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Compactly generated rectifiable spaces or paratopological groups*

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Abstract. A rectifiable space (or a paratopological group) G is compactly generated if $G = \langle K\rangle$ for some compact subset K of G. In this paper, we mainly discuss compactly generated rectifiable spaces or paratopological groups. The main results are that: (1) each σ-compact metrizable rectifiable space containing a dense compactly generated rectifiable subspace is compactly generated; (2) a metriable rectifiable space is compactly generated if and only if it is σ-compact and finitely generated modulo open sets; (3) any σ-compact paratopological group can be embedded as a closed paratopological subgroup in some compactly generated paratopological group. Finally, we consider generalized metric properties of compactly generated rectifiable spaces.

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1. Introduction

Recall that a *topological group* G is a group G with a (Hausdorff) topology such that the product map from $G \times G$ onto G is jointly continuous and the inverse map of G onto itself associating x^{-1} with arbitrary $x \in G$ is continuous. A *paratopological group* G is a group G with a topology such that the product maps of $G \times G$ into G is jointly continuous. A topological space G is said to be a *rectifiable space* [4] provided that there are a surjective homeomorphism $\varphi : G \times G \to G \times G$ and an element $e \in G$ such that $\pi_1 \circ \varphi = \pi_1$ and for every $x \in G$ we have $\varphi(x, x) = (x, e)$, where $\pi_1 : G \times G \to G$ is the projection to the first coordinate. If G is a rectifiable space, then φ is called a *rectification* on G. It is well known that rectifiable spaces and paratopological groups are all good generalizations of topological groups. It is easy to see that a topological group G with the neutral element e has a rectification $\varphi(x, y) = (x, x^{-1}y)$. However, there exists a paratopological group which is not a rectifiable space; Sorgenfrey line ([8, Example 1.2.2]) is such an example. Also, the 7-dimensional sphere S_7 is rectifiable but not a topological group [21, § 3]. In fact, it

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is even not a semitopological group, because each (locally) compact semitopological group is a topological group [7]. Further, it is easy to see that both paratopological groups and rectifiable spaces are homogeneous.

Recently, the study of rectifiable spaces has become an interesting topic in topological algebra, see [1, 11, 13, 14, 15, 16, 20, 21].

2. Preliminaries

The following theorem was announced for the first time in [4], and the readers can see the proof in [5, 11, 20].

Theorem 1 (see [4])**.** *A topological space G is rectifiable if and only if there exist an element $e \in G$ and two continuous maps $p : G^2 \to G$, $q : G^2 \to G$ such that for any $x \in G, y \in G$ the next identities hold:*

$$p(x, q(x, y)) = q(x, p(x, y)) = y, q(x, x) = e.$$

In fact, we can assume that $p = \pi_2 \circ \varphi^{-1}$ and $q = \pi_2 \circ \varphi$ in Theorem 1. If we fix a point $x \in G$, then $f_x, g_x : G \to G$ defined with $f_x(y) = p(x, y)$ and $g_x(y) = q(x, y)$, for each $y \in G$, are homeomorphisms. We denote f_x, g_x by $p(x, G), q(x, G)$, respectively.

If G is a rectifiable space, then we shall call the map p the multiplication on G. Moreover, sometimes we shall write $x \cdot y$ instead of $p(x, y)$ and $A \cdot B$ instead of $p(A, B)$ for any $A, B \subset G$. Therefore, $q(x, y)$ is an element such that $x \cdot q(x, y) = y$; since $x \cdot e = x \cdot q(x, x) = x$ and $x \cdot q(x, e) = e$, it follows that e is a right neutral element for G and $q(x, e)$ is a right inverse for x. Hence a rectifiable space G is a topological algebraic system with binary operations p, q, 0-ary operation e and identities as above. It is easy to see that this algebraic system need not satisfy the associative law about the multiplication operation p. Clearly, every topological loop is rectifiable.

