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Stochastic perturbations method for a system of Riemann invariants*

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Abstract. Based on our results [1] on a representation of solutions to the Cauchy problem for a multidimensional non-viscous Burgers equation obtained by a method of stochastic perturbation of the associated Langevin system, we deduce an explicit asymptotic formula for smooth solutions to the Cauchy problem for any genuinely nonlinear hyperbolic system of equations written in the Riemann invariants.

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1. Introduction

We consider a system of the following form:

$$\frac{\partial r_i}{\partial t} + f_i(r_1, \dots, r_n)\frac{\partial r_i}{\partial x} = 0, \qquad i = 1, \dots, n, \tag{1}$$

written for the functions $r_i = r_i(t, x)$, $t \in \overline{\mathbb{R}}_+$, $x \in \mathbb{R}$, subject to initial data

$$r(0, x) = r^0(x) = (r_1^0(x), \dots, r_n^0(x)). \tag{2}$$

We assume that the vector-function $f(r) \in C^1(\mathbb{R}^n)$ is real-valued and

$$\det\left(\frac{\partial f_i(r)}{\partial r_j}\right) \neq 0, \qquad i, j = 1, ..., n. \tag{3}$$

It is well known that any quasilinear hyperbolic system $u_t + A(u)u_x = 0$, where $u = u(t, x)$ is an n-vector $\mathbb{R}^2 \to \mathbb{R}^n$, $A(u)$ is an $(n \times n)$ matrix with smooth coefficients, can be written in the Riemann invariants if it consists of two equations.

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For three equations the system can be written in the Riemann invariants if and only if the left eigenvectors l_k of A satisfy either $\operatorname{rot} l_k = 0$ or $l_k \cdot \operatorname{rot} l_k = 0$ [3]. Among physically meaningful systems admitting the Riemann invariants one can name the equations of isentropic gas dynamics, equations of shallow water, equations of electrophoresis and equations of chromatography [8, 10, 4].

2. Extended system and stochastic differential equation associated with the equations of characteristics

Based on (1), we consider a system

$$\frac{\partial q_i}{\partial t} + f_k(q_1,\dots,q_n)\frac{\partial q_i}{\partial x_k} = 0, \qquad i,k = 1,\dots,n, \tag{4}$$

written for functions $q_i(t,x)$, $x \in \mathbb{R}^n$, together with initial data

$$q(0,x) = q^0(x) = (q_1^0(x),\dots,q_n^0(x)). \tag{5}$$

Let us suppose that x_i, $i = 1,...,n$, are in turn functions of one variable $\bar{x} \in \mathbb{R}$. In this case, system (4) should be rewritten as

$$\frac{\partial q_i}{\partial t} + f_k(q_1,\dots,q_n)\frac{\partial q_i}{\partial x_k}\frac{dx_k(\bar{x})}{d\bar{x}} = 0, \qquad i,k = 1,\dots,n. \tag{6}$$

It can be readily checked that if we set

$$q_i(t,0,\dots,\underbrace{\bar{x}}_{i-\text{th place}},\dots,0) := r_i(t,\bar{x}), \tag{7}$$

such that $\frac{\partial q_i}{\partial x_m} = 0$, $i \neq m$, and

$$q_i^0(0,\dots,\underbrace{\bar{x}}_{i-\text{th place}},\dots,0) := r_i^0(\bar{x}), \tag{8}$$

$i,m = 1,\dots,n$, then the vector-function $r(t,\bar{x})$ solves the problem (1), (2).

