\Big\{\frac{\frac{2}{\sqrt{N+1}}\bar{\gamma}_1 - N - 1}{\sum_{k=1}^{N}\sup_{|t|\leq\bar{\gamma}_1} F(k,t)}, \frac{\frac{2}{\sqrt{N+1}}\bar{\gamma}_2 - N - 1}{\sum_{k=1}^{N}\sup_{|t|\leq\bar{\gamma}_2} F(k,t)}\Big\}\Big),$$

and for every sign-changing function $g : [1,N]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ *satisfying in* (6), *in the case* $\lambda = \mu$ *problem* (1) *admits at least two solutions* $u_{1,\lambda}$ *and* $u_{2,\lambda}$ *such that* $\max_{k\in[1,N]_{\mathbb{Z}}} |u_{1,\lambda}(k)| < \bar{\gamma}_1$ *and* $\max_{k\in[1,N]_{\mathbb{Z}}} |u_{2,\lambda}(k)| < \bar{\gamma}_2$.

Proof. Put

$$\bar{f}(k,t) = \begin{cases} f(k,-\bar{\gamma}_1), & \text{if } (k,t) \in [1,N]_{\mathbb{Z}} \times (-\infty,-\bar{\gamma}_1) \\ f(t,x), & \text{if } (k,t) \in [1,N]_{\mathbb{Z}} \times [-\bar{\gamma}_1,\bar{\gamma}_2] \\ f(k,\bar{\gamma}_2), & \text{if } (k,t) \in [1,N]_{\mathbb{Z}} \times (\bar{\gamma}_2,\infty). \end{cases}$$

Clearly, $\bar{f} : [1,N]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Now put $\bar{F}(k,t) = \int_0^t \bar{f}(k,\xi)d\xi$ for all $(k,\xi) \in [1,N]_{\mathbb{Z}} \times \mathbb{R}$ and take X and Φ as (2) and (20), respectively, and

$$\Psi(u) = -\sum_{k=1}^{N}\Big(\bar{F}(k,u(k)) + G(k,u(k))\Big)$$

for all $u \in X$. Our goal is to apply Theorem 4 to Φ and Ψ. It is well known that $\lim_{\|u\|_X\to\infty} \Phi(u) = \infty$ and Ψ is a differentiable functional whose differential at the point $u \in X$ is

$$\Psi'(u)v = -\sum_{k=1}^{N}\Big(\bar{f}(k,u(k))v(k) + g(k,u(k))v(k)\Big)$$

for any $v \in X$, and it is also sequentially weakly lower semicontinuous. Furthermore, $\Psi' : X \to X^*$ is a compact operator. Thus, it is enough to show that Φ and Ψ satisfy conditions (n_1), (n_2) and (n_3) in Theorem 4. Let

$$\bar{r}_1 = \frac{2}{\sqrt{N+1}}\bar{\gamma}_1 - N - 1, \quad \bar{r}_2 = \frac{2}{\sqrt{N+1}}\bar{\gamma}_2 - N - 1$$

and let $\omega \in X$ as in the proof of Theorem 4. Due to the assumptions (23) and (21), we have $\bar{r}_1 < \Phi(\omega) < \bar{r}_2$ and $\inf_X \Phi < \bar{r}_1 < \bar{r}_2$. Moreover, arguing as in the proof of Theorem 7 and taking Remark 2 into account, we obtain

$$\varphi(\bar{r}_1) \leq \frac{\sum_{k=1}^{N} \sup_{|t| \leq \bar{\gamma}_1} F(k,t)}{\frac{2}{\sqrt{N+1}}\bar{\gamma}_1 - N - 1},$$

$$\varphi(\bar{r}_2) \leq \frac{\sum_{k=1}^{N} \sup_{|t| \leq \bar{\gamma}_2} F(k,t)}{\frac{2}{\sqrt{N+1}}\bar{\gamma}_2 - N - 1}$$

and

$$\varphi_2^*(\bar{r}_2, \bar{r}_2) \geq \frac{\sum_{k=1}^{N} F(k,\eta)}{2(\sqrt{1+\bar{\eta}^2} - 1)}.$$