If G is a rectifiable space (or a paratopological group) and $X \subset G$, then we use $\langle X \rangle$ to denote the smallest rectifiable subspace of G which contains X. A set X algebraically generates G if $G = \langle X \rangle$.

Recall that a rectifiable space G (a paratopological group) is:
(1) σ-compact if $G = \bigcup\{K_n : n \in \mathbb{N}\}$, where each K_n is compact, and
(2) compactly generated if $G = \langle K \rangle$ for some compact subset K of G.

Note 1. *(a): Obviously, each compactly generated rectifiable space is σ-compact. However, there exists a compactly generated paratopological group which is not σ-compact. Indeed, let X be an uncountable compact space, and let $AP(X)$ be a free Abelian paratopological group. Then $-X$ is closed discrete in $AP(X)$ [17], which implies that $AP(X)$ is not σ-compact. Moreover, $AP(X)$ is not a topological group.*

(b): There exists a countable, metrizable, and compactly generated paratopological group which is not a topological group. Indeed, let the rational number $\mathbb{Q}$ with the subspace topology of Sorgenfrey line. Then $\mathbb{Q}$ is a countable, metrizable paratopological group which is not a topological group. Put $\mathcal{S} = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$; then $Q = \langle \mathcal{S} \rangle$. Therefore, $\mathbb{Q}$ is compactly generated.

(c): Sorgenfrey line is not a compactly generated paratopological group since each compact subset of Sorgenfrey line is countable [2, 3.3.b].

All spaces considered in this paper are supposed to be T_1 and regular unless stated otherwise. The notation $\mathbb{N}$ denotes the set of all positive integer numbers. The letter e denotes the neutral element of a group or the right neutral element of a rectifiable space. Readers may refer to [2, 8, 10] for notations and terminology not explicitly given here.

3. Compactly generated rectifiable spaces

In this section, we mainly discuss compactly generated rectifiable spaces. Firstly, we give some technical lemmas.

Lemma 1 (see [9])**.** *Let $\{U_n : n \in \mathbb{N}\}$ be a local base at point e of a topological space G such that $\overline{U_{n+1}} \subset U_n$ for all $n \in \mathbb{N}$. Assume that $\{F_n : n \in \mathbb{N}\}$ is a sequence of subsets of G such that*

1. *each F_n is compact, and*
2. *$F_n \subset \overline{U_n}$.*

Then $K = \bigcup\{F_n : n \in \mathbb{N}\} \cup \{e\}$ is compact. Moreover, if each F_n is finite, then for each enumeration $i : \mathbb{N} \to K$ a sequence $\{i(n) : n \in \mathbb{N}\}$ converges to e.

Let A be a subspace of a rectifiable space G. Then A is called *a rectifiable subspace [14] of G* if we have $p(A, A) \subset A$ and $q(A, A) \subset A$.

Lemma 2 (see [14])**.** *Let G be a rectifiable space. If V is an open rectifiable subspace of G, then V is closed in G.*

Lemma 3. *Let H be a dense rectifiable subspace of a rectifiable space G. Then for each open rectifiable subspace E of H there exists an open rectifiable subspace E' of G such that $E' \cap H = E$.*

Proof. Let

$$E' = \bigcup\{V : V \text{ is open in } G \text{ and } \mathrm{cl}_G(V) \cap H \subset E\}.$$

Obviously, E' is an open subset of G and $E' \cap H = E$. Now, we shall prove that E' is a rectifiable subspace of G.