We associate with (4) the following system of stochastic differential equations:

$$\begin{aligned} dX_i(t) &= f_i(Q_1(t),...,Q_n(t))dt + \sigma_1 d(W_i^1)_t, \\ dQ_i(t) &= \sigma_2\, d(W_i^2)_t, \quad i = 1,...,n, \\ X_i(0) &= x_i, \quad Q_i(0) = q_i, \quad t > 0, \end{aligned} \tag{9}$$

where $X(t)$ and $Q(t)$ are random values with given initial data, $(X(t),Q(t))$ takes values in the phase space $\mathbb{R}^n \times \mathbb{R}^n$, σ_1 and σ_2 are positive constants, $|\sigma| \neq 0$ ($\sigma = (\sigma_1,\sigma_2)$), $(W^j)_t = (W_1^j,\dots,W_n^j)_t$, $j = 1,2$, are independent Brownian motions.

Let $P(t, dq_1,...,dq_n, dx_1,...,dx_n)$, $t \in \mathbb{R}_+$, $x_i \in \mathbb{R}$, $q_i \in \mathbb{R}$, $i = 1,..,n$, be the joint probability distribution of random variables (Q,X), subject to the initial data

$$P_0(dr,dx) = \prod_{i=1}^{n} \delta_q(q_i^0(x_i))\rho_i^0(x_i)\,dx = \delta_q(q^0(s))\,\rho^0(s), \tag{10}$$

where $\rho_i^0(x_i)$ are bounded nonnegative functions from $C(\mathbb{R})$ and dx is a Lebesgue measure on $\mathbb{R}^n$, and δ_q is a Dirac measure concentrated on q.

We look at $P = P(t, dq, dx)$ as a generalized function (distribution) with respect to the variable q. It satisfies the Fokker-Planck equation

$$P_t + \sum_{i=1}^{n} f_i(q) P_{x_i} = \frac{\sigma_1^2}{2} \sum_{i=1}^{n} P_{x_i\, x_i} + \frac{\sigma_2^2}{2} \sum_{i=1}^{n} P_{q_i\, q_i} \tag{11}$$

with initial data (10).

There is a standard procedure to find the fundamental solution for (11) (see, e.g. [5]). This procedure consists of a reduction of the equation to a Fredholm integral equation, the solution of which can be found in the form of series. We are going to show that one can also find an explicit solution to the Cauchy problem (11), (10).

Let us introduce, still in the general case, the functions for $t \in \mathbb{R}_+$, $x \in \mathbb{R}^n$, depending on $\sigma = (\sigma_1, \sigma_2)$:

$$\rho_{i,\sigma}(t, x_i) = \int\limits_{\mathbb{R}^{2n-1}} P(t, x, dr)\, d\breve{x}_i, \tag{12}$$

$$\rho_\sigma(t, x) = \int\limits_{\mathbb{R}^n} P(t, x, dq), \tag{13}$$

$$q_{i,\sigma}(t, x) = \frac{\int\limits_{\mathbb{R}^n} q_i\, P(t, x, dq)}{\int\limits_{\mathbb{R}^n} P(t, x, dq)}, \tag{14}$$

$$f_{i,\sigma}(t, x) = \frac{\int\limits_{\mathbb{R}^n} f_i(q)\, P(t, x, dq)}{\int\limits_{\mathbb{R}^n} P(t, x, dq)}, \tag{15}$$

where

$$d\breve{x}_i \;=\; dx_1 \,..\, dx_{i-1}\, dx_{i+1} \,...\, dx_n, \quad dr \;=\; dr_1 ... dr_n.$$

We can consider these values if the integrals exist in the Lebesgue sense.

It will readily be observed that $q_{i,\sigma}(0, x) = q_i^0(x)$ and $f_{i,\sigma}(0, x) = f(q_i^0(x))$.

We denote

$$\bar\rho_i(t, \bar x) = \lim_{\sigma \to 0} \rho_{i,\sigma}(t, \bar x), \quad \bar\rho(t, x) = \lim_{\sigma \to 0} \rho_\sigma(t, x),$$
$$\bar q_{i,\sigma}(t, x) = \lim_{\sigma \to 0} q_{i,\sigma}(t, x), \quad \bar f_i(t, x) = \lim_{\sigma \to 0} f_{i,\sigma}(t, x),$$

provided these limits exist, $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2}$.