Hence, from (A_8), the conditions (n_2) and (n_3) of Theorem 4 hold. Therefore, from Theorem 4 we obtain that for each $\lambda \in \Lambda$, the problem

$$\begin{cases} -\Delta\Big(\frac{\Delta u(k-1)}{\sqrt{1+(\Delta u(k-1))^2}}\Big) = \lambda\Big(\bar{f}(k,u(k)) + g(k,u(k))\Big), \ k \in [1,N]_{\mathbb{Z}} \\ u(0) = u(N+1) = 0, \end{cases}$$

admits at least two solutions $u_{1,\lambda}$ and $u_{2,\lambda}$ such that $\max_{k\in[1,N]_{\mathbb{Z}}} |u_{1,\lambda}(k)| < \bar{\gamma}_1^2$ and $\max_{k\in[1,N]_{\mathbb{Z}}} |u_{2,\lambda}(k)| < \bar{\gamma}_2^2$. Observing that these solutions are also the solutions for problem (1) in the case $\lambda = \mu$, the conclusion follows. □

We now consider the case where variables of function f are separated, in which case problem (1) is written as follows:

$$\begin{cases} -\Delta\Big(\frac{\Delta u(k-1)}{\sqrt{1+(\Delta u(k-1))^2}}\Big) = \lambda\Big(\theta(k)f(u(k)) + g(k,u(k))\Big), \ k \in [1,N]_{\mathbb{Z}} \\ u(0) = u(N+1) = 0, \end{cases}$$

where $\theta : [1,N]_{\mathbb{Z}} \to \mathbb{R}$ is a non-negative and non-zero function such that $\theta \in l^1([1,N]_{\mathbb{Z}})$ and $f : \mathbb{R} \to \mathbb{R}$ is a non-negative and continuous function.

Put

$$F(\xi) = \int_0^{\xi} f(t)dt, \ \ for\ all\ \ \xi \in \mathbb{R}.$$

We apply the results of Theorems 7 and 8 to the case when $f(k,t) = \theta(k)f(t)$ ($\forall$ $(k,t) \in [1,N]_{\mathbb{Z}} \times \mathbb{R}$).

Theorem 9. *Assume that there exist two positive constants $\bar{\gamma} > \frac{(N+1)\sqrt{N+1}}{2}$ and $\bar{\eta}$ with the property*

$$\frac{\sqrt{N+1}}{2}(2\sqrt{1+\bar{\eta}^2} + N - 1) > \bar{\gamma}$$

and let the assumption (A_2) and

(A_8) $\frac{F(\bar\gamma)}{\frac{2}{\sqrt{N+1}}\bar\gamma-N-1} < \frac{F(\eta)}{2(\sqrt{1+\bar\eta^2}-1)}$ *hold.*

Then, for every

$$\lambda \in \Big(\frac{2(\sqrt{1+\bar\eta^2}-1)}{\|\theta\|_{l^1([1,N]_{\mathbb{Z}})}F(\eta)}, \frac{\frac{2}{\sqrt{N+1}}\bar\gamma - N - 1}{\|\theta\|_{l^1([1,N]_{\mathbb{Z}})}F(\bar\gamma)}\Big)$$

and for every sign-changing function $g : [1,N]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ *satisfying* (6)*, in the case* $\lambda = \mu$*, problem* (1) *admits at least three solutions in* X*.*

Theorem 10. *Assume that there exist three positive constants* $\bar\gamma_1 > \frac{(N+1)\sqrt{N+1}}{2}$*,* $\bar\eta$ *and* $\bar\gamma_2 > \frac{(N+1)\sqrt{N+1}}{2}$ *with the property*

$$\frac{2}{\sqrt{N+1}}\bar\gamma_1 - N < 2\sqrt{1+\bar\eta^2} - 1 < \frac{2}{\sqrt{N+1}}\bar\gamma_2 - N$$

and

(A_9)

$$\max\Big\{\frac{F(\bar\gamma_1)}{\frac{2}{\sqrt{N+1}}\bar\gamma_1 - N - 1}, \frac{F(\bar\gamma_2)}{\frac{2}{\sqrt{N+1}}\bar\gamma_2 - N - 1}\Big\} < \frac{F(\eta)}{2(\sqrt{1+\bar\eta^2}-1)}.$$