Indeed, suppose that $a, b \in E'$. It follows from the definition of E' that there exist open sets U and V in G such that $a \in U, b \in V, \mathrm{cl}_G(U) \cap H \subset E$ and $\mathrm{cl}_G(V) \cap H \subset E$. By the density of H in G, we have $\mathrm{cl}_G(U \cap H) = \mathrm{cl}_G(U)$ and $\mathrm{cl}_G(V \cap H) = \mathrm{cl}_G(V)$. Therefore, it follows from the continuity of p in G that

$$p(U, V) \subset p(\mathrm{cl}_G(U), \mathrm{cl}_G(V)) = p(\mathrm{cl}_G(U \cap H), \mathrm{cl}_G(V \cap H)) \subset \mathrm{cl}_G(p(U \cap H, V \cap H)).$$

Then we have $\mathrm{cl}_G(p(U, V)) = \mathrm{cl}_G(p(U \cap H, V \cap H))$, and

$$\mathrm{cl}_G(p(U, V)) \cap H = \mathrm{cl}_G(p(U \cap H, V \cap H)) \cap H \subset \mathrm{cl}_G(p(E, E)) \cap H = \mathrm{cl}_G(E) \cap H = E$$

since E is closed in H by Lemma 2. Therefore, $p(a, b) \in p(U, V) \subset E'$.

Suppose that $c, d \in E'$. Then there exist open sets O, W in G such that $c \in O, d \in W$, $\mathrm{cl}_G(O) \cap H \subset E$ and $\mathrm{cl}_G(W) \cap H \subset E$. Obviously, $q(O, W)$ is open in G. Moreover, it also easy to see that $\mathrm{cl}_G(q(O, W)) = \mathrm{cl}_G(q(O \cap H, W \cap H))$. Since

$$\mathrm{cl}_G(q(O, W)) \cap H = \mathrm{cl}_G(q(O \cap H, W \cap H)) \cap H \subset \mathrm{cl}_G(q(E, E)) \cap H \subset \mathrm{cl}_G(E) \cap H = E,$$

it follows that $q(c, d) \in q(O, W) \subset E'$. □

Corollary 1. *A dense rectifiable subspace of a connected rectifiable space has no proper open rectifiable subspaces.*

Proof. By Lemma 2, each open rectifiable subspace of a rectifiable space is closed, and therefore, a connected rectifiable space cannot have proper open rectifiable subspaces. Now the result follows from Lemma 3. □

Lemma 4 (see [14])**.** *Let G be a rectifiable space. If Y is a dense subset of G and U is an open neighborhood of the right neutral element e of G, then $G = Y \cdot U$.*

Theorem 2. *If a σ-compact metrizable rectifiable space G contains a dense compactly generated rectifiable subspace, then G is also compactly generated.*

Proof. Let H be a dense rectifiable subspace of G such that H is generated by some compact set E, and let $G = \bigcup\{K_n : n \in \mathbb{N}\}$, where each K_n is compact. Since G is metrizable, the point e has a countable local base $\{U_n : n \in \mathbb{N}\}$, where $\overline{U_{n+1}} \subset U_n$ for each $n \in \mathbb{N}$. By the density of H in G, it follows from Lemma 4 that $p(H, U_n) = G$ for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, there exists a finite subset F_n of H such that $K_n \subset p(F_n, U_n)$, and put $L_n = \overline{U_n} \cap q(F_n, K_n)$, then each $K_n \subset p(F_n, L_n)$ since $K_n \subset p(F_n, q(F_n, K_n))$. Obviously, each L_n is compact and, by Lemma 1, $L = \bigcup\{L_n : n \in \mathbb{N}\}$ is also compact. Therefore,

$$G = \bigcup\{K_n : n \in \mathbb{N}\} \subset \bigcup\{p(F_n, L_n) : n \in \mathbb{N}\} \subset \bigcup\{p(H, L_n) : n \in \mathbb{N}\} \subset p(H, L).$$

Since H is generated by E, G is generated by the compact set $E \cup L$. Therefore, G is compactly generated. □

Corollary 2. *If a σ-compact metrizable rectifiable space G contains a dense finitely generated rectifiable subspace, then G is also compactly generated.*

Next, we define the notion of finitely generated modulo open sets in rectifiable spaces which contains all compactly generated rectifiable spaces.

