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Second order parameter-uniform convergence for a finite difference method for a partially singularly perturbed linear parabolic system

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Abstract. A linear system of n second order differential equations of parabolic reaction-diffusion type with initial and boundary conditions is considered. The first k equations are singularly perturbed. Each of the leading terms of the first m equations, $m \leq k$, is multiplied by a small positive parameter and these parameters are assumed to be distinct. The leading terms of the next $k-m$ equations are multiplied by the same perturbation parameter ε_m. Since the components of the solution exhibit overlapping layers, Shishkin piecewise-uniform meshes are introduced, which are used in conjunction with a classical finite difference discretisation, to construct a numerical method for solving this problem. It is proved that in the maximum norm the numerical approximations obtained with this method are first order convergent in time and essentially second order convergent in the space variable, uniformly with respect to all of the parameters.
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1. Introduction

The following parabolic initial-boundary value problem is considered for a partially singularly perturbed linear system of second order differential equations

$$\frac{\partial \vec{u}}{\partial t} - E\frac{\partial^2 \vec{u}}{\partial x^2} + A\vec{u} = \vec{f}, \text{ on } \Omega, \ \vec{u} \text{ given on } \Gamma, \tag{1}$$

where $\Omega = \{(x,t) : 0 < x < 1, 0 < t \leq T\}$, $\overline{\Omega} = \Omega \cup \Gamma$, $\Gamma = \Gamma_L \cup \Gamma_B \cup \Gamma_R$ with $\vec{u}(0,t) = \vec{\phi}_L(t)$ on $\Gamma_L = \{(0,t) : 0 \leq t \leq T\}$, $\vec{u}(x,0) = \vec{\phi}_B(x)$ on $\Gamma_B = \{(x,0) : 0 \leq x \leq 1\}$, $\vec{u}(1,t) = \vec{\phi}_R(t)$ on $\Gamma_R = \{(1,t) : 0 \leq t \leq T\}$. Here, for all $(x,t) \in \overline{\Omega}$, $\vec{u}(x,t)$ and $\vec{f}(x,t)$ are column $n-$vectors, E and $A(x,t)$ are $n \times n$ matrices, $E = \mathrm{diag}(\vec{\varepsilon})$, $\vec{\varepsilon} = (\varepsilon_1, \cdots, \varepsilon_n)$ with the assumption that

$$0 < \varepsilon_1 < \varepsilon_2 \ldots < \varepsilon_m = \varepsilon_{m+1} = \ldots = \varepsilon_k < \varepsilon_{k+1} = \ldots = \varepsilon_n = 1. \tag{2}$$

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Problem (1) can also be written in the operator form

$$\vec{L}\vec{u} \;=\; \vec{f} \;\text{ on }\; \Omega, \;\; \vec{u} \;\text{ given }\; \text{on }\; \Gamma,$$

where the operator $\vec{L}$ is defined by

$$\vec{L} \;=\; I\frac{\partial}{\partial t} - E\frac{\partial^2}{\partial x^2} + A,$$

where I is the identity matrix. The reduced problem corresponding to (1) is defined by:
for $i = 1, ..., k$,

$$\frac{\partial u_{0i}}{\partial t}(x,t) + \sum_{j=1}^{n} a_{ij}(x,t)u_{0j}(x,t) = f_i(x,t), u_{0i} = u_i \;\text{ on }\; \Gamma_B,$$

for $i = k+1, ..., n$,

$$\frac{\partial u_{0i}}{\partial t}(x,t) - \frac{\partial^2 u_{0i}}{\partial x^2}(x,t) + \sum_{j=1}^{n} a_{ij}(x,t)u_{0j}(x,t) = f_i(x,t), u_{0i} = u_i \;\text{ on }\; \Gamma.$$

Since the components of the solution are weakly coupled, it follows from the ordering of the ε_i that they exhibit twin sets of overlapping layers on each of the two boundaries Γ_L and Γ_R. On each of these boundaries the layers are now described. A layer corresponding to ε_i is said to be a sublayer of a layer of the same component corresponding to ε_j, if $\varepsilon_i < \varepsilon_j$. Each of the components $u_1, \cdots, u_m$ has a layer of width $O(\sqrt{\varepsilon_m})$, while u_{m-1} has an additional sublayer of width $O(\sqrt{\varepsilon_{m-1}})$; u_{m-2} has additional sublayers of width $O(\sqrt{\varepsilon_{m-1}})$ and $O(\sqrt{\varepsilon_{m-2}})$; and so on. Finally, u_1 has additional sublayers of width $O(\sqrt{\varepsilon_{m-1}}), \cdots, O(\sqrt{\varepsilon_1})$. The components $u_{m+1}, \cdots, u_k$ have the same sets of layers as the component u_m. The components $u_{k+1}, \ldots, u_n$ have less severe boundary layers of width $O(\sqrt{\varepsilon_1}), \ldots, O(\sqrt{\varepsilon_m})$, in the sense that the layers occur in their derivatives and not the components.

For a general introduction to parameter - uniform numerical methods for singular perturbation problems see [1, 9, 12]. Parameter-uniform numerical methods for singularly perturbed problems of parabolic type are dealt with in [2, 3, 4, 8]. In [7] and [6], a method for a partially singularly perturbed system of two equations of reaction-diffusion type is considered. In [10], second order convergence of a numerical method for a partially singularly perturbed system of n equations of reaction - diffusion type is established. Paper [2] contains a parameter-uniform numerical method of second order convergence for solving a singularly perturbed system of parabolic type, whereas the earlier paper [3] gives a first order convergent method for the same type of problem.

Motivated by [10] and [2], a partially singularly perturbed parabolic problem of type (1) is considered in the present paper. A parameter - uniform numerical method, which is proved to be essentially second order convergent in space and first order convergent in time for this system is presented.

2. Solutions of the continuous problem

For all $(x,t) \in \overline{\Omega}$, it is assumed that the components $a_{ij}(x,t)$ of $A(x,t)$ satisfy the inequalities

$$a_{ii}(x,t) > \sum_{\substack{j\neq i\\ j=1}}^{n} |a_{ij}(x,t)| \text{ for } 1 \leq i \leq n, \text{ and } a_{ij}(x,t) \leq 0 \text{ for } i \neq j \tag{3}$$

and, for some α,

$$0 < \alpha < \min_{\substack{(x,t)\in\overline{\Omega}\\ 1\leq i\leq n}} \left(\sum_{j=1}^{n} a_{ij}(x,t)\right). \tag{4}$$

It is also assumed, without loss of generality, that

$$\sqrt{\varepsilon}_m \leq \frac{\sqrt{\alpha}}{6}. \tag{5}$$

The norms $\parallel \vec{V} \parallel = \max_{1\leq k\leq n} |V_k|$ for any n-vector $\vec{V}$, $\parallel y \parallel_D = \sup\{|y(x,t)| : (x,t) \in D\}$ for any scalar-valued function y and domain D, and $\parallel \vec{y} \parallel = \max_{1\leq k\leq n} \parallel y_k \parallel$ for any vector-valued function $\vec{y}$ are introduced. When $D = \overline{\Omega}$ or Ω, the subscript D is usually dropped. Further, $\vec{f}$ and A are assumed to be sufficiently smooth and sufficient compatibility conditions are assumed such that

$$\vec{u} \in C^4_\lambda(\overline{\Omega}) \text{ for } \vec{f}, A \in C^2_\lambda(\overline{\Omega}), \quad \lambda \in (0,1). \tag{6}$$

See Section 2 in [2] for compatibility results in detail. Here

$$C^k_\lambda(D) = \{u : \frac{\partial^{l+m} u}{\partial x^l \partial t^m} \in C^0_\lambda(D) \text{ for } l, m \geq 0 \text{ and } 0 \leq l + 2m \leq k\}.$$

It is assumed throughout the paper that all of the assumptions (3)–(6) of this section are fulfilled. Furthermore, C denotes a generic positive constant, which is independent of x, t and of all singular perturbation and discretization parameters. Inequalities between vectors are understood in the componentwise sense.

3. Standard analytical results

Here, in the following, we state the analytical results - the maximum principle and its consequence, the stability result, without proof. Proofs are similar to those found in [2]. The operator $\vec{L}$ satisfies the following maximum principle

Lemma 1. *Let assumptions* (3) - (6) *hold. Let* $\vec{\psi}$ *be any vector-valued function in the domain of* $\vec{L}$ *such that* $\vec{\psi} \geq \vec{0}$ *on* Γ. *Then* $\vec{L}\vec{\psi}(x,t) \geq \vec{0}$ *on* Ω *implies that* $\vec{\psi}(x,t) \geq \vec{0}$ *on* $\overline{\Omega}$.

Let $\tilde{A}(x,t)$ be any principal sub-matrix of $A(x,t)$ and $\vec{\tilde{L}}$ the corresponding operator. To see that any $\vec{\tilde{L}}$ satisfies the same maximum principle as $\vec{L}$, it suffices to observe that the elements of $\tilde{A}(x,t)$ satisfy *a fortiori* the same inequalities as those of $A(x,t)$.

Lemma 2. *Let assumptions* (3) - (6) *hold. If $\vec{\psi}$ is any vector-valued function in the domain of $\vec{L}$, then, for each i, $1 \leq i \leq n$ and $(x,t) \in \overline{\Omega}$,*

$$|\psi_i(x,t)| \leq \max\left\{ \| \vec{\psi} \|_\Gamma, \frac{1}{\alpha} \| \vec{L}\vec{\psi} \| \right\}.$$

Standard estimates of the derivatives of the solution of (1) are contained in the following lemma.