3. Explicit probability density function

Equation (11) can be solved explicitly. Moreover, for the sake of simplicity, we set $\sigma_2 = 0$ and denote $\sigma_1 = \sigma$.

Proposition 1. *Problem* (11), (10) *has the following solution:*

$$P(t,x,dq) = \frac{1}{(\sqrt{2\pi t}\sigma)^n} \tag{16}$$

$$\times \int\limits_{\mathbb{R}^n} \delta_r(r^0(s))\rho^0(s) \exp\left(-\frac{\sum\limits_{i=1}^{n}(f_i(r_1^0(s_1),\dots,r_n^0(s_n))\,t + (s_i - x_i))^2}{2\sigma^2 t}\right) ds,$$

for $t \ge 0$, $x \in \mathbb{R}^n$, *therefore*

$$\int\limits_{\mathbb{R}^n} \phi(q) P(t,x,dq) = \frac{1}{(\sqrt{2\pi t}\sigma)^n} \tag{17}$$

$$\times \int\limits_{\mathbb{R}^n} \phi(r^0(s))\rho^0(s) \exp\left(-\frac{\sum\limits_{i=1}^{n}(f_i(r_1^0(s_1),\dots,r_n^0(s_n))t + (s_i - x_i))^2}{2\sigma^2 t}\right) ds,$$

for all $\phi(r) \in C_0(\mathbb{R}^n)$.

Proof. Let us apply the Fourier transform to $P(t,x,dq)$ in (11), (10) with respect to the variable x and obtain the Cauchy problem for the Fourier transform $\tilde{P} = \tilde{P}(t,\lambda,dq)$ of $P(t,x,dq)$:

$$\frac{\partial \tilde{P}}{\partial t} = -(\frac{1}{2}\sigma^2|\lambda|^2 + i(\lambda, f(q)))\tilde{P}, \tag{18}$$

$$\tilde{P}(0,\lambda,dq) = \int\limits_{\mathbb{R}^n} e^{-i(\lambda,s)} \delta_q(q^0(s))\rho^0(s) ds, \qquad \lambda \in \mathbb{R}^n. \tag{19}$$

Equation (18) can be easily integrated and we obtain the solution given by the following formula:

$$\tilde{P}(t,\lambda,dq) = \tilde{P}(0,\lambda,dq) e^{-\frac{1}{2}\sigma^2|\lambda|^2 t + i(\lambda, f(q))\,t}. \tag{20}$$

The inverse Fourier transform (in the distributional sense) allows to find the density function $P(t,x,dq)$, $t > 0$:

$$\begin{aligned}
P(t,x,dq) &= \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} e^{i(\lambda,x)} \tilde{P}(t,\lambda,dq)\, d\lambda \\
&= \frac{1}{(2\pi)^{2n}} \int\limits_{\mathbb{R}^n} e^{i(\lambda,x)} \left(\int\limits_{\mathbb{R}^n} e^{-i(\lambda,s)} e^{-i(\lambda,f(q))\,t}\, \delta_q(q^0(s))\, \rho^0(s) ds \right) e^{-\frac{1}{2}\sigma^2|\lambda|^2 t} d\lambda \\
&= \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} \delta_q(q^0(s))\, \rho^0(s) \int\limits_{\mathbb{R}^n} e^{-\frac{1}{2}\sigma^2 t\left(\lambda - \frac{i|f(q)t+s-x|}{\sigma^2 t}\right)^2 - \frac{|f(q)t+s-x|^2}{2\sigma^2 t}} d\lambda ds \\
&= \frac{1}{(\sqrt{2\pi t}\sigma)^n} \int\limits_{\mathbb{R}^n} \delta_q(q^0(s))\, \rho^0(s)\, e^{-\frac{|f(q^0(s))t+s-x|^2}{2\sigma^2 t}} ds, \qquad t \ge 0,\ x \in \mathbb{R}^n.
\end{aligned}$$