Definition 1. *We will say that a rectifiable space (or a paratopological group) G is finitely generated modulo open sets if for each non-empty open rectifiable subspace H of G there exists a finite subset F of G such that $G = \langle F \cup H \rangle$.*

Proposition 1. *Let G be a rectifiable space. Then the following conditions are equivalent:*

1. *G is finitely generated modulo open sets;*

2. for each non-empty open subset V of G there exists a finite subset F of G such that $G = \langle F \cup V\rangle$.

Proof. Obviously, (2)⇒(1).

(1)⇒(2). Let V be a non-empty open subset V of G. Let H be the rectifiable subspace generated by V, that is, $H = \langle V\rangle$. Obviously, H is open in G, and so by (2) there is a finite set $F \subset G$ such that

$$G = \langle F \cup H\rangle = \langle F \cup \langle V\rangle\rangle = \langle F \cup V\rangle.$$

□

Theorem 3. *If a rectifiable space G is compactly generated, then it is finitely generated modulo open sets.*

Proof. Assume that $G = \langle K\rangle$, where K is a compact set. Let H be an open rectifiable subspace of G. Then $\mathscr{H} = \{g \cdot H : g \in G\}$ is an open covering of G. Since K is compact, there exist finitely many elements of $\mathscr{H}$, say $g_1 \cdot H, \cdots , g_n \cdot H$, which cover K. Put $F = \{g_1, \cdots , g_n\}$. Then $G = \langle F \cup H\rangle$. □

Theorem 4. *Let G be a metrizable rectifiable space G and A a countable subset of G. Suppose that G is finitely generated modulo open sets. Then G contains a sequence $\mathcal{S}$ converging to e of G such that $A \subset \langle \mathcal{S}\rangle$.*

Proof. Let $A = \{a_n : n \in \omega\}$. Since G is metrizable, let $\{U_n : n \in \omega\}$ be a local base at e such that

$$G = U_0 \supseteq U_1 \supseteq \cdots \supseteq U_n \supseteq \cdots .$$

Since G is finitely generated modulo open sets, for each $n \in \omega$ we can fix a finite set F_n such that $G = \langle F_n \cup U_{n+1}\rangle$.

By induction on n, we will define a sequence $\{B_n : n \in \omega\}$ of finite subsets of G with the following properties:

(a) $B_n \subset U_n$;

(b) $G = \langle B_0 \cup B_1 \cup \cdots \cup B_n \cup U_{n+1}\rangle$, and

(c) $a_n \in \langle B_0 \cup B_1 \cup \cdots \cup B_n\rangle$.

To begin with, let $B_0 = F_0 \cup \{a_0\}$; then B_0 satisfies all three conditions (a)-(c). Assume that we have already defined finite sets $B_0, B_1, \cdots , B_{n-1}$ satisfying all three conditions (a)-(c). By (b), $F_n \cup \{a_n\} \subset \langle B_0 \cup B_1 \cup \cdots \cup B_{n-1} \cup U_n\rangle$. Since F_n is finite, we can find a finite set $B_n \subset U_n$ such that

$$F_n \cup \{a_n\} \subset \langle B_0 \cup B_1 \cup \cdots \cup B_{n-1} \cup B_n\rangle.$$

Clearly, (a)-(c) are satisfied.

Put $\mathcal{S} = \cup\{B_n : n \in \omega\}$. By (c), $A \subset \langle \mathcal{S}\rangle$. By Lemma 1 and (a), $\mathcal{S}$ can be enumerated as a sequence converging to e. □

Theorem 5. *Let G be a σ-compact metrizable rectifiable space G. Then G is compactly generated if and only if G is finitely generated modulo open sets.*

Proof. By Theorem 3, we only need to prove the sufficiency. Suppose that for each open rectifiable subspace H of G there exists a finite set F such that $G = \langle F \cup H \rangle$. Obviously, G is separable, and let D be a countable dense subset of G. By Theorem 4, G has a dense compactly generated rectifiable subspace, and by Theorem 2, G is compactly generated. □

Corollary 3. *A metrizable rectifiable space G is compactly generated if and only if G is σ-compact and finitely generated modulo open sets.*

A rectifiable space without proper open rectifiable subspaces trivially satisfies condition (2) of Proposition 1. Therefore, we have the following corollary.