Then, for each

$$\lambda \in \Lambda = \Big(\frac{2(\sqrt{1+\bar\eta^2}-1)}{\|\theta\|_{l^1([1,N]_{\mathbb{Z}})}F(\eta)}, \min\Big\{\frac{\frac{2}{\sqrt{N+1}}\bar\gamma_1 - N - 1}{\|\theta\|_{l^1([1,N]_{\mathbb{Z}})}F(\bar\gamma_1)}, \frac{\frac{2}{\sqrt{N+1}}\bar\gamma_2 - N - 1}{\|\theta\|_{l^1([1,N]_{\mathbb{Z}})}F(\bar\gamma_2)}\Big\}\Big)$$

and for every sign-changing function $g : [1,N]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ *satisfying* (6)*, in the case* $\lambda = \mu$*, problem* (1) *admits at least two solutions* $u_{1,\lambda}$ *and* $u_{2,\lambda}$ *such that* $\max_{k\in[1,N]_{\mathbb{Z}}} |u_{1,\lambda}(k)| < \bar\gamma_1$ *and* $\max_{k\in[1,N]_{\mathbb{Z}}} |u_{2,\lambda}(k)| < \bar\gamma_2$*.*

References

[1] D. AVERNA, G. BONANNO, *A three critical point theorem and its applications to the ordinary Dirichlet problem,* Topol. Methods Nonlinear Anal. TMA **22**(2003), 93–103.

[2] C. BEREANU, P. JEBELEAN, J. MAWHIN, *Radial solutions for Neumann problems involving mean curvature operators in Euclidean and Minkowski spaces*, Math. Nachr. **283**(2010), 379–391.

[3] C. BEREANU, P. JEBELEAN, J. MAWHIN, *Multiple solutions for Neumann and periodic problems with singular ϕ-Laplacian*, J. Funct. Anal. **261**(2011), 3226–3246.

[4] C. BEREANU, P. JEBELEAN, J. MAWHIN, *Radial solutions of Neumann problems involving mean extrinsic curvature and periodic nonlinearities*, Calc. Var. Partial Differential Equations **46**(2013), 113–122.

[5] C. BEREANU, P. JEBELEAN, P. J. TORRES, *Positive radial solutions for Dirichlet problems with mean curvature operators in Minkowski space*, J. Funct. Anal. **264**(2013), 270–287.

[6] M. BOHNER, G. CARISTI, A. GHOBADI, S. HEIDARKHANI, *Three solutions for discrete anisotropic Kirchhoff-type problems*, Demonstr. Math. **56**(2023), Paper No. 20220209, 13pp.

[7] G. BONANNO, *A critical points theorem and nonlinear differential problems*, J. Global Optim. **28**(2004), 249–258.

[8] G. BONANNO, *Multiple critical points theorems without the Palais-Smale condition*, J. Math. Anal. Appl. **299**(2004), 600–614.

[9] G. BONANNO, *A critical point theorem via the Ekeland variational principle*, Nonlinear Anal. **75**(2012), 2992–3007.

[10] G. BONANNO, *Relations between the mountain pass theorem and local minima*, Adv. Nonlinear Anal. **1**(2012), 205–220.

[11] G. BONANNO, P. CANDITO, *Infinitely many solutions for a class of discrete nonlinear boundary value problems*, Appl. Anal. **88**(2009), 605–616.

[12] G. BONANNO, P. CANDITO, *Nonlinear difference equations investigated via critical point methods*, Nonlinear Anal. **70**(2009), 3180–3186.

[13] G. BONANNO, P. CANDITO, G. D'AGUÌ, *Variational methods on finite dimensional Banach spaces and discrete problems*, Adv. Nonlinear Stud. **14**(2014), 915–939.

[14] G. BONANNO, A. CHINNÌ, *Existence and multiplicity of weak solutions for elliptic Dirichlet problems with variable exponent*, J. Math. Anal. Appl. **418**(2014), 812–827.

[15] G. BONANNO, R. LIVREA, J. MAWHIN, *Existence results for parametric boundary value problems involving the mean curvature operator*, Nonlinear Differ. Equ. Appl. **22**(2015), 411–426.

[16] G. BONANNO, D. O'REGAN, *A boundary value problem on the half-line via critical point methods*, Dynam. Systems Appl. **15**(2006), 395–408.

[17] P. CANDITO, G. D'AGUÌ, *Three solutions to a perturbed nonlinear discrete Dirichlet problem*, J. Math. Anal. Appl. **375**(2011), 594–601.

