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On operator equilibrium problems*

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Abstract. This paper is devoted to the study of operator equilibrium problems. By using the KKM theorem, we give sufficient conditions for the existence of solutions of these problems. As a consequence, the existence of solutions for operator variational inequalities and operator minimization problems are derived.

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1. Introduction

The operator variational inequalities were introduced by Domokos and Kolumbán [2]. Inspired by their work, Kum and Kim [6, 7] developed the scheme of operator variational inequalities from the single-valued case into the multi-valued one. The weak operator equilibrium problems were studied by Kazmi and Raouf [4] and Kum and Kim [8].

Kum, Kim and Lee [5] introduced the parametric form of the generalized operator equilibrium problems. They analyzed lower and upper semicontinuity of the solution map. The closedness and Hadamard well-posedness of parametric operator equilibrium problems were studied in [10].

The problem under consideration is the following:

Let X, Y and Z be Hausdorff topological vector spaces; $L(X, Y)$ the space of all continuous linear operators from X to Y and $K \subset L(X, Y)$ a nonempty convex set. Let $C : K \to 2^Z$ be a set-valued mapping such that for each $f \in K$, $C(f)$ is a convex, closed cone with nonempty interior and the apex at the origin of Z. Let a vector-valued mapping $F : K \times K \to Z$ be given.

The weak operator equilibrium problem (OEP) is to find $f \in K$ such that

$$F(f, g) \notin -\operatorname{Int} C(f), \forall g \in K.$$

The goal of this paper is to study the existence of solutions for operator equilibrium problems. The starting point of this paper was Theorem 2.1 obtained by Kazmi and Raouf in [4]. We will prove that the statement of the theorem holds even if we omit the hemicontinuity and $C(f)$-pseudo monotonicity conditions.

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The paper is organized as follows. In Section 2, we recall some necessary notions and introduce a new definition of upper semicontinuity. In Section 3, we obtain our main result, the existence of solutions for operator equilibrium problems by using the KKM theorem. In the final section, the existence of solutions for operator variational inequalities and operator minimization problems are derived.

We shall use the following notations. For any subset A of a topological space Z, by A^c we denote the complement of A in Z. By the notation $\overline{A}$ we mean the closure of A. The $\mathrm{co}\,(A)$ denotes the convex hull of a set A.

2. Preliminaries

Now we give some definitions and preliminary results needed in the later sections.

Denote $P := \cap_{f\in K} C(f)$.

Definition 1 (See [4]). *A mapping $G : K \to Z$ is said to be natural quasi P-convex if for any $f, g \in K$ and $\lambda \in [0,1]$,*

$$G\left(\lambda g + (1-\lambda) f\right) \in \mathrm{co}\left\{G(g), G(f)\right\} - P.$$

The following definition extends the notion of upper semicontinuity introduced by Luc [9] to cones depending on an operator.

Definition 2. *A mapping $G : K \to Z$ is said to be C_f-upper semicontinuous if for each $f_0 \in K$ there exists an open neighborhood U of f_0 such that for any $f \in U \cap K$ and $v \in \mathrm{Int}\, C\left(f\right)$*

$$G\left(f\right) \in G\left(f_0\right) + v - \mathrm{Int}\, C\left(f\right).$$

Definition 3 (See [2]). *Let B be a subset of K. A set-valued mapping $C : K \to 2^Z$ is said to have an closed graph with respect to B if for every net $\{f_\alpha\}_{\alpha\in\Gamma} \subset K$ and $\{z_\alpha\}_{\alpha\in\Gamma} \subset Z$ such that $z_\alpha \in C(f_\alpha)$, f_α converges to $f \in B$ with respect to the topology of pointwise convergence (w.r.t.p.c. for short) and z_α converges to $z \in Z$, then $z \in C(f)$.*

We introduce the notion of an h-closed graph.

Definition 4. *Let B be a subset of K. A set-valued mapping $C : K \to 2^Z$ is said to have an h-closed graph with respect to B if for every net $\{f_\alpha\}_{\alpha\in\Gamma} \subset K$ and $z \in C(f_\alpha)$, f_α converges to $f \in B$ w.r.t.p.c, then $z \in C(f)$.*

Remark 1. *If a set valued mapping $C : K \to 2^Y$ has a closed graph with respect to $B \subset K$, then C has an h-closed graph with respect B. The inverse relation does not take place in general. Indeed, let $C : K \to 2^Y$ be defined by $C\left(f\right) = D$, for every $f \in K$, where D is an open convex cone. Then C has an h-closed graph with respect to $B = K$, but the graph of mapping C is not closed with respect to $B = K$.*

Definition 5. *Let B be a convex compact (w.r.t.p.c.) subset of K. A mapping $F : K \times K \to Z$ is said to be coercive with respect to B, if there exists $g_0 \in B$ such that*

$$F(f, g_0) \in -\mathrm{Int}\, C(f), \forall f \in K \backslash B.$$

The starting point of this paper is the following theorem obtained by Kazmi and Raouf in [4].