Corollary 4. *A σ-compact metrizable rectifiable space G without proper open rectifiable subspaces is compactly generated.*

By Corollaries 1 and 4, we also have the following corollary.

Corollary 5. *A σ-compact dense rectifiable subspace of a connected metrizable rectifiable space G is compactly generated.*

Corollary 6. *A σ-compact connected metrizable rectifiable space G is compactly generated.*

By Theorems 3 and 4, it is easy to prove the following theorem.

Theorem 6. *A countable metrizable rectifiable space is compactly generated if and only if it is compactly generated by a sequence converging to the right neutral element e.*

4. Compactly generated paratopological groups

In this section, we mainly discuss compactly generated paratopological groups.

Lemma 5. *Let G be a paratopological group. If Y is a dense subset of G and U is an open neighborhood of the neutral element e of G, then $G = Y^{-1} \cdot U$.*

Proof. For arbitrary $g \in G$, since Y is a dense subset of G, we have $Ug^{-1} \cap Y \neq \emptyset$. Take $x \in Ug^{-1} \cap Y$. Then $g \in x^{-1}U \subset Y^{-1} \cdot U$. □

The proof of the following theorem is similar to that of Theorem 2.

Theorem 7. *If a σ-compact first-countable paratopological group G contains a dense compactly generated subgroup, then G is also compactly generated.*

Proof. Let H be a dense subgroup of G such that H is generated by some compact set E, and let $G = \bigcup\{K_n : n \in \mathbb{N}\}$, where each K_n is compact. Since G is first-countable, the point e has a countable local base $\{U_n : n \in \mathbb{N}\}$, where $\overline{U_{n+1}} \subset U_n$ for each $n \in \mathbb{N}$. By the density of H in G, it follows from Lemma 5 that $HU_n = G$ for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, there exists a finite subset F_n of H such that $K_n \subset F_nU_n$, and put $L_n = \overline{U_n} \cap (F_n)^{-1}K_n$, then each $K_n \subset F_nL_n$ since $K_n \subset$

$F_n(F_n)^{-1}K_n$. Obviously, each L_n is compact and, by Lemma 1, $L = \bigcup\{L_n : n \in \mathbb{N}\}$ is also compact. Therefore,

$$G = \bigcup\{K_n : n \in \mathbb{N}\} \subset \bigcup\{F_nL_n : n \in \mathbb{N}\} \subset \bigcup\{HL_n : n \in \mathbb{N}\} \subset HL.$$

Since H is generated by E, G is generated by the compact set $E \cup L$. Therefore, G is compactly generated. □

Note 2. *Under the class of paratopological groups, we can obtain all results from Proposition 1 to Theorem 5 and Corollary 4 to Theorem 6 in Section 3 by similar proofs. In fact, the respective counterparts also hold for first-countable paratopological groups and this condition is weaker than the metrizability.*

Since a compactly generated rectifiable space G is σ-compact, G has Souslin property, see [18] or [19]. Moreover, E.A. Reznichenko showed that every σ-compact Hausdorff paratopological group has Souslin property, see [2, Theorem 5.7.12]. However, the following question is still open.