[18] G. CARISTI, S. HEIDARKHANI, A. SALARI, *Variational approaches to Kirchhoff-type second-order impulsive differential equations on the half-Line*, Results Math. **73**(2018), 44.

[19] Y. CHEN, Z. ZHOU, *Existence of three solutions for a nonlinear discrete boundary value problem with φ_c-Laplacian*, Symmetry **12**(2020), 1839.

[20] J. M. CUSHING, S. LEVARGE, N. CHITNIS, S. M. HENSON, *Some discrete competition models and the competitive exclusion principle*, J. Difference Equ. Appl. **10**(2004), 1139–1151.

[21] S. DU, Z. ZHOU, *On the existence of multiple solutions for a partial discrete Dirichlet boundary value problem with mean curvature operator*, Adv. Nonlinear Anal. **11**(2022), 198–211.

[22] R. FINN, *Equilibrium Capillary Surfaces*, Springer, New York, 1986.

[23] J. R. GRAEF, L. KONG, M. WANG, *Multiple solutions to a periodic boundary value problem for a nonlinear discrete fourth order equation*, Adv. Dyn. Syst. Appl. **8**(2013), 203–215.

[24] J. R. GRAEF, L. KONG, M. WANG, *Two nontrivial solutions for a discrete fourth order periodic boundary value problem*, Commun. Appl. Anal. **19**(2015), 487–496.

[25] G. HALDAR, *On entire solutions of system of Fermat-type difference and partial differential-difference equations in $\mathbb{C}^n$*, Rend. Circ. Mat. Palermo Ser. II, 2024, DOI: 10.1007/s12215-023-00997-y.

[26] T. S. HE, Y. W. ZHOU, Y. T. XU, C. Y. CHEN, *Sign-changing solutions for discrete second-order periodic boundary value problems*, Bull. Malays. Math. Sci. Soc. **38**(2015), 181–195.

[27] S. HEIDARKHANI, G. A. AFROUZI, G. CARISTI, J. HENDERSON, S. MORADI, *A varia-*

tional approach to difference equations, J. Difference Equ. Appl. **22**(2016), 1761–1776.

[28] L. Jiang, Z. Zhou, *Three solutions to Dirichlet boundary value problems for p-Laplacian difference equations*, Adv. Difference Equ. **2008**(2007), 1–10.

[29] A. Kurganov, P. Rosenau, *On reaction processes with saturating diffusion*, Nonlinearity **19**(2006), 171–193.

[30] G. H. Lin, Z. Zhou, *Homoclinic solutions of discrete ö-Laplacian equations with mixed nonlinearities*, Commun. Pure Appl. Anal. **17**(2018), 1723–1747.

[31] Y. H. Long, J. L. Chen, *Existence of multiple solutions to second-order discrete Neumann boundary value problems*, Appl. Math. Lett. **83**(2018), 7–14.

[32] J. Mawhin, *Periodic solutions of second order nonlinear difference systems with ϕ-Laplacian: A variational approach*, Nonlinear Anal. **75**(2012), 4672–4687.

[33] A. Nastasi, S. Tersian, C. Vetro, *Homoclinic solutions of nonlinear Laplacian difference equations without Ambrosetti-Rabinowitz condition,* Acta Math. Sci. Ser. B (Engl. Ed.) **41**(2021), 712–718.

[34] F. Obersnel, P. Omari, *Positive solutions of the Dirichlet problem for the prescribed mean curvature equation*, J. Differential Equations **249**(2010), 1674–1725.

[35] F. Obersnel, P. Omari, S. Rivetti, *Existence, regularity and stability properties of periodic solutions of a capillarity equation in the presence of lower and upper solutions*, Nonlinear Anal. Real World Appl. **13**(2012), 2830–2852.

[36] O. Rey, *Heat flow for the equation of surfaces with prescribed mean curvature*, Math. Ann. **291**(1991), 123–146.

[37] Z. Tan, G. Wu, *On the heat flow equation of surfaces of constant mean curvature in higher dimensions*, Acta Math. Sci. Ser. B (Engl. Ed.) **31**(2011), 1741–1748.

[38] Z. Zhou, J. Ling, *Infinitely many positive solutions for a discrete two point nonlinear boundary value problem with ϕ_c-Laplacian*, Appl. Math. Lett. **91**(2019), 28–34.