Lemma 3. *Let assumptions* (3) - (6) *hold and let $\vec{u}$ be the exact solution of (1). Then, for all $(x,t) \in \overline{\Omega}$ and for each $i = 1, \ldots, n$,*

$$|\frac{\partial^l u_i}{\partial t^l}(x,t)| \leq C(||\vec{u}||_\Gamma + \sum_{q=0}^{l} ||\frac{\partial^q \vec{f}}{\partial t^q}||), \ \ l = 0,1,2$$

$$|\frac{\partial^l u_i}{\partial x^l}(x,t)| \leq C\varepsilon_i^{\frac{-l}{2}}(||\vec{u}||_\Gamma + ||\vec{f}|| + ||\frac{\partial \vec{f}}{\partial t}||), \ \ l = 1,2$$

$$|\frac{\partial^l u_i}{\partial x^l}(x,t)| \leq C\varepsilon_i^{-1}\varepsilon_1^{\frac{-(l-2)}{2}}(||\vec{u}||_\Gamma + ||\vec{f}|| + ||\frac{\partial \vec{f}}{\partial t}|| + ||\frac{\partial^2 \vec{f}}{\partial t^2}|| + \varepsilon_1^{\frac{l-2}{2}}||\frac{\partial^{l-2} \vec{f}}{\partial x^{l-2}}||), \ \ l = 3,4$$

$$|\frac{\partial^l u_i}{\partial x^{l-1}\partial t}(x,t)| \leq C\varepsilon_i^{\frac{1-l}{2}}(||\vec{u}||_\Gamma + ||\vec{f}|| + ||\frac{\partial \vec{f}}{\partial t}|| + ||\frac{\partial^2 \vec{f}}{\partial t^2}||), \ \ l = 2,3.$$

Proof. The bound on $\vec{u}$ is an immediate consequence of Lemma 2. Differentiating (1) partially with respect to time once and twice, and applying Lemma 2, the bounds on $\frac{\partial \vec{u}}{\partial t}$ and $\frac{\partial^2 \vec{u}}{\partial t^2}$ are obtained. To bound $\frac{\partial u_i}{\partial x}$, for each $i = 1, ..., k$ and (x,t), consider an interval $I = [a, a + \sqrt{\varepsilon_i}]$, $a \geq 0$ such that $x \in I$.
Then for some y such that $a < y < a + \sqrt{\varepsilon_i}$ and $t \in (0,T]$

$$\frac{\partial u_i}{\partial x}(y,t) = \frac{u_i(a + \sqrt{\varepsilon_i}, t) - u_i(a,t)}{\sqrt{\varepsilon_i}}$$

$$|\frac{\partial u_i}{\partial x}(y,t)| \leq C\varepsilon_i^{\frac{-1}{2}}||\vec{u}||. \tag{7}$$

Then for any $x \in I$

$$\frac{\partial u_i}{\partial x}(x,t) = \frac{\partial u_i}{\partial x}(y,t) + \int_y^x \frac{\partial^2 u_i(s,t)}{\partial x^2} ds$$

$$\frac{\partial u_i}{\partial x}(x,t) = \frac{\partial u_i}{\partial x}(y,t) + \varepsilon_i^{-1} \int_y^x \left(\frac{\partial u_i(s,t)}{\partial t} - f_i(s,t) + \sum_{j=1}^{n} a_{ij}(s,t)u_j(s,t) \right) ds$$

$$|\frac{\partial u_i}{\partial x}(x,t)| \leq |\frac{\partial u_i}{\partial x}(y,t)| + C\varepsilon_i^{-1} \int_y^x (||\vec{u}||_\Gamma + ||\vec{f}|| + ||\frac{\partial \vec{f}}{\partial t}||) ds.$$

Using (7) in the above equation

$$|\frac{\partial u_i}{\partial x}(x,t)| \le C\varepsilon_i^{\frac{-1}{2}}(||\vec{u}||_\Gamma + ||\vec{f}|| + ||\frac{\partial \vec{f}}{\partial t}||).$$

For $i = k+1, ..., n$, choosing an interval $I = [a, a+s],\ a \ge 0,\ s > 0$ such that $x \in I$ and following the same steps above the bound for $|\frac{\partial u_i}{\partial x}(x,t)|$ is obtained.

Rearranging the terms in (1), it is easy to get

$$\begin{aligned}
|\frac{\partial^2 u_i}{\partial x^2}| &\le C\varepsilon_i^{-1}(||\vec{u}||_\Gamma + ||\vec{f}|| + ||\frac{\partial \vec{f}}{\partial t}||),\ \ i = 1, ..., m,\\
|\frac{\partial^2 u_i}{\partial x^2}| &\le C\varepsilon_m^{-1}(||\vec{u}||_\Gamma + ||\vec{f}|| + ||\frac{\partial \vec{f}}{\partial t}||),\ \ i = m+1, ..., k,\\
|\frac{\partial^2 u_i}{\partial x^2}| &\le C(||\vec{u}||_\Gamma + ||\vec{f}|| + ||\frac{\partial \vec{f}}{\partial t}||),\ \ i = k+1, ..., n.
\end{aligned}$$

Analogous steps are used to get the rest of the estimates. □

The Shishkin decomposition of the exact solution $\vec{u}$ of (1) is $\vec{u} = \vec{v} + \vec{w}$, where the smooth component $\vec{v}$ is the solution of

$$\vec{L}\vec{v} = \vec{f} \ \text{in}\ \Omega,\ \vec{v} = \vec{u}_0 \ \text{on}\ \Gamma \tag{8}$$

and the singular component $\vec{w}$ is the solution of

$$\vec{L}\vec{w} \ = \ \vec{0}\ \text{in}\ \Omega,\ \vec{w} = \vec{u} - \vec{v} \ \text{on}\ \Gamma. \tag{9}$$

Section 2 in [2] ensures that $\vec{v}, \vec{w} \in C^4_\lambda(\overline{\Omega})$. For convenience the left and right boundary layers of $\vec{w}$ are separated using the further decomposition $\vec{w} = \vec{w}^L + \vec{w}^R$, where $\vec{L}\vec{w}^L = \vec{0}$ on Ω, $\vec{w}^L = \vec{w}$ on Γ_L, $\vec{w}^L = \vec{0}$ on $\Gamma_B \cup \Gamma_R$ and $\vec{L}\vec{w}^R = \vec{0}$ on Ω, $\vec{w}^R = \vec{w}$ on Γ_R, $\vec{w}^R = \vec{0}$ on $\Gamma_L \cup \Gamma_B$.
Bounds on the smooth component and its derivatives are contained in

Lemma 4. *Let assumptions* (3)–(6) *hold. Then the smooth component* $\vec{v}$ *and its derivatives satisfy, for each* $(x,t) \in \overline{\Omega}$ *and* $i = 1, \dots, n$,

$(a)\ \ |\frac{\partial^l v_i}{\partial t^l}(x,t)| \le C$ for $l = 0, 1, 2$

$(b)\ \ |\frac{\partial^l v_i}{\partial x^l}(x,t)| \le C(1 + \varepsilon_i^{1-\frac{l}{2}})$ for $l = 0, 1, 2, 3, 4$

$(c)\ \ |\frac{\partial^l v_i}{\partial x^{l-1}\partial t}(x,t)| \le C$ *for* $l = 2, 3.$

Proof. The proof is as given in [2]. □

4. Improved estimates

The layer functions $B_i^L,\ B_i^R,\ B_i,\ i=1,\ \dots,\ m,$, associated with the solution $\vec{u}$, are defined on $[0,1]$ by

$$B_i^L(x)=e^{-x\sqrt{\alpha/\varepsilon_i}},\ B_i^R(x)=B_i^L(1-x),\ B_i(x)=B_i^L(x)+B_i^R(x).$$

The following elementary properties of these layer functions, for all $1\le i<j\le m$ and $0\le x<y\le 1$, should be noted:

$$B_i(x)=B_i(1-x), B_i^L(x)\ <\ B_j^L(x),\ \ B_i^L(x)>\ B_i^L(y),\ \ 0\ <\ B_i^L(x)\ \le\ 1,$$
$$B_i^R(x)\ <\ B_j^R(x),\ \ B_i^R(x)\ <\ B_i^R(y),\ \ 0\ <\ B_i^R(x)\ \le\ 1.$$

$B_i(x)$ is monotone decreasing for increasing $x\in[0,\frac{1}{2}]$.

$B_i(x)$ is monotone increasing for increasing $x\in[\frac{1}{2},1]$.

$$B_i(x)\le 2B_i^L(x) \text{ for } x\in[0,\frac{1}{2}],\ B_i(x)\le 2B_i^R(x) \text{ for } x\in[\frac{1}{2},1]. \tag{10}$$

$$B_i^L(2\sqrt{\frac{\varepsilon_i}{\alpha}}\ln N)=N^{-2}. \tag{11}$$

The points $x_{i,j}^{(s)}$ are now defined. They satisfy interesting ordering properties, which are established in the subsequent lemma.

Definition 1. *For B_i^L, B_j^L, each $i,j,\ \ 1\le i\ne j\le m$ and each $s, s>0$, the point $x_{i,j}^{(s)}$ is defined by*

$$\frac{B_i^L(x_{i,j}^{(s)})}{\varepsilon_i^s}=\frac{B_j^L(x_{i,j}^{(s)})}{\varepsilon_j^s}. \tag{12}$$

It is remarked that

$$\frac{B_i^R(1-x_{i,j}^{(s)})}{\varepsilon_i^s}=\frac{B_j^R(1-x_{i,j}^{(s)})}{\varepsilon_j^s}. \tag{13}$$

In the next lemma, the existence and uniqueness of the points $x_{i,j}^{(s)}$ are shown. Various properties are also established.