Question 1. *Let G be a compactly generated paratopological group. Does G have Souslin property?*

Theorem 8. *Any σ-compact paratopological group G can be embedded as a closed paratopological subgroup in some compactly generated paratopological group.*

Proof. Let $\sigma\Pi = \sigma\Pi\{G_n : n \in \mathbb{Z}\}$ be the σ-product of copies of G with the topology induced from Tikhonov power $G^{\mathbb{Z}}$, where $\sigma\Pi$ is a σ-product with the neutral element e as a distinguished point. Then $\sigma\Pi$ is also a paratopological group. For each $n \in \mathbb{Z}$, let $i_n : G \to G_n$ be a topological isomorphism, and we can identify G_n with its image in $\sigma\Pi$ under the natural embedding. Suppose that $G = \bigcup\{K_n : n \in \mathbb{Z}\}$, where each K_n is compact. Let K denote the subspace $\bigcup_{n\in\mathbb{Z}} i_n(K_n)$ of the paratopological group $\sigma\Pi$. Since K is closed in the compact subspace $\Pi\{K_n : n \in \mathbb{Z}\}$ of the paratopological group $G^{\mathbb{Z}}$ under the natural embedding $\sigma\Pi \to G^{\mathbb{Z}}$, K is compact in $\sigma\Pi$.

The group $\mathbb{Z}$ of integers with the discrete topology acts on the paratopological group $\sigma\Pi$ by shifting coordinates: for $x = (x_n)_{n\in\mathbb{Z}} \in \sigma\Pi$ and $k \in \mathbb{Z}$, $k \cdot x$ is the element of $\sigma\prod$ whose n-th coordinate is x_{n+k}. Let G' denote the semidirect product $\sigma\Pi \rtimes \mathbb{Z}$. Assume 1_z is the smallest positive element of $\mathbb{Z}$. Then the space $K \cup \{1_z\}$ is a compact subspace of G' and $G' = \langle K \cup \{1_z\}\rangle$. Indeed, let $H = \langle K \cup \{1_z\}\rangle$ in G'. Clearly, $\mathbb{Z} \subset H$. Next, we shall prove that, for each $m \in \mathbb{Z}$, $G_m \subset H$. Take arbitrary $x \in G_m$. Then $i_m^{-1}(x) \in K_n$ for some $n \in \mathbb{Z}$. Let a be the element $(i_n i_m^{-1}(x), 0)$ and b the element $(e, m-n)$ of the semidirect product G'. Clearly, $a \in K \subset H$ and $b \in H$, and hence ba belongs to H. However, it is easy to see that $ba = x$. □

If G be countable, then each of the sets K_n can be assumed finite. A simple analysis of the topological structure of the space $K \cup \{1_z\}$ enables us to obtain

Theorem 9. *Any countable paratopological group G can be embedded as a closed paratopological subgroup in some paratopological group algebraically generated by a subspace homeomorphic to the one-point compactification $\partial\mathbb{N}$ of a countable discrete space.*

Question 2. *Can any σ-compact rectifiable space G be embedded as a closed rectifiable subspace in some compactly generated rectifiable space?*

5. Generalized metrizability properties of compactly generated rectifiable spaces

A closed mapping f is called *perfect* if each fiber is compact.

Proposition 2. *Suppose that F is a compact subspace of a rectifiable space G. Then the restriction p and q to the subspace $F \times G$ is a perfect and open mapping of $F \times G$ onto G.*

Proof. We firstly prove that the restriction p to the subspace $F \times G$ is a perfect and open mapping of $F \times G$ onto G.