Theorem 1. *Let $K \subset L(X,Y)$ be a nonempty closed convex set. Let $F : K \times K \to Y$ satisfying the following conditions:*

i) $F(f,f) = 0$ *for every* $f \in K$*;*

ii) $C(f)-$*pseudo monotone in the first argument;*

iii) hemicontinuous in the first argument;

iv) natural quasi $P-$*convex in the second argument;*

v) coercive with respect to the compact convex set $B \subset K$*;*

vi) for each $g \in K$*,* $F(g,\cdot)$ *is upper semicontinuous on* B*.*

If for each $f \in K$*, the graph of* $Y \backslash \{-C(f)\}$ *is closed with respect to* B*, then* (OEP) *has a solution.*

The incompleteness of article [4] is that upper semicontinuity of a vector valued mapping is not given. Upper semicontinuity of a vector valued mapping is not obvious; it depends on the definition of the ordering relation. For more details, see also [11] and [1]. We presume that they used the definition of upper semicontinuity defined for set-valued mappings, but upper semicontinuity of set-valued mappings becomes continuity for vector valued mappings.

3. Main result

This section is devoted to deriving some existence results for operator equilibrium problems. The techniques are based on KKM mappings and the KKM theorem stated below.

Definition 6 (See [3]). *Let K be a nonempty subset of a vector space E. A set-valued mapping $G : K \to 2^E$ is called a KKM-mapping, if for any finite subset $\{x_1, ..., x_n\}$ of K,* $\mathrm{co}(\{x_1, ..., x_n\}) \subset \cup_{i=1}^{n} G(x_i)$

Theorem 2 (See [3]). *Let E be a topological vector space, K a nonempty subset of E and $G : K \to 2^E$ a KKM map such that $G(x)$ is closed for each $x \in K$ and compact for at least one $x \in K$; then $\cap_{x \in K} G(x) \neq \emptyset$.*

Now we state our main result.

Theorem 3. *Let $K \subset L(X,Y)$ be a nonempty closed convex set. Let $F : K \times K \to Z$ satisfying the following conditions:*

i) $F(f,f) \notin -\operatorname{Int} C(f)$, $\forall f \in K$*;*

ii) natural quasi P-convex in the second argument;

iii) coercive with respect to a compact convex set $B \subset K$;

iv) C_f-upper semicontinuous in the first argument on B.

If the graph of $Z \backslash \{-\operatorname{Int} C(\cdot)\}$ is h-closed with respect to B, then (OEP) has a solution.

Proof. First, we note that the coercivity of F guarantees that if $f \in K$ is a solution of (OEP), then $f \in B$.

We define the set-valued mapping $S : K \to 2^B$ by

$$S(g) := \{f \in B : F(f,g) \notin -\operatorname{Int} C(f)\}.$$

First, we prove that S is a KKM mapping, that is, for any finite subset $\{g_1, g_2, ..., g_n\} \subset K$ if $g \in \operatorname{co}\{g_1, g_2, ..., g_n\}$, then $g \in \bigcup_{i=1}^{n} S(g_i)$. Let us suppose the contrary, i.e., that there exists a $g \notin \bigcup_{i=1}^{n} S(g_i)$, i.e.,

$$F(g, g_i) \in -\operatorname{Int} C(g), \forall i = \overline{1,n}.$$

Since F is natural quasi P-convex in the second argument, we have that there exist $\mu_i \in [0,1]$ such that $\sum_{i=1}^{n} \mu_i = 1$ and

$$\begin{aligned} F(g,g) &\in \sum_{i=1}^{n} \mu_i F(g, g_i) - P \subset -\sum_{i=1}^{n} \mu_i \operatorname{Int} C(g) - (\cap_{g \in K} C(g)) \\ &\subseteq -\operatorname{Int} C(g) - C(g) \subset -\operatorname{Int} C(g) \end{aligned}$$

contradicting assumption i).

Thus $\operatorname{co}\{g_1, g_2, ..., g_n\} \subseteq \bigcup_{i=1}^{n} S(g_i)$.

Next, we show that $S(g)$ is closed for every $g \in K$. For every net $\{f_\alpha\}_{\alpha \in \Gamma} \subset S(g)$, i.e.,

$$F(f_\alpha, g) \notin -\operatorname{Int} C(f_\alpha), \tag{1}$$

such that f_α converges (w.r.t.p.c.) to $f \in B$ we have $f \in S(g)$. Indeed, since F is C_f-upper semicontinuous on B in the first argument, we obtain that there exists an $\alpha_0 \in \Gamma$ such that for any $\alpha \geq \alpha_0$ and $v \in \operatorname{Int} C(f_\alpha)$

$$F(f_\alpha, g) \in F(f,g) + v - \operatorname{Int} C(f_\alpha). \tag{2}$$

We have to prove that

$$F(f,g) \notin -\operatorname{Int} C(f_\alpha) \text{ for all } \alpha \geq \alpha_0. \tag{3}$$

Assume the contrary, that there exists $\alpha_* > \alpha_0$ such that $F(f,g) \in -\operatorname{Int} C(f_{\alpha_*})$. Let $v = F(f,g)$ in (2)

$$F(f_{\alpha_*}, g) \in -v + v - \operatorname{Int} C(f_{\alpha_*}) = -\operatorname{Int} C(f_{\alpha_*})$$

contradicting (1).