Lemma 5. *For all i,j, such that $1\le i<j\le m$ and $0<s\le 3/2$, the points $x_{i,j}^{(s)}$ exist, are uniquely defined and satisfy the following inequalities*

$$\frac{B_i^L(x)}{\varepsilon_i^s}>\frac{B_j^L(x)}{\varepsilon_j^s},\ \ x\in[0,x_{i,j}^{(s)}),\ \ \frac{B_i^L(x)}{\varepsilon_i^s}<\frac{B_j^L(x)}{\varepsilon_j^s},\ x\in(x_{i,j}^{(s)},1]. \tag{14}$$

Moreover,

$$x_{i,j}^{(s)}<x_{i+1,j}^{(s)},\ \text{if}\ \ i+1<j\ \ \text{and}\ \ x_{i,j}^{(s)}<x_{i,j+1}^{(s)},\ \ \text{if}\ \ i<j. \tag{15}$$

Also

$$x^{(s)}_{i,j} < 2s\sqrt{\frac{\varepsilon_j}{\alpha}} \ \ \textit{and} \ \ x^{(s)}_{i,j} \in (0, \frac{1}{2}) \ \text{ if } \ i < j. \tag{16}$$

Analogous results hold for the B^R_i, B^R_j *and the points* $1 - x^{(s)}_{i,j}$.

Proof. The proof is as given in [11]. □

Bounds on the singular components $\vec{w}^L$, $\vec{w}^R$ of $\vec{u}$ and their derivatives are contained in

Lemma 6. *Let assumptions* (3) - (6) *hold. Then there exist constants* C_1 *and* C_2 *such that, for each* $(x,t) \in \overline{\Omega}$,

for $i=1,\ \dots,\ m$,

$$|\frac{\partial^l w^L_i}{\partial t^l}(x,t)| \le C_1 B^L_m(x) + C_2\varepsilon_m(1 - B^L_m(x)), \text{ for } l = 0,1,2,$$

$$|\frac{\partial^l w^L_i}{\partial x^l}(x,t)| \le C \textstyle\sum_{q=i}^m \dfrac{B^L_q(x)}{\varepsilon_q^{l/2}}, \text{ for } l = 1,2,$$

$$|\frac{\partial^3 w^L_i}{\partial x^3}(x,t)| \le C \textstyle\sum_{q=1}^m \dfrac{B^L_q(x)}{\varepsilon_q^{3/2}}, \ \ |\dfrac{\partial^4 w^L_i}{\partial x^4}(x,t)| \le C\dfrac{1}{\varepsilon_i} \textstyle\sum_{q=1}^m \dfrac{B^L_q(x)}{\varepsilon_q},$$

for $i=m+1,\dots,\ k$,

$$|\frac{\partial^l w^L_i}{\partial t^l}(x,t)| \le C_1 B^L_m(x) + C_2\varepsilon_m(1 - B^L_m(x)), \text{ for } l = 0,1,2,$$

$$|\frac{\partial^l w^L_i}{\partial x^l}(x,t)| \le C\frac{B^L_m(x)}{\varepsilon_m^{l/2}}, \text{ for } l = 1,2, \tag{17}$$

$$|\frac{\partial^3 w^L_i}{\partial x^3}(x,t)| \le C \textstyle\sum_{q=1}^m \dfrac{B^L_q(x)}{\varepsilon_q^{3/2}}, \ \ |\dfrac{\partial^4 w^L_i}{\partial x^4}(x,t)| \le C\dfrac{1}{\varepsilon_m} \textstyle\sum_{q=1}^m \dfrac{B^L_q(x)}{\varepsilon_q},$$

for $i=k+1,\dots,\ n$,

$$|\frac{\partial^l w^L_i}{\partial t^l}(x,t)| \le C_2\varepsilon_m(1 - B^L_m(x)), \text{ for } l = 0,1,2,$$

$$|\frac{\partial^l w^L_i}{\partial x^l}(x,t)| \le C_1 B^L_m(x) + C_2\varepsilon_m(1 - B^L_m(x)), \text{ for } l = 1,2,$$

$$|\frac{\partial^3 w^L_i}{\partial x^3}(x,t)| \le C \textstyle\sum_{q=1}^m \dfrac{B^L_q(x)}{\varepsilon_q^{1/2}}, \quad |\dfrac{\partial^4 w^L_i}{\partial x^4}(x,t)| \le C \textstyle\sum_{q=1}^m \dfrac{B^L_q(x)}{\varepsilon_q}.$$

Analogous results hold for the w^R_i *and their derivatives.*

Proof. The lemma is to be proved by induction. The case for $n = 2$ and $m = 1$ is considered now. To obtain the bound for w^L_i $i = 1, 2$, define the functions

$\psi_1{}^{\pm}(x,t) \;=\; C_1 e^{\alpha t} B_1^L(x) + C_2\varepsilon_1 e^{\alpha t}(1 - B_1^L(x)) \;\pm\; w_1^L(x,t)$,
$\psi_2{}^{\pm}(x,t) \;=\; C_2\varepsilon_1 e^{\alpha t}(1-B_1^L(x)) \pm w_2^L(x,t)$. It is easy to find that $(\vec{L}\vec{\psi}^{\pm})_1(x,t) \geq 0$ and $(\vec{L}\vec{\psi}^{\pm})_2(x,t) \geq 0$ for the choice of C_1 and C_2 such that $C_2\alpha > C_1|a_{21}|$. Then for $C_1 > \max\limits_t |\phi_L(t)|$, the bound for w_1^L and w_2^L follows from Lemma 1.

Differentiating $(\vec{L}\vec{w}^L)_i(x,t) = 0$ for $i = 1, 2$ partially with respect to t and applying Lemma 2 for $\dfrac{\partial w_i^L}{\partial t}(x,t)$, the bound for $\dfrac{\partial w_i^L}{\partial t}(x,t)$ is obtained for the proper choice of C_1 and C_2.

Rearranging the equation of the system satisfied by $w_1^L(x,t)$ and $w_2^L(x,t)$ yields

$$\varepsilon_1 \frac{\partial^2 w_1^L}{\partial x^2}(x,t) = \frac{\partial w_1^L}{\partial t}(x,t) + \sum_{j=1}^{2} a_{ij}(x,t) w_j^L(x,t)$$

$$\frac{\partial^2 w_2^L}{\partial x^2}(x,t) = \frac{\partial w_2^L}{\partial t}(x,t) + \sum_{j=1}^{2} a_{ij}(x,t) w_j^L(x,t).$$

Then the bounds of $\dfrac{\partial w_i^L}{\partial t}(x,t)$ and $w_i^L(x,t),\ i = 1,2$ give the bound of $\dfrac{\partial^2 w_i^L}{\partial x^2}(x,t)$ for $i = 1, 2$. To bound $\dfrac{\partial w_i^L}{\partial x}(x,t)$, for $i = 1, 2$, consider an interval $I = [a, a+\sqrt{\varepsilon_1}] \subset [0,1]$, $a \geq 0$ and $x \in I$. Then for some y such that $a < y < a + \sqrt{\varepsilon_1}$, applying the mean value theorem yields

$$|\frac{\partial w_1^L}{\partial x}(y,t)| \leq C\varepsilon_1^{-\frac{1}{2}}||\vec{w}||. \tag{18}$$

Then for $x \in I$ such that $y < \eta < x$, $\dfrac{\partial w_1^L}{\partial x}(x,t) = \dfrac{\partial w_1^L}{\partial x}(y,t) + (x-y)\dfrac{\partial^2 w_1^L}{\partial x^2}(\eta,t)$.
Using (18) and the bound of $\dfrac{\partial^2 w_1^L}{\partial x^2}$ in the above $|\dfrac{\partial w_1^L}{\partial x}(x,t)| \leq C\varepsilon_1^{\frac{-1}{2}} B_1^L(x)$.
Considering the interval $I = [a, a+s] \subset [0,1]$, $s, a \geq 0$ and $x \in I$, the same arguments lead to the bound of $\dfrac{\partial w_2^L}{\partial x}(x,t)$.

Differentiating twice the equation satisfied by $w_i^L(x,t),\ i = 1,2$ with respect to t and rearranging, the bound of $(\vec{L}\dfrac{\partial^2 \vec{w}^L}{\partial t^2})_i(x,t)$ is obtained. Then Lemma 2 gives the bound of $\dfrac{\partial^2 w_i^L}{\partial t^2}(x,t),\ i = 1,2$ for a proper choice of C_1 and C_2 . Similarly, differentiating the equation satisfied by $w_i^L(x,t),\ i = 1,2$ with respect to x and using Lemma 2 the bound for $\dfrac{\partial^2 w_i^L}{\partial x \partial t}(x,t)$ follows. To bound $\dfrac{\partial^3 w_i^L}{\partial x^3}(x,t)$, the defining equation of $w_i^L(x,t),\ i = 1,2$ is differentiated with respect to x. Then using the bounds which are already obtained leads to the required bound. Similar steps lead to the bound of $\dfrac{\partial^4 w_i^L}{\partial x^4}(x,t)$.