Let $f : F \times G \to F \times G$ be defined by $f(x, y) = (x, p(x, y))$ for each $(x, y) \in F \times G$. Obviously, f is continuous, one-to-one, and $f(F \times G) = F \times G$. Moreover, $f^{-1}(x, y) = (x, q(x, y))$. Therefore, f^{-1} is also continuous. Thus f is a homeomorphism. For $i = 1, 2$, denote by π_i the projection of $F \times G$ onto the i-th factor. Since $p(x, y) = \pi_2(x, p(x, y)) = \pi_2 f(x, y)$ for all $x \in F$ and $y \in G$, p is the composition of f and π_2, that is, $p = \pi_2 \circ f$. Since F is compact, it follows from [8, Theorem 3.1.16] that π_2 is closed. Then p is closed since f is a homeomorphism and π_2 is closed. For each $y \in G$, $p^{-1}(y) = f^{-1}(F \times \{y\}) = \bigcup\{(x, q(x, y)) : x \in F\}$ is closed in the compact subspace $F \times q(F, y)$. Indeed, let $(x, q(z, y)) \in (F \times q(F, y)) \setminus p^{-1}(y)$, where $x, z \in F$. Then $q(x, y) \neq q(z, y)$, and thus there exist two open sets U and V in G such that $q(x, y) \in U$, $q(z, y) \in V$ and $U \cap V = \emptyset$. Since q is continuous, there exists an open neighborhood W of e such that $q(x \cdot W, y \cdot W) \subset U$ and $q(z \cdot W, y \cdot W) \subset V$. Then $(x \cdot W, V)$ is an open neighborhood of $(x, q(z, y))$. However, since $q(x \cdot w, y) \subset U$ for each $w \in W$, it follows that $(x \cdot W, V) \cap p^{-1}(y) = \emptyset$. Therefore, $p^{-1}(y)$ is closed in $F \times q(F, y)$, and thus it is compact. Then p is perfect.

Let O be an open subset of $F \times G$. Put $O' = \pi_1(O)$. For each $x \in O'$, let $U_x = \{y \in G : (x, y) \in O\}$; then O_x is open in G as the projection of the open subset $O \cap \pi_1^{-1}(x)$ of $\{x\} \times G$ onto the second factor. Therefore, $p(O) = \bigcup_{x \in O'} p(x, O_x)$ is open in G, which implies that p is an open mapping.

As for the mapping q, we only redefine the mapping f by $(x, y) = (x, q(x, y))$ for each $(x, y) \in F \times G$, and the rest of the proof is immediate. $\square$

Corollary 7. *Suppose that F is a compact subspace of a rectifiable space G, and that M is a closed subspace of G. Then $p(F, M)$ and $q(F, M)$ are all closed in G.*

Note 3. *Corollary 7 gives an affirmative answer to the following question. Recently, L.X. Peng and S.J. Guo [16] have also obtained Corollary 7. However, we prove Corollary 7 by a different method.*

Question 3 (see [15])**.** *Let G be a rectifiable. If F, P are compact and closed subsets of G, respectively, is $P \cdot F$ or $F \cdot P$ closed in G?*

Since the restriction of a perfect mapping to a closed subspace is again a perfect mapping, it follows from Corollary 7 and Proposition 2 that we have the following corollary.

Corollary 8. *Suppose that F is a compact subspace of a rectifiable space G, and that M is a closed subspace of G. Then the restriction p and q to the subspace $F \times M$ is a perfect mapping of $F \times M$ onto a closed subspace of G.*

A space G is of *countable tightness* if for each subset A of G and each point $x \in \mathrm{cl}(A)$ there exists a countable subset D of A such that $x \in \mathrm{cl}(D)$.

Theorem 10. *Suppose that F is a compact subspace of a rectifiable space G and that M is a closed subspace of G. Suppose also that both F and M have countable tightness. Then both spaces $p(F, M)$ and $q(F, M)$ have countable tightness, too.*

Proof. Since perfect mappings do not increase the tightness and the tightness of the product $F \times M$ is countable by [8, 3.12.8(a)], it follows from Corollary 8 that both spaces $p(F, M)$ and $q(F, M)$ have countable tightness, too. □

Theorem 11. *Suppose that F is a compact metrizable subspace of a rectifiable space G, and that M is a closed metrizable subspace of G. Then both spaces $p(F, M)$ and $q(F, M)$ are metrizable, too.*

Proof. By Corollary 7, $p(F, M)$ and $q(F, M)$ are closed in G. Since perfect mappings preserve the metrizability [8, Theorem 4.4.15], it follows from Corollary 8 that $p(F, M)$ and $q(F, M)$ are metrizable. □

A *network* for a space X is a collection $\mathscr{F}$ of subsets of X such that whenever $x \in U$ with U open, there exists $F \in \mathscr{F}$ with $x \in F \subset U$.