The third equality is satisfied by Fubini's theorem, which can be applied by the absolute integrability and the bound on the function involved. Thus, the proposition is proved. □

Remark 1. *In the general case $\sigma_2 \neq 0$, an analogous formula can be obtained in a similar way (see [6] in this context).*

Corollary 1. *The functions ρ_σ, q_σ and f_σ defined in* (13) – (15) *can be represented by the following formulae:*

$$\rho_\sigma(t,x) = \frac{1}{(\sqrt{2\pi t}\sigma)^n}\int\limits_{\mathbb{R}^n} \rho^0(s)\, e^{-\frac{\sum\limits_{i=1}^{n}|f_i(r^0(s))t+s_i-x_i|^2}{2\sigma^2 t}} ds, \tag{21}$$

$$q_{i,\sigma}(t,x) = \frac{\int\limits_{\mathbb{R}^n} r^0(s)\rho^0(s)\, e^{-\frac{\sum\limits_{i=1}^{n}|f_i(r^0(s))t+s_i-x_i|^2}{2\sigma^2 t}} ds}{\int\limits_{\mathbb{R}^n} \rho^0(s)\, e^{-\frac{\sum\limits_{i=1}^{n}|f_i(r^0(s))t+s_i-x_i|^2}{2\sigma^2 t}} ds}, \qquad i = 1,...,n, \tag{22}$$

$$f_{i,\sigma}(t,x) = \frac{\int\limits_{\mathbb{R}^n} f(r^0(s))\rho^0(s)\, e^{-\frac{\sum\limits_{i=1}^{n}|f_i(r^0(s))t+s_i-x_i|^2}{2\sigma^2 t}} ds}{\int\limits_{\mathbb{R}^n} \rho^0(s)\, e^{-\frac{\sum\limits_{i=1}^{n}|f_i(r^0(s))t+s_i-x_i|^2}{2\sigma^2 t}} ds}, \qquad i = 1,...,n. \tag{23}$$

Proof. The result is obtained by substitution of $P(t,x,dq)$ as given by (16) in (13), (14), (15). □

4. Representation of a smooth solution to (1), (2)

Now we are going to prove that if ρ_i and $\bar{q}_i$ are continuous, then

$$\bar{r}_i(\bar{x}) = \bar{q}_i(t,x)|_{\{x_j=0,\, x_i=\bar{x}\}}, \qquad j \neq i,$$

tend to the solution $r_i(t,\bar{x})$ of problem (1), (2) as $\sigma \to 0$.

Namely, the following theorem holds.

Theorem 1. *Let $r(t,\bar{x})$ be a solution to the Cauchy problem* (1), (2), *$r^0 \in C_b^1(\mathbb{R})$ and $t_*(r^0)$ the supremum of t such that this solution is smooth. Then for $t \in [0, t_*(r^0))$,*

$$r_i(t,\bar{x}) = \bar{q}_i(t,x)|_{\{x_j=0,\, x_i=\bar{x}\}} = \lim_{\sigma\to 0} q_{\sigma,i}(t,x)|_{\{x_j=0,\, x_i=\bar{x}\}}, \qquad j \neq i,$$

where $q_\sigma(t,x)$ is given by (22) and the limit exists pointwise.

Proof. The easiest way to prove the theorem is to reduce (4) to the multidimensional non-viscous Burgers equation and use the representation from [1]. Namely, (4) has the form

$$\partial_t q + (f(q), \nabla) q = 0, \tag{24}$$

where $q(t,x) = (q_1, ..., q_n)$ is a vector-function $\mathbb{R}^{n+1} \to \mathbb{R}^n$, $f(q)$ is a non-degenerate differential mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$, such that its Jacobian satisfies the condition $\det \frac{\partial f_i(q)}{\partial q_j} \neq 0$, $i, j = 1, ..., n$, due to (3).