It is worth to remark on the case $n = 3$, there may two subcases (i) $m = 1$ and $k = 2$, (ii) $m = k = 2$. In both cases, defining suitable barrier functions and using

arguments similar to those used for the case $n = 2$, one can get the lemma for $n = 3$. Assume the lemma to be true for a partially singularly perturbed system of $n-1$ equations, where k, $1 \leq k \leq n-2$, equations are singularly perturbed among which the first m equations have distinct perturbation parameters $\varepsilon_1 < \varepsilon_2 \ldots \varepsilon_{m-1} < \varepsilon_m$. Now consider $\vec{L}\vec{w}^L = \vec{0}$ on Ω, where $\vec{w}^L$ is an $n-$ vector and the corresponding $\varepsilon_i, 1 \leq i \leq n$, satisfy (2). Define
$\psi_i{}^{\pm}(x,t) \;=\; C_1 e^{\alpha t} B^L_m(x) + C_2 \varepsilon_m e^{\alpha t}(1 - B^L_m(x)) \;\pm\; w^L_i(x,t) \quad i = 1, ..., k,$
$\psi_i{}^{\pm}(x,t) \;=\; C_2 \varepsilon_m e^{\alpha t}(1 - B^L_m(x)) \;\pm\; w^L_i(x,t), \quad i = k+1, ..., n.$
Choosing $C_2 > \frac{C_1}{\alpha}|\sum_{\substack{j=1\\ i\neq j}}^{n} a_{ij}|$, it is found that $(\vec{L}\vec{\psi}^{\pm})_i(x,t) > 0$. So choosing $C_1 > \max_t |\phi_L(t)|$, the bound for $w^L_i(x,t),\ i = 1, ..., n$, follows from Lemma 1. To bound $\frac{\partial w^L_i}{\partial t}(x,t)$, the equation $(\vec{L}\vec{w}^L)_i \;=\; 0,\ i = 1, ..., n$ is differentiated partially with respect to t. Then Lemma 2 gives the result. Considering $I = [a, a+s]$ for $a, s \geq 0$, applying the mean value theorem it is not hard to prove the estimate $|\frac{\partial w^L_n}{\partial x}(x,t)| \leq C||\vec{w}||$. Rearranging the terms of the equation $(\vec{L}\vec{w}^L)_n = 0$ and using estimates w_i and $\frac{\partial w^L_n}{\partial t}$ found already,

$$|\frac{\partial^2 w^L_n}{\partial x^2}(x,t)| \leq C_1 B^L_m(x) + C_2 \varepsilon_m (1 - B^L_m(x))$$

Differentiating $(\vec{L}\vec{w}^L)_n = 0$ once and twice with respect x and using the relevant previously found estimates,

$$|\frac{\partial^l w^L_n}{\partial x^l}(x,t)| \leq C \sum_{q=1}^{m} \frac{B^L_q(x)}{\varepsilon_q^{(l-2)/2}}, \quad l = 3, 4.$$

Now the first $n-1$ equations satisfied by $\vec{\tilde{w}}^L$,

$$\frac{\partial \vec{\tilde{w}}^L}{\partial t} - \tilde{E}\frac{\partial^2 \vec{\tilde{w}}^L}{\partial x^2} + \tilde{A}\vec{\tilde{w}}^L = \vec{g}, \tag{19}$$

are considered. Here, $\tilde{E}$ and $\tilde{A}$ are the matrices obtained by deleting the last row and column from E, A, respectively, the components of $\vec{g}$ are $g_i = -a_{i,n} w^L_n$ for $1 \leq i \leq n-1$. It is noted that $|\frac{\partial^l g_i}{\partial x^l}(x,t)| \;=\; |\frac{\partial^l w^L_n}{\partial x^l}(x,t)|,\ \ l = 0, ..., 4$ are known. Boundary conditions for $\vec{\tilde{w}}^L$ are

$$\tilde{w}_i = w_i \text{ on } \Gamma, i = 1, ..., n-1. \tag{20}$$

Now decompose $\vec{\tilde{w}}^L$ into smooth and singular components $\vec{q}, \vec{r}$, respectively, where $\vec{L}\vec{q} = \vec{g},\ \vec{q} = \vec{\tilde{w}}_0$ on Γ, $\vec{L}\vec{r} = \vec{0},\ \vec{r} = \vec{\tilde{w}}^L - \vec{q}$ on Γ. Here $\vec{\tilde{w}}_0$ is the solution of the reduced problem of (19) and (20). Using the bounds of $\vec{g}$ and its derivatives, Lemma 4 leads to the bounds of $\vec{q}$ and its derivatives. To bound the singular component $\vec{r}$ and its derivatives, the following cases are considered.

Case 1 : All $n-1$ equations are singularly perturbed with $m = k = n-1$:

The estimates for $i = 1, ..., n-1$ are found in Lemma 4.3 of [2].

Case 2 : All $n-1$ equations are singularly perturbed with $m < k = n-1$:
With a slight modification in the barrier function used in Lemma 4.3 of [2], it is not hard to deduce the result.

Case 3 : The $n-1$ equations are partially singularly perturbed, where only the first m, $m \le k \le n-2$, equations have distinct perturbation parameters:

By induction, the estimates of $\vec{r}$ are obtained.
Combining the bounds for the derivatives of q_i and r_i, $i = 1, ..., n-1$, the bounds of $w_i^L(x,t)$, $i = 1, ...n-1$, and its derivatives follow. Recalling the bounds of the derivatives of w_n^L completes the proof of the lemma for the system of n equations. A similar proof of analogous results for the boundary layer functions w_i^R holds. □

In the following lemma, sharper estimates of the smooth component are presented.

Lemma 7. *Let assumptions* (3)–(6) *hold. Then the smooth component* $\vec{v}$ *of the solution* $\vec{u}$ *of* (1) *satisfies for all* $(x,t) \in \overline{\Omega}$

$$|\frac{\partial^l v_i}{\partial x^l}(x,t)| \le C\left(1 + \sum_{q=i}^{m} \frac{B_q(x)}{\varepsilon_q^{\frac{l}{2}-1}}\right) \text{ for } l = 0,1,2, \text{ and } i = 1, ..., n,$$

$$|\frac{\partial^3 v_i}{\partial x^3}(x,t)| \le C\left(1 + \sum_{q=i}^{m} \frac{B_q(x)}{\varepsilon_q^{1/2}}\right) \text{ for } i = 1, \cdots, k,$$

$$|\frac{\partial^3 v_i}{\partial x^3}(x,t)| \le C\left(1 + B_m(x)\right) \text{ for } i = k+1, \cdots, n.$$

Proof. Define barrier functions

$$\vec{\psi}^{\pm}(x,t) = C[1 + B_m(x)]\vec{e} \pm \frac{\partial^l \vec{v}}{\partial x^l}(x,t), \quad l = 0,1,2 \quad \text{and} \quad (x,t) \in \overline{\Omega}.$$

Using Lemma 4, it follows that, for a proper choice of C, with $\vec{v} = \vec{u}_0$ on Γ,

$$\begin{aligned} \psi_i^{\pm}(0,t) &= C \pm \frac{\partial^l v_i}{\partial x^l}(0,t) \ge 0, \\ \psi_i^{\pm}(1,t) &= C \pm \frac{\partial^l v_i}{\partial x^l}(1,t) \ge 0 \\ \psi_i^{\pm}(x,0) &= C[1 + B_m(x)] \pm \frac{\partial^l v_i}{\partial x^l} \ge 0 \end{aligned}$$

and $(\vec{L}\vec{\psi}^{\pm})_i(x,t) \ge 0$.
By Lemma 1

$$|\frac{\partial^l v_i}{\partial x^l}(x,t)| \le C[1 + B_m(x)] \text{ for } l = 0,1,2. \tag{21}$$

Using Lemma 4 and Lemma 6 and proceeding on the same lines as in Lemma 4.4 in [2], the required bound for $\frac{\partial^3 v_i}{\partial x^3}$, $i = 1, ..., n$ follows. □

5. The Shishkin mesh

A piecewise uniform Shishkin mesh with $M \times N$ mesh-cells is now constructed. Let $\Omega_t^M = \{t_k\}_{k=1}^M$, $\Omega_x^N = \{x_j\}_{j=1}^{N-1}$, $\overline{\Omega}_t^M = \{t_k\}_{k=0}^M$, $\overline{\Omega}_x^N = \{x_j\}_{j=0}^N$, $\Omega^{M,N} = \Omega_t^M \times \Omega_x^N$, $\overline{\Omega}^{M,N} = \overline{\Omega}_t^M \times \overline{\Omega}_x^N$ and $\Gamma^{M,N} = \Gamma \cap \overline{\Omega}^{M,N}$. The mesh $\overline{\Omega}_t^M$ is chosen to be a uniform mesh with M mesh-intervals on $[0,T]$. The mesh $\overline{\Omega}_x^N$ is a piecewise-uniform mesh on $[0,1]$ obtained by dividing $[0,1]$ into $2m+1$ mesh-intervals as follows

$$[0,\sigma_1] \cup \cdots \cup (\sigma_{m-1},\sigma_m] \cup (\sigma_m, 1-\sigma_m] \cup (1-\sigma_m, 1-\sigma_{m-1}] \cup \cdots \cup (1-\sigma_1, 1].$$

The m parameters σ_r, which determine the points separating the uniform meshes, are defined by $\sigma_0 = 0$, $\sigma_{m+1} = \frac{1}{2}$, $\sigma_m = \min\left\{\frac{1}{4}, 2\sqrt{\frac{\varepsilon_m}{\alpha}}\ln N\right\}$, and for $r = m-1, \ldots, 1$,

$$\sigma_r = \min\left\{\frac{r\sigma_{r+1}}{r+1}, 2\sqrt{\frac{\varepsilon_r}{\alpha}}\ln N\right\}. \tag{22}$$

Clearly

$$0 < \sigma_1 < \ldots < \sigma_m \leq \frac{1}{4}, \qquad \frac{3}{4} \leq 1-\sigma_m < \ldots < 1-\sigma_1 < 1.$$

Then, on the sub-interval $(\sigma_m, 1-\sigma_m]$ a uniform mesh with $\frac{N}{2}$ mesh-intervals is placed and on each of the sub-intervals $(\sigma_r, \sigma_{r+1}]$ and $(1-\sigma_{r+1}, 1-\sigma_r]$, $r = 0, 1, \ldots, m-1$, a uniform mesh of $\frac{N}{4m}$ mesh-intervals is placed. In practice, it is convenient to take

$$N = 2mq, \quad q \geq 3, \tag{23}$$

where m is the number of distinct singular perturbation parameters involved in problem (1). This construction leads to a class of 2^m piecewise uniform Shishkin meshes $\overline{\Omega}_x^N$ on $[0,1]$ and hence 2^m piecewise uniform Shishkin meshes $\overline{\Omega}^{M,N}$ on $[0,1] \times [0,T]$.