Theorem 12. *Let G be a rectifiable space, and let H be a rectifiable subspace of G compactly generated by a compact metrizable space F. Suppose further that $G = p(H, M)$, where M is a closed metrizable subspace of G. Then G is the union of a countable family of closed metrizable subspaces.*

Proof. By induction on n, we can define a sequence $\{A_n : n \in \omega\}$ of subsets of G such that:

(1) $A_0 = F \cup p(F, F) \cup q(F, F)$;

(2) $A_1 = p(A_0, A_0) \cup q(A_0, A_0)$;

(3) $A_n = p(A_{n-1}, A_{n-1}) \cup q(A_{n-1}, A_{n-1})$.

Obviously, each $p(A_n, A_n), q(A_n, A_n), A_n$ are compact. Since compact space with a countable network is metrizable [10], it follows from Theorem 11 that each A_n is also metrizable. Since $H = \langle F \rangle$, $H = \bigcup_{n \in \omega} A_n$. Since $G = p(H, M)$, it follows from Theorem 11 again that G is the union of a countable family of closed metrizable subspaces. □

A *neighborhood assignment* for a space X is a function φ from X to the topology of X such that $x \in \varphi(x)$ for each point $x \in X$. A space X is a *D-space*[6], if for any neighborhood assignment φ for X there is a closed discrete subset D of X such that $X = \bigcup_{d \in D} \varphi(d)$.

Corollary 9. *Let G be a rectifiable space, and let H be a rectifiable subspace of G compactly generated by a compact metrizable space F. Suppose further that $G = p(H, M)$, where M is a closed metrizable subspace of G. Then G is a D-space.*

Proof. It is well known that each metrizable space is a D-space. Hence M is a D-space, and then each $p(h, M)$ is a D-space, too. Since a countable infinite union of closed D-subspaces is D [3], it follows that $G = p(H, M) = \bigcup_{h\in H} p(h, M)$ is a D-space. $\square$

Recall that a space X has a *quasi-G_δ-diagonal* provided there is a sequence $\{\mathcal{G}(n) : n \in \mathbb{N}\}$ of collections of open subsets of X such that for any distinct points $x, y \in X$ there is a number n with $x \in st(x, \mathcal{G}(n)) \subset X \setminus \{y\}$.

Theorem 13. *Let G be a compactly generated Tychonoff rectifiable space, and $Y = bG \setminus G$ have locally quasi-G_δ-diagonal, where bG is a Hausdorff compactification of G. Then G satisfies one of the following conditions:*

(1) G is locally compact;

(2) G is separable and metrizable.

Proof. Suppose that G is nowhere locally compact. Since G is σ-compact, it follows from [14, Theorem 7.3] that G is separable and metrizable. $\square$

A space X is said to have a *regular G_δ-diagonal* if the diagonal $\Delta = \{(x, x) : x \in X\}$ can be represented as the intersection of the closures of a countable family of open neighborhoods of Δ in $X \times X$.

Since a rectifiable space with a countable pseudocharacter has a regular G_δ-diagonal [14] and a paracompact space with a G_δ-diagonal is submetrizable [10], we have the following proposition.

Proposition 3. *If G is a compactly generated rectifiable space with a countable pseudocharacter, then G is submetrizable.*

Proposition 4. *Let G be a compactly generated Tychonoff rectifiable space with a countable pseudocharacter, and let $Y = bG \setminus G$ be Lindelöf, where bG is a Hausdorff compactification of G. Then G is separable and metrizable.*

Proof. Since $Y = bG \setminus G$ is Lindelöf, G is countable type [12], and thus G is a p-space [1]. Then G is a σ-compact p-space with a G_δ-diagonal, hence it is separable and metrizable [10, Corollaries 3.8 and 3.20]. $\square$

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