We multiply (24) by $\nabla_q f_i(q)$, $i = 1, ..., n$, to get

$$\partial_t f(q) + (f(q), \nabla) f(q) = 0. \tag{25}$$

Thus, we can introduce a new vectorial variable $u = f(q)$ to reduce the Cauchy problem for (25) to

$$\partial_t u + (u, \nabla) u = 0,\ t > 0, \qquad u(x, 0) = u_0(x) \in C^1(\mathbb{R}^n) \cap C_b(\mathbb{R}^n), \tag{26}$$

with $u_0(x) = f(q_0(x))$. As follows from [1], the solution to the non - viscous Burgers equation (26) before the moment t_* of a singularity formation can be obtained as a pointwise limit as $\sigma \to 0$ of

$$u_\sigma(t,x) = \frac{\int\limits_{\mathbb{R}^n} u_0(s) \rho^0(s) e^{-\frac{|u_0(s)t + s - x|^2}{2\sigma^2 t}} ds}{\int\limits_{\mathbb{R}^n} \rho^0(s) e^{-\frac{|u_0(s)t + s - x|^2}{2\sigma^2 t}} ds}. \tag{27}$$

Therefore, we find the representation of the solution to the stochastically perturbed along the characteristic equation (25) using formula (27) with $f(q_0(x))$ instead of $u_0(x)$.

Now we can go back to the vector-function $q(t,x)$. As follows from Proposition 1, $\bar{q}_i(t,x) = \lim\limits_{\sigma \to 0} q_{\sigma,i}(t,x)$. At last, we use the redesignation (7) to obtain the statement of theorem 1. $\square$

Further, we can find the maximal time $t_*(r^0)$ of the existence of a smooth solution to problem (1), (2).

Theorem 2. *If at least one derivative*

$$\frac{\partial f_i(q)}{\partial q_i} \frac{dr_i^0(x)}{dx}, \qquad i = 1, \ldots, n, \tag{28}$$

is negative, then the time $t_*(r^0)$ of the existence of a smooth solution to (1), (2) *is finite and*

$$t_*(r^*) = \min_i \left\{ -\left(\frac{\partial f_i(q)}{\partial q_i} \frac{dr_i^0(x)}{dx} \right)^{-1} \right\}. \tag{29}$$

Otherwise, $t_*(r^0) = \infty$.

Proof. Let us come back to problem (26) and denote by $J(u_0(x))$ the Jacobian matrix of the map $x \mapsto u_0(x)$. As it was shown in [7] (Theorem 1), if $J(u_0(x))$ has at least one eigenvalue which is negative for a certain point $x \in \mathbb{R}^n$, then the classical solution to (26) fails to exist beyond a positive time $t_*(u_0)$. Otherwise, $t_*(u_0) = \infty$. The matrix $C(t,x) = (I + tJ(u_0(x)))$, where I is the identity matrix, fails to be invertible for $t = t_*(u_0)$. Thus, due to representation (25), if $J(f(q^0(x)))$ has at least one eigenvalue which is negative for a certain point $x \in \mathbb{R}^n$, then the classical solution to (4) fails to exist beyond a positive time $t_*(q^0)$. For problem (1), (2) this means that at least one derivative (28) is negative. Value (29) can be found from the condition of invertibility of the matrix $(I + tJ(f(q_0(x))))$. $\square$

Remark 2. *It easy to see that Theorem 2 can be considered as a version of [4, Theorem 7.8.2], with a specification of the blow - up time obtained by a different method.*

5. Balance laws associated with systems written in Riemann invariants

Now we get one more corollary of results of [1]. Let us denote

$$g_{i,\sigma}(t,\bar{x}) = f_{i,\sigma}(t,x)|_{\{x_j=0,\, x_i=\bar{x}\}}, \quad \bar{g}_i(t,\bar{x}) = \lim_{\sigma\to 0} g_{i,\sigma}(t,\bar{x}).$$