In particular, when all the parameters σ_r, $r = 1, ..., m$ are with left choice, the Shishkin mesh $\overline{\Omega}_x^N$ becomes a classical uniform mesh with transition parameters $\sigma_r = \frac{r}{4m}$, $r = 1, ..., m$ and with the stepsize N^{-1} from 0 to 1. The Shishkin mesh suggested here is different from the meshes in [3], [2] and [5]. This mesh has the features of an ideal Shishkin mesh that (i) when all the parameters have the left choice it is the classical uniform mesh and (ii) it is coarse in the outer region and becomes finer and finer towards the boundary. From the above construction it is clear that the transition points $\{\sigma_r, 1-\sigma_r\}_{r=1}^m$ are the only points at which the step-size can change and that it does not necessarily change at each of these points. The following notation is introduced: if $x_j = \sigma_r$, then $h_r^- = x_j - x_{j-1}, h_r^+ = x_{j+1} - x_j$, $J = \{\sigma_r : h_r^+ \neq h_r^-\}$. In general, for each point x_j in the mesh-interval $(\sigma_{r-1}, \sigma_r]$,

$$x_j - x_{j-1} = 4mN^{-1}(\sigma_r - \sigma_{r-1}). \tag{24}$$

Also, for $x_j \in (\sigma_m, \frac{1}{2}]$, $x_j - x_{j-1} = 2N^{-1}(1-2\sigma_m)$ and for $x_j \in (0, \sigma_1]$, $x_j - x_{j-1} = 4mN^{-1}\sigma_1$. Thus, for $1 \leq r \leq m-1$, the change in the step-size at the point $x_j = \sigma_r$ is

$$h_r^+ - h_r^- = 4mN^{-1}((r+1)d_r - rd_{r-1}), \tag{25}$$

where

$$d_r = \frac{r\sigma_{r+1}}{r+1} - \sigma_r \tag{26}$$

with the convention $d_0 = 0$. Notice that $d_r \geq 0$, that $\Omega^{M,N}$ is a classical uniform mesh mentioned above when $d_r = 0$ for all $r = 1 \ \dots \ m$ and, from (22), that

$$\sigma_r \leq C\sqrt{\varepsilon_r} \ln N, \quad 1 \leq r \leq m. \tag{27}$$

It follows from (24) and (27) that for $r = 1, \ \dots \ , m-1$,

$$h_r^- + h_r^+ \leq C\sqrt{\varepsilon_{r+1}} N^{-1} \ln N. \tag{28}$$

Also

$$\sigma_r = \frac{r}{s}\sigma_s, \text{ when } d_r = \cdots = d_s = 0, \ 1 \leq r < s \leq m. \tag{29}$$

The results in the following lemma are used later.

Lemma 8. *Assume that $d_r > 0$ for some $r, 1 \leq r \leq m$. Then the following inequalities hold*

$$B_r^L(1-\sigma_r) \leq B_r^L(\sigma_r) = N^{-2}. \tag{30}$$

$$x_{r-1,r}^{(s)} \ \leq \ \sigma_r - h_r^- \text{ for } 0 < s \leq 2, 1 < r \leq m. \tag{31}$$

$$B_q^L(\sigma_r - h_r^-) \leq CB_q^L(\sigma_r) \ \text{ for } \ 1 \leq r \leq q \leq m. \tag{32}$$

$$\frac{B_q^L(\sigma_r)}{\sqrt{\varepsilon_q}} \leq C\frac{1}{\sqrt{\varepsilon_r} \ln N} \ \text{ for } \ 1 \leq q \leq m, \ \ 1 \leq r \leq m. \tag{33}$$

Analogous results hold for B_r^R.

Proof. Using the definitions of $B_r^L(x)$ and σ_r, (30) follows.
By Lemma 5,

$$x_{r-1,r}^{(s)} < 2s\sqrt{\frac{\varepsilon_r}{\alpha}} = \frac{s\sigma_r}{\ln N} \leq \frac{\sigma_r}{2}.$$

Also, by (23) and (24),

$$h_r^- = 4mN^{-1}(\sigma_r - \sigma_{r-1}) = \frac{(\sigma_r - \sigma_{r-1})}{q} < \frac{\sigma_r}{2}.$$

It follows that $x_{r-1,r}^{(s)} + h_r^- \leq \sigma_r$ as required.
To verify (32), note from (24) that

$$h_r^- = 4mN^{-1}(\sigma_r - \sigma_{r-1}) \leq 4mN^{-1}\sigma_r = 2^3 mN^{-1}\sqrt{\frac{\varepsilon_r}{\alpha}} \ln N.$$

But

$$e^{2^3 mN^{-1}\sqrt{\frac{\varepsilon_r}{\alpha}}\ln N} \le (N^{\frac{1}{N}})^{8m} \le C.$$

Since $r \le q$,

$$\sqrt{\frac{\alpha}{\varepsilon_q}}h_r^- \le \sqrt{\frac{\varepsilon_r}{\varepsilon_q}}4mN^{-1}\sigma_r \le 8mN^{-1}\ln N\sqrt{\frac{\varepsilon_r}{\alpha}}.$$

It follows that

$$B_q^L(\sigma_r - h_r^-) = B_q^L,(\sigma_r)e^{\sqrt{\frac{\alpha}{\varepsilon_q}}h_r^-} \le CB_q^L(\sigma_r)$$

as required.

To verify (33), if $q \ge r$, the result is trivial. On the other hand, if $q < r$,

$$B_q^L(\sigma_r) = e^{-\sqrt{\frac{\alpha}{\varepsilon_q}}\sigma_r} = e^{-2\sqrt{\frac{\varepsilon_r}{\varepsilon_q}}\ln N} \le \frac{C}{\ln N}\sqrt{\frac{\varepsilon_q}{\varepsilon_r}},$$

where the inequality is obtained by using the result $e^{-t} \le \frac{1}{t}$ for all $t \ge 0$. □

6. The discrete problem

In this section, a classical finite difference operator with an appropriate Shishkin mesh is used to construct a numerical method for (1), which is shown later to be first order parameter-uniform in time and essentially second order parameter-uniform in the space variable.

The discrete initial-boundary value problem is now defined on any mesh by the finite difference method

$$D_t^-\vec{U} - E\delta_x^2\vec{U} + A\vec{U} = \vec{f} \text{ on } \Omega^{M,N}, \ \vec{U} = \vec{u} \text{ on } \Gamma^{M,N}. \tag{34}$$

This is used to compute numerical approximations to the exact solution of (1). It is assumed henceforth that the mesh is a Shishkin mesh, as defined in the previous section. Note that (34), can also be written in the operator form

$$\vec{L}^{M,N}\vec{U} \ = \ \vec{f} \text{ on } \Omega^{M,N}, \ \vec{U} = \vec{u} \text{ on } \Gamma^{M,N},$$

where

$$\vec{L}^{M,N} \ = \ ID_t^- - E\delta_x^2 + A$$

and D_t^-, δ_x^2, D_x^+ and D_x^- are the difference operators

$$\begin{aligned}
D_t^-\vec{U}(x_j,t_k) \ &= \ \frac{\vec{U}(x_j,t_k) - \vec{U}(x_j,t_{k-1})}{t_k - t_{k-1}},\\
\delta_x^2\vec{U}(x_j,t_k) \ &= \ \frac{D_x^+\vec{U}(x_j,t_k) - D_x^-\vec{U}(x_j,t_k)}{(x_{j+1} - x_{j-1})/2},\\
D_x^+\vec{U}(x_j,t_k) \ &= \ \frac{\vec{U}(x_{j+1},t_k) - \vec{U}(x_j,t_k)}{x_{j+1} - x_j},\\
D_x^-\vec{U}(x_j,t_k) \ &= \ \frac{\vec{U}(x_j,t_k) - \vec{U}(x_{j-1},t_k)}{x_j - x_{j-1}}.
\end{aligned}$$

For any function $\vec{Z}$ defined on the Shishkin mesh $\overline{\Omega}^{M,N}$, we define
$||\vec{Z}|| = \max_i \max_{j,k} |Z_i(x_j, t_k)|$.
The following discrete results are analogous to those for the continuous case.

Lemma 9. *Let assumptions* (3) - (6) *hold. Then, for any vector-valued mesh function* $\vec{\Psi}$*, the inequalities* $\vec{\Psi} \ \geq \ \vec{0}$ on $\Gamma^{M,N}$ *and* $\vec{L}^{M,N}\vec{\Psi} \ \geq \ \vec{0}$ *on* $\Omega^{M,N}$ *imply that* $\vec{\Psi} \geq \vec{0}$ *on* $\overline{\Omega}^{M,N}$.