Since the graph of $Z\backslash\{-\operatorname{Int} C(\cdot)\}$ is h-closed with respect to B from inclusion (3), it follows that $F(f,g) \notin -\operatorname{Int} C\,(f)$. Hence $S\,(g)$ is closed for every $g \in K$.

From the coercivity of F with respect to a compact convex set $B \subset K$ it follows that there exists $g_0 \in B$ such that $S(g_0) \subset B$. Consequently, $S(g_0)$ is compact (w.r.t.p.c.). Thus, Theorem 2 implies that $\cap_{g\in K} S(g) \neq \emptyset$ which means that (OEP) has a solution. ☐

Remark 2. *The obtained theorem gives a better existence result for* (OEP) *than Theorem 1. The hemicontinuity and the* $C\,(f)$*-pseudo monotonicity of mapping* F *in the first argument are not required and we ask for weaker conditions for closedness and assumption* i)*.*

4. Particular cases

In this section, we give some existence results for operator variational inequalities and operator minimization problems.

Let X and Y be Hausdorff topological vector spaces and $K \subset L(X,Y)$ a nonempty convex set. Let $C : K \to 2^Y$ be a set-valued mapping such that for each $f \in K$, $C(f)$ is a convex, closed cone with nonempty interior and the apex at the origin of Y. Let $< l, x >$ the value of operator $l \in L\,(X,Y)$ at $x \in X$ and let $T : K \to X$ be a given mapping.

The operator variational inequality (OVI) studied by Domokos and Kolumbán [2] is to find $f \in K$ such that

$$< g - f, T\,(f) > \notin \operatorname{Int} C(f), \forall g \in K.$$

The following corollary gives the existence result for (OVI).

Corollary 1. *Let* $K \subset L(X,Y)$ *be a nonempty closed convex set. Let* $T : K \to X$ *be a mapping. Assume that:*

i) there is a nonempty, convex, compact subset $B \subset K$ *such that there exists* $g_0 \in B$ *such that*

$$< g_0 - f, T\,(f) > \in \operatorname{Int} C(f), \forall f \in K\backslash B;$$

ii) for each $g, f_0 \in B$ *there exists an open neighborhood* U *of* f_0 *such that for any* $f \in U \cap K$ *and* $v \in \operatorname{Int} C\,(f)$

$$< g - f, T\,(f) > \in < g - f_0, T\,(f_0) > + v - \operatorname{Int} C\,(f)\,;$$

iii) the graph of $Y\backslash\{-\operatorname{Int} C(\cdot)\}$ *is h-closed with respect to* B*.*

Then (OVI) *has a solution.*

Proof. Let $F : K \times K \to Y$ be defined by $F(f,g) =< g - f, T(f) >$. Since $0 \in P$, we have

$$\begin{aligned} F(f, \lambda g_1 + (1-\lambda) g_2) &=< \lambda g_1 + (1-\lambda) g_2 - f, T(f) > \\ &=< \lambda g_1 + (1-\lambda) g_2 - \lambda f - (1-\lambda) f, T(f) > \\ &= \lambda < g_1 - f, T(f) > + (1-\lambda) < g_2 - f, T(f) > \\ &= \lambda F(f, g_1) + (1-\lambda) F(f, g_2) \\ &\in \operatorname{co}\{F(f,g_1), F(f,g_2)\} - P, \forall f, g_1, g_2 \in K, \forall \lambda \in [0,1]. \end{aligned}$$

From

$$F(f,f) =< f - f, T(f) >= 0 \notin -\operatorname{Int} C(f), \forall f \in K$$

we obtain that the assumptions of Theorem 3 are satisfied. Thus, Theorem 3 yields the conclusion. □

Let X, Y and Z be Hausdorff topological vector spaces and $K \subset L(X,Y)$ a nonempty convex set. Let $C : K \to 2^Z$ be a set-valued mapping such that for each $f \in K$, $C(f)$ is a convex, closed cone with nonempty interior and the apex at the origin of Z. Let $\phi : K \to Z$ be a given mapping.

The weak operator minimization problem (OMP) is to find $f \in K$ such that

$$\phi(f) - \phi(g) \notin -\operatorname{Int} C(f), \forall g \in K.$$

The existence result for (OMP) is the following:

Corollary 2. *Let $K \subset L(X,Y)$ be a nonempty closed convex set and $\phi : K \to Z$ a mapping. Assume that:*

i) $-\phi$ is natural quasi P-convex;

ii) there is a nonempty compact convex set $B \subset K$ such that there exists $g_0 \in B$ such that

$$\phi(f) - \phi(g_0) \in -\operatorname{Int} C(f), \forall f \in K \backslash B;$$

iii) ϕ is C_f-upper semicontinuous on B;

iv) the graph of $Z \backslash \{-\operatorname{Int} C(\cdot)\}$ is h-closed with respect to B.

Then (OMP) has a solution.

Proof. Let $F : K \times K \to Z$ be defined by $F(f,g) = \phi(f) - \phi(g)$. It can be easily checked that every assumption of Theorem 3 is satisfied. This completes the proof. □

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