Theorem 3. *The functions $\rho_{i,\sigma}$ and $g_{i,\sigma}$ satisfy the following system of $2n$ equations:*

$$\frac{\partial \rho_{i,\sigma}}{\partial t} + \partial_x(\rho_{i,\sigma} g_{i,\sigma}) = \frac{1}{2}\sigma^2 \frac{\partial^2 \rho_{i,\sigma}}{\partial x^2}, \tag{30}$$

$$\begin{aligned}\frac{\partial(\rho_{\sigma,i} g_{\sigma,i})}{\partial t} + \partial_x(\rho_{\sigma,i} g^2_{\sigma,i}) = {}& \frac{1}{2}\sigma^2 \frac{\partial^2(\rho_{\sigma,i} g_{\sigma,i})}{\partial x^2} \\ & - \int\limits_{\mathbb{R}^{2n-1}} (g_{\sigma,i} - \bar{g}(r)_i)\big((g_\sigma - \bar{g}(r)), \nabla_x P(t,x,dq)\big) d\check{x}.\end{aligned} \tag{31}$$

For $t \in (0, t_*(r^0))$ its limit functions $\bar{\rho}_i$ and $\bar{g}_i$ satisfy the system of $2n$ conservation laws:

$$\frac{\partial \bar{\rho}_i}{\partial t} + \partial_x(\bar{\rho}_i \bar{g}_i) = 0, \tag{32}$$

$$\frac{\partial(\bar{\rho}_i \bar{g}_i)}{\partial t} + \partial_x(\bar{\rho}_i \bar{g}_i^2) = 0, \tag{33}$$

$i = 1, .., n$, $t \geq 0$.

Proof. The statement follows from Theorem 2.1 [1]. Namely, the theorem implies that the scalar function $\rho_\sigma(t,x)$ and the vector-function $f_\sigma(t,x)$ solve the following system:

$$\frac{\partial \rho_\sigma}{\partial t} + \operatorname{div}_x(\rho_\sigma f_\sigma) = \frac{1}{2}\sigma^2 \sum_{k=1}^{n} \frac{\partial^2 \rho_\sigma}{\partial x_k^2}, \tag{34}$$

$$\begin{aligned}\frac{\partial(\rho_\sigma f_{i,\sigma})}{\partial t} + \nabla(\rho_\sigma\, f_{i,\sigma} f_\sigma) = {}& \frac{1}{2}\sigma^2 \sum_{k=1}^{n} \frac{\partial^2(\rho_\sigma f_{\sigma,i})}{\partial x_k^2} \\ & - \int\limits_{\mathbb{R}^n} (f_{i,\sigma} - \bar{f}_i)\big((f_\sigma - \bar{f}), \nabla_x P(t,x,dq)\big),\end{aligned} \tag{35}$$

with $i = 1, .., n$, $t \geq 0$, and the integral term vanishes as $\sigma \to 0$. To obtain the statement of Theorem 3, it is sufficient to set $x_j = 0$, $x_i = \bar{x}$ and $\rho_j(x_j) = 1$, $j \neq i$, for every fixed i. □

Remark 3. *System* (32), (33) *constitutes n systems of the so - called "pressureless" gas dynamics, the simplest model introduced to describe the formation of large structures in the Universe, see, e.g. [9].*

Remark 4. *The method of stochastic perturbations allows to study solutions to quasilinear systems written in Riemann invariants at the moment of the singularity formation and the shock waves evolution as well (in this context, see [2], [1], [6] for simpler cases). In particular, it is possible to prove that after the moment* $t_*(r^0)$ *of singularity formation in the solution to problem* (1), (2) *the limit system for* $\rho_{i,\sigma}$ *and* $g_{i,\sigma}$ *differs from* (32), (33) *and contains an additional integral term in the group of equations* (33). *This term does not vanish as* $\sigma \to 0$ *and can be considered as a gradient of a specific pressure term.*

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