Proof. Let i^*, j^*, k^* be such that $\Psi_{i^*}(x_{j^*}, t_{k^*}) = \min_i \min_{j,k} \Psi_i(x_j, t_k)$ and assume that the lemma is false. Then $\Psi_{i^*}(x_{j^*}, t_{k^*}) < 0$. From the hypotheses we have $j^* \neq 0, \ N$ and $\Psi_{i^*}(x_{j^*}, t_{k^*}) - \Psi_{i^*}(x_{j^*}, t_{k^*-1}) \leq 0$, $\Psi_{i^*}(x_{j^*}, t_{k^*}) - \Psi_{i^*}(x_{j^*-1}, t_{k^*}) \leq 0$, $\Psi_{i^*}(x_{j^*+1}, t_{k^*}) - \Psi_{i^*}(x_{j^*}, t_{k^*}) \geq 0$, so $D_t^- \Psi_{i^*}(x_{j^*}, t_{k^*}) \leq 0$, $\delta_x^2 \Psi_{i^*}(x_{j^*}, t_{k^*}) \ > 0$. It follows that

$$\begin{aligned} \left(\vec{L}^{M,N}\vec{\Psi}\right)_{i^*}(x_{j^*}, t_{k^*}) \ &= \ D_t^- \Psi_{i^*}(x_{j^*}, t_{k^*}) - \varepsilon_{i*}\delta_x^2 \Psi_{i^*}(x_{j^*}, t_{k^*}) \\ &\quad + \sum_{q=1}^{n} a_{i^*,\,q}(x_{j^*}, t_{k^*})\Psi_q(x_{j^*}, t_{k^*}) \ < \ 0, \end{aligned}$$

which is a contradiction, as required. □

An immediate consequence of this is the following discrete stability result.

Lemma 10. *Let assumptions* (3) - (6) *hold. Then, for any vector-valued mesh function* $\vec{\Psi}$ *on* $\overline{\Omega}^{M,N}$ *and* $i = 1, \ldots, n$,

$$|\Psi_i(x_j, t_k)| \ \leq \ \max\left\{ ||\vec{\Psi}||_{\Gamma^{M,N}}, \frac{1}{\alpha}||\vec{L}^{M,N}\vec{\Psi}|| \right\}.$$

Proof. Define the two functions

$$\vec{\Theta}^{\pm}(x_j, t_k) = \max\{||\vec{\Psi}||_{\Gamma^{M,N}}, \frac{1}{\alpha}||L^{\vec{M,N}}\vec{\Psi}||\}\vec{e} \pm \vec{\Psi}(x_j, t_k),$$

where $\vec{e} = (1, \ \ldots \ , 1)$. Using the properties of A it is not hard to verify that $\vec{\Theta}^{\pm} \geq \vec{0}$ on $\Gamma^{M,N}$ and $\vec{L}^{M,N}\vec{\Theta}^{\pm} \geq \vec{0}$ on $\Omega^{M,N}$. It follows from Lemma 9 that $\vec{\Theta}^{\pm} \geq \vec{0}$ on $\overline{\Omega}^{M,N}$. □

The following comparison principle will be used in the proof of the error estimate.

Lemma 11. *Assume that for each* $i = 1, \ \ldots \ , n$*, the vector-valued mesh functions* $\vec{\Phi}$ *and* $\vec{Z}$ *satisfy*

$$|Z_i| \leq \Phi_i \ \text{ on } \ \Gamma^{M,N} \ \text{ and } \ |(\vec{L}^{M,N}\vec{Z})_i| \leq (\vec{L}^{M,N}\vec{\Phi})_i \ \text{ on } \ \Omega^{M,N}.$$

Then, for each $i = 1, \ \ldots \ , n$,

$$|Z_i| \leq \Phi_i \ \text{ on } \ \overline{\Omega}^{M,N}.$$

Proof. Define the two mesh functions $\vec{\Psi}^{\pm}$ by

$$\vec{\Psi}^{\pm} = \vec{\Phi} \pm \vec{Z}.$$

Then, for each $i = 1, \ \ldots \ , n, \quad \Psi_i^{\pm}$ satisfies

$$\Psi_i^{\pm} \geq 0 \ \text{ on } \ \Gamma^{M,N} \ \text{ and } \ (\vec{L}^{M,N}\vec{\Psi}^{\pm})_i \geq 0 \ \text{ on } \ \Omega^{M,N}.$$

The result follows from an application of Lemma 9. □

7. The local truncation error

From Lemma 10, it is obvious that in order to bound the error $\vec{U} - \vec{u}$, it suffices to bound $\vec{L}^{M,N}(\vec{U} - \vec{u})$. But, for $(x_j, t_k) \in \Omega^{M,N}$, his expression satisfies

$$\begin{aligned}&\vec{L}^{M,N}(\vec{U} - \vec{u}) = \vec{L}^{M,N}(\vec{U}) - \vec{L}^{M,N}(\vec{u}) = \\ &\vec{f} - \vec{L}^{M,N}(\vec{u}) = \vec{L}(\vec{u}) - \vec{L}^{M,N}(\vec{u}) = (\vec{L} - \vec{L}^{M,N})\vec{u}.\end{aligned}$$

It follows that

$$\vec{L}^{M,N}(\vec{U} - \vec{u}) = (\frac{\partial}{\partial t} - D_t^-)\vec{u} - E(\frac{\partial^2}{\partial x^2} - \delta_x^2)\vec{u}.$$

Let $\vec{V}, \vec{W}^L, \vec{W}^R$ be the discrete analogues of $\vec{v}, \vec{w}^L, \vec{w}^R$, respectively. Then for each $i = 1, \ \ldots \ , n,$

$$|(\vec{L}^{M,N}(\vec{V} - \vec{v}))_i| \leq |(\frac{\partial}{\partial t} - D_t^-)v_i| + |\varepsilon_i(\frac{\partial^2}{\partial x^2} - \delta_x^2)v_i|, \tag{35}$$

$$|(\vec{L}^{M,N}(\vec{W^L} - \vec{w}^L))_i| \leq |(\frac{\partial}{\partial t} - D_t^-)w_i^L| + |\varepsilon_i(\frac{\partial^2}{\partial x^2} - \delta_x^2)w_i^L|, \tag{36}$$

$$|(\vec{L}^{M,N}(\vec{W^R} - \vec{w}^R))_i| \leq |(\frac{\partial}{\partial t} - D_t^-)w_i^R| + |\varepsilon_i(\frac{\partial^2}{\partial x^2} - \delta_x^2)w_i^R|. \tag{37}$$

Thus, the smooth and singular components of the local truncation error can be treated separately. Note that for any smooth function ψ and for each $(x_j, t_k) \in \Omega^{M,N}$, the following distinct estimates of the local truncation error hold:

$$|(\frac{\partial}{\partial t} - D_t^-)\psi(x_j, t_k)| \ \leq \ C(t_k - t_{k-1}) \max_{s \ \in \ [t_{k-1}, t_k]} |\frac{\partial^2 \psi}{\partial t^2}(x_j, s)|, \tag{38}$$

$$|(\frac{\partial}{\partial x} - D_x^+)\psi(x_j, t_k)| \ \leq \ C(x_{j+1} - x_j) \max_{s \ \in \ [x_j, x_{j+1}]} |\frac{\partial^2 \psi}{\partial x^2}(s, t_k)|, \tag{39}$$

$$|(\frac{\partial^2}{\partial x^2} - \delta_x^2)\psi(x_j, t_k)| \ \leq \ C \max_{s \ \in \ I_j} |\frac{\partial^2 \psi}{\partial x^2}(s, t_k)|, \tag{40}$$

$$|(\frac{\partial^2}{\partial x^2} - \delta_x^2)\psi(x_j, t_k)| \ \leq \ C(x_{j+1} - x_{j-1}) \max_{s \in I_j} |\frac{\partial^3 \psi}{\partial x^3}(s, t_k)|. \tag{41}$$

Furthermore, if $x_j \notin J$, then

$$|(\frac{\partial^2}{\partial x^2} - \delta_x^2)\psi(x_j, t_k)| \ \leq \ C(x_{j+1} - x_{j-1})^2 \max_{s \in I_j} |\frac{\partial^4 \psi}{\partial x^4}(s, t_k)|. \tag{42}$$

Here $I_j = [x_{j-1}, x_{j+1}]$.

8. Error estimate

The proof of the error estimate is broken down into two parts. In the first, a theorem concerning the smooth part of the error is proved. Then the singular part of the error is considered. A barrier function is now constructed, which is used in both parts of the proof.
For each $x_j = \sigma_r \in J$, introduce a piecewise linear polynomial θ_r on $\overline{\Omega}$, defined by

$$\theta_r(x) = \begin{cases} \dfrac{x}{\sigma_r}, & 0 \le x \le \sigma_r. \\ 1, & \sigma_r < x < 1-\sigma_r. \\ \dfrac{1-x}{\sigma_r}, & 1-\sigma_r \le x \le 1. \end{cases}$$

It is not hard to verify that for any $x_j \in \Omega^{M,N}$

$$(\vec{L}^{M,N}\theta_r\vec{e})_i(x_j) \ge \begin{cases} \alpha\theta_r(x_j), & \text{if } x_j \notin J \\ \alpha + \dfrac{2\varepsilon_i}{\sigma_r(h_r^- + h_r^+)}, & \text{if } x_j \in J,\ x_j \in \{\sigma_r, 1-\sigma_r\}, \end{cases} \tag{43}$$

where $\vec{e}$ is a unit - column n- vector.
Now, define the barrier function $\vec{\Phi}$ by

$$\vec{\Phi}(x_j, t_k) = C[M^{-1} + (N^{-1}\ln N)^2(1 + \sum_{\{r:\ \sigma_r \in J\}} \theta_r(x_j))]\vec{e}, \tag{44}$$

where C is any sufficiently large constant.
Then, on $\Omega^{M,N}$, $\vec{\Phi}$ satisfies

$$0 \le \Phi_i(x_j, t_k) \le C(M^{-1} + (N^{-1}\ln N)^2), \quad 1 \le i \le n. \tag{45}$$

Also, for $x_j \notin J$,

$$(\vec{L}^{M,N}\vec{\Phi})_i(x_j, t_k) \ge C(M^{-1} + (N^{-1}\ln N)^2) \tag{46}$$

and for $x_j \in J,\ x_j \in \{\sigma_r, 1-\sigma_r\}$, using (27), (28) and (43),

$$(\vec{L}^{M,N}\vec{\Phi})_i(x_j, t_k) \ge C(M^{-1} + (N^{-1}\ln N)^2 + \frac{\varepsilon_i}{\sqrt{\varepsilon_r\varepsilon_{r+1}}}N^{-1}). \tag{47}$$

The following theorem gives the estimate for the smooth component of the error.

Theorem 1. *Let assumptions* (3) - (6) *hold. Let $\vec{v}$ denote the smooth component of the exact solution from (1) and $\vec{V}$ the smooth component of the discrete solution from (34). Then*

$$||\vec{V} - \vec{v}|| \le C(M^{-1} + (N^{-1}\ln N)^2). \tag{48}$$

Proof. By the comparison principle in Lemma 11 it suffices to show that for all i, j, k and some C,

$$|(\vec{L}^{M,N}(\vec{V} - \vec{v}))_i(x_j, t_k)| \le (\vec{L}^{M,N}\vec{\Phi})_i(x_j, t_k). \tag{49}$$

For each mesh point x_j there are two possibilities: either $x_j \notin J$ or $x_j \in J$.
If $x_j \notin J$, apply Lemma 4(a) with $l = 2$ and (38) to the t-derivative and apply Lemma 4(b) with $l = 4$ and (42) to the x- derivative to get

$$\begin{aligned} |(\vec{L}^{M,N}(\vec{V}-\vec{v}))_i(x_j,t_k)| &\le C[t_k - t_{k-1} + (x_{j+1}-x_{j-1})^2] \\ &\le C(M^{-1} + (N^{-1}\ln N)^2). \end{aligned} \tag{50}$$

Then (46) and (50) imply (49).
On the other hand, if $x_j \in J$, then $x_j \in \{\sigma_r, 1-\sigma_r\}$, for some $r, 1 \le r \le m$. Here the argument for $x_j = \sigma_r$ is given. For $x_j = 1-\sigma_r$ it is analogous.
If $x_j = \sigma_r \in J$, apply Lemma 4(a) with $l = 2$ and (38) to the t-derivative, and apply Lemma 7 with $l = 3$ and (41) to the x- derivative to get

$$|(\vec{L}^{M,N}(\vec{V}-\vec{v}))_i(x_j,t_k)| \le C[t_k - t_{k-1} + \varepsilon_i(x_{j+1}-x_{j-1})(1+\sum_{q=i}^{m}\frac{B_q(x_{j-1})}{\sqrt{\varepsilon_q}})].$$

So, since $x_{j-1} = \sigma_r - h_r^-$,

$$|(\vec{L}^{M,N}(\vec{V}-\vec{v}))_i(x_j,t_k)| \le C[M^{-1} + \varepsilon_i N^{-1}(1+\sum_{q=i}^{m}\frac{B_q(\sigma_r - h_r^-)}{\sqrt{\varepsilon}_q})]. \tag{51}$$

For each r, $1 \le r \le m$ there are at most two possibilities: either $i \ge r$ or $i \le r-1$.
If $i \ge r$, then $\sum_{q=i}^{m}\frac{B_q(\sigma_r-h_r^-)}{\sqrt{\varepsilon_q}} \le \frac{C}{\sqrt{\varepsilon_i}} \le \frac{C}{\sqrt{\varepsilon_r}}$. Substituting this into (51) gives

$$|(\vec{L}^{M,N}(\vec{V}-\vec{v}))_i(x_j,t_k)| \le C[M^{-1} + \frac{\varepsilon_i}{\sqrt{\varepsilon_r}}N^{-1}]. \tag{52}$$

(47) and (52) imply (49).
If $i \le r-1$, which arises only if $r \ge 1$, there are two possibilities: either $d_r > 0$ or $d_r = 0$ and $d_{r-1} > 0$, because the case $d_r = d_{r-1} = 0$ cannot occur for $x_j = \sigma_r \in J$. Since $x_{j-1} = \sigma_r - h_r^-$ and $\sigma_r - h_r^- < \frac{1}{2}$, $B_q(x_{j-1}) = B_q(\sigma_r - h_r^-) = B_q^L(\sigma_r - h_r^-) + B_q^R(\sigma_r - h_r^-) \le 2B_q^L(\sigma_r - h_r^-)$. Then $\sum_{q=i}^{m}\frac{B_q(\sigma_r-h_r^-)}{\sqrt{\varepsilon_q}} \le 2\sum_{q=i}^{m}\frac{B_q^L(\sigma_r-h_r^-)}{\sqrt{\varepsilon_q}}$.
If $d_r > 0$, then using (14) in Lemma 5 and (31) in Lemma 8 gives $\frac{B_q^L(\sigma_r-h_r^-)}{\sqrt{\varepsilon_q}} \le \frac{B_r^L(\sigma_r-h_r^-)}{\sqrt{\varepsilon_r}}$ for $1 \le q \le r$. Hence $\sum_{q=i}^{m}\frac{B_q(\sigma_r-h_r^-)}{\sqrt{\varepsilon_q}} \le \frac{C}{\sqrt{\varepsilon_r}}$. Substituting this into (51) gives

$$|(\vec{L}^{M,N}(\vec{V}-\vec{v}))_i(x_j,t_k)| \le C[M^{-1} + \frac{\varepsilon_i}{\sqrt{\varepsilon}_r}N^{-1}]. \tag{53}$$

(47) and (53) imply (49).
If $d_r = 0$ and $d_{r-1} > 0$, then using (14) and the fact that $\sigma_r - h_r^- \ge \sigma_{r-1} \ge x_{q,r-1}$, $1 \le q \le r-2$ give $\frac{B_q^L(\sigma_r-h_r^-)}{\sqrt{\varepsilon_q}} \le \frac{B_{r-1}^L(\sigma_r-h_r^-)}{\sqrt{\varepsilon_{r-1}}}$ for $1 \le q \le r-1$. Hence

$$\begin{aligned} \sum_{q=i}^{m}\frac{B_q^L(\sigma_r - h_r^-)}{\sqrt{\varepsilon_q}} \le C\sum_{q=r-1}^{m}\frac{B_q^L(\sigma_{r-1})}{\sqrt{\varepsilon_q}} &\le C[\tfrac{B_{r-1}^L(\sigma_{r-1})}{\sqrt{\varepsilon_{r-1}}} + \tfrac{1}{\sqrt{\varepsilon_r}}] \\ &\le C[\tfrac{N^{-2}}{\sqrt{\varepsilon_{r-1}}} + \tfrac{1}{\sqrt{\varepsilon_r}}]. \end{aligned}$$

Substituting this into (51) gives

$$\begin{aligned}|(L^{M,N}(\vec{V}-\vec{v}))_i(x_j,t_k)| &\le C[M^{-1}+\tfrac{\varepsilon_i}{\sqrt{\varepsilon_r}}N^{-1}+\tfrac{\varepsilon_i}{\sqrt{\varepsilon_{r-1}}}N^{-3}]\\ &\le C[M^{-1}+\tfrac{\varepsilon_i}{\sqrt{\varepsilon_r}}N^{-1}].\end{aligned} \tag{54}$$

(47) and (54) imply (49). This completes the proof. □

In order to estimate the singular component of the error the following four lemmas are required.

Lemma 12. *Assume that $x_j \notin J$. Let assumptions* (3) - (6) *hold. Then, on $\Omega^{M,N}$, for each $1 \le i \le n$, the following estimates hold*

$$|(\vec{L}^{M,N}(\vec{W}^L-\vec{w}^L))_i(x_j,t_k)| \le C(M^{-1}+\frac{(x_{j+1}-x_{j-1})^2}{\varepsilon_1}). \tag{55}$$

An analogous result holds for $\vec{W}^R-\vec{w}^R$.

Proof. Since $x_j \notin J$, from (42) and Lemma 6, it follows that

$$\begin{aligned}|(\vec{L}^{M,N}(\vec{W}^L-\vec{w}^L))_i(x_j,t_k)| &= |(((\tfrac{\partial}{\partial t}-D_t^-)-E(\tfrac{\partial^2}{\partial x^2}-\delta_x^2))\vec{w}^L)_i(x_j,t_k)|\\ &\le C(M^{-1}+(x_{j+1}-x_{j-1})^2\max_{s\,\in\, I_j}\sum_{q\,=\,1}^{m}\frac{B_q^L(s)}{\varepsilon_q})\\ &\le C(M^{-1}+\tfrac{(x_{j+1}-x_{j-1})^2}{\varepsilon_1}),\end{aligned}$$

as required. □

The following decompositions of the singular components w_i^L are used in the next lemma with $d_r > 0$ for some r, $1 \le r \le m$.

$$w_i^L = \sum_{l=1}^{r+1} w_{i,l}, \tag{56}$$

where the components $w_{i,l}$ are defined by

$$w_{i,r+1} = \begin{cases} p_i^{(s)} & \text{on } [0, x_{r,r+1}^{(s)}) \\ w_i^L & \text{otherwise}\end{cases}$$

and for each l, $r \ge l \ge 2$,

$$w_{i,l} = \begin{cases} p_i^{(s)} & \text{on } [0, x_{l-1,l}^{(s)}) \\ w_i^L - \displaystyle\sum_{q=l+1}^{r+1} w_{i,q} & \text{otherwise}\end{cases}$$

and

$$w_{i,1} = w_i^L - \sum_{q=2}^{r+1} w_{i,q} \quad \text{on } [0,1].$$

Here the polynomials $p_i^{(s)}$, for $s = 3/2$ and $s = 1$, are defined by

$$p_i^{(3/2)}(x,t) = \sum_{q=0}^{3} \frac{\partial^q w_i^L}{\partial x^q}(x_{r,r+1}^{(3/2)}, t)\frac{(x - x_{r,r+1}^{(3/2)})^q}{q!}$$

and

$$p_i^{(1)}(x,t) = \sum_{q=0}^{4} \frac{\partial^q w_i^L}{\partial x^q}(x_{r,r+1}^{(1)}, t)\frac{(x - x_{r,r+1}^{(1)})^q}{q!}.$$

Notice that decomposition (56) depends on the choice of the polynomials $p_i^{(s)}$ and that the $x_{i,j}^{(s)}$ are defined by (12). The following lemma provides estimates of the derivatives of the components in decomposition (56).

Lemma 13. *Assume that $d_r > 0$ for some r, $1 \le r \le m$. Let assumptions* (3) - (6) *hold. Then, for each l and r, $1 \le l \le r$, and all $(x_j, t_k) \in \Omega^{M,N}$, the components in the decomposition* (56) *satisfy the following estimates*
for each $1 \le i \le m$,

$$\begin{aligned}
|\frac{\partial^2 w_{i,l}}{\partial x^2}(x_j,t_k)| &\le C\min\{\frac{1}{\varepsilon_l}, \frac{1}{\varepsilon_i}\}B_l^L(x_j),\\
|\frac{\partial^3 w_{i,l}}{\partial x^3}(x_j,t_k)| &\le C\min\{\frac{1}{\varepsilon_i\sqrt{\varepsilon_l}}, \frac{1}{\varepsilon_l^{3/2}}\}B_l^L(x_j),\\
|\frac{\partial^3 w_{i,r+1}}{\partial x^3}(x_j,t_k)| &\le C\min\{\sum_{q=r+1}^{m}\frac{B_q^L(x_j)}{\varepsilon_i\sqrt{\varepsilon_q}}, \sum_{q=r+1}^{m}\frac{B_q^L(x_j)}{\varepsilon_q^{3/2}}\},\\
|\frac{\partial^4 w_{i,l}}{\partial x^4}(x_j,t_k)| &\le C\frac{B_l^L(x_j)}{\varepsilon_i\varepsilon_l},\\
|\frac{\partial^4 w_{i,r+1}}{\partial x^4}(x_j,t_k)| &\le C\sum_{q=r+1}^{m}\frac{B_q^L(x_j)}{\varepsilon_i\varepsilon_q},
\end{aligned}$$

for each $m+1 \le i \le k$,

$$\begin{aligned}
|\frac{\partial^2 w_{i,l}}{\partial x^2}(x_j,t_k)| &\le C\min\{\frac{1}{\varepsilon_l}, \frac{1}{\varepsilon_m}\}B_l^L(x_j),\\
|\frac{\partial^3 w_{i,l}}{\partial x^3}(x_j,t_k)| &\le C\min\{\frac{1}{\varepsilon_m\sqrt{\varepsilon_l}}, \frac{1}{\varepsilon_l^{3/2}}\}B_l^L(x_j),\\
|\frac{\partial^3 w_{i,r+1}}{\partial x^3}(x_j,t_k)| &\le C\min\{\sum_{q=r+1}^{m}\frac{B_q^L(x_j)}{\varepsilon_m\sqrt{\varepsilon_q}}, \sum_{q=r+1}^{m}\frac{B_q^L(x_j)}{\varepsilon_q^{3/2}}\},\\
|\frac{\partial^4 w_{i,l}}{\partial x^4}(x_j,t_k)| &\le C\frac{B_l^L(x_j)}{\varepsilon_m\varepsilon_l},\\
|\frac{\partial^4 w_{i,r+1}}{\partial x^4}(x_j,t_k)| &\le C\sum_{q=r+1}^{m}\frac{B_q^L(x_j)}{\varepsilon_m\varepsilon_q},
\end{aligned}$$

for each $1 \le i \le n$*, and for each* $k+1 \le i \le n$*,*

$$|\frac{\partial^2 w_{i,l}}{\partial x^2}(x_j,t_k)| \le C B_l^L(x_j),$$
$$|\frac{\partial^3 w_{i,l}}{\partial x^3}(x_j,t_k)| \le C \frac{B_l^L(x_j)}{\sqrt{\varepsilon_l}},$$
$$|\frac{\partial^3 w_{i,r+1}}{\partial x^3}(x_j,t_k)| \le C \sum_{q=r+1}^{m} \frac{B_q^L(x_j)}{\sqrt{\varepsilon_q}},$$
$$|\frac{\partial^4 w_{i,l}}{\partial x^4}(x_j,t_k)| \le C \frac{B_l^L(x_j)}{\varepsilon_l},$$
$$|\frac{\partial^4 w_{i,r+1}}{\partial x^4}(x_j,t_k)| \le C \sum_{q=r+1}^{m} \frac{B_q^L(x_j)}{\varepsilon_q}.$$

Analogous results hold for the w_i^R *and their derivatives.*

Proof. Consider first decomposition (56) corresponding to the polynomials $p_i^{(3/2)}$. From the above definitions it follows that for each l, $1 \le l \le r$, $w_{i,l} = 0$ on $[x_{l,l+1}^{(3/2)}, 1]$. To establish the bounds on the third derivatives, for $i = 1, ..., k$, it is obvious that for $x \in [x_{r,r+1}^{(3/2)}, 1]$ Lemma 6 and $x \ge x_{r,r+1}^{(3/2)}$ imply that

$$|\frac{\partial^3 w_{i,r+1}}{\partial x^3}(x,t)| = |\frac{\partial^3 w_i^L}{\partial x^3}(x,t)| \le C \sum_{q=1}^{m} \frac{B_q^L(x)}{\varepsilon_q^{3/2}} \le C \sum_{q=r+1}^{m} \frac{B_q^L(x)}{\varepsilon_q^{3/2}};$$

for $x \in [0, x_{r,r+1}^{(3/2)}]$, Lemma 6 and $x \le x_{r,r+1}^{(3/2)}$ imply that

$$\begin{aligned}|\frac{\partial^3 w_{i,r+1}}{\partial x^3}(x,t)| &= |\frac{\partial^3 w_i^L}{\partial x^3}(x_{r,r+1}^{(3/2)},t)| \\ &\le C \sum_{q=1}^{m} \frac{B_q^L(x_{r,r+1}^{(3/2)})}{\varepsilon_q^{3/2}} \le C \sum_{q=r+1}^{m} \frac{B_q^L(x_{r,r+1}^{(3/2)})}{\varepsilon_q^{3/2}} \le C \sum_{q=r+1}^{m} \frac{B_q^L(x)}{\varepsilon_q^{3/2}};\end{aligned}$$

and for each $l = r, \ \ldots \ , 2$, it follows that for $x \in [x_{l,l+1}^{(3/2)}, 1]$,

$$\frac{\partial^3 w_{i,l}}{\partial x^3} = 0;$$

for $x \in [x_{l-1,l}^{(3/2)}, x_{l,l+1}^{(3/2)}]$, Lemma 6 implies that

$$\begin{aligned}|\frac{\partial^3 w_{i,l}}{\partial x^3}(x,t)| &\le |\frac{\partial^3 w_i^L}{\partial x^3}(x,t)| + \sum_{q=l+1}^{r+1} |\frac{\partial^3 w_{i,q}}{\partial x^3}(x,t)| \\ &\le C \sum_{q=1}^{m} \frac{B_q^L(x)}{\varepsilon_q^{3/2}} \le C \frac{B_l^L(x)}{\varepsilon_l^{3/2}}, \text{ using (14)};\end{aligned}$$

for $x \in [0, x_{l-1,l}^{(3/2)}]$, Lemma 6 and $x \leq x_{l-1,l}^{(3/2)}$ imply that

$$|\frac{\partial^3 w_{i,l}}{\partial x^3}(x,t)| = |\frac{\partial^3 w_i^L}{\partial x^3}(x_{l-1,l}^{(3/2)},t)|$$
$$\leq C\sum_{q=1}^{m}\frac{B_q^L(x_{l-1,l}^{(3/2)})}{\varepsilon_q^{3/2}} = C\frac{B_l^L(x_{l-1,l}^{(3/2)})}{\varepsilon_l^{3/2}} \leq C\frac{B_l^L(x)}{\varepsilon_l^{3/2}}, \text{ using (12) and (14);}$$

for $x \in [x_{1,2}^{(3/2)}, 1]$,

$$\frac{\partial^3 w_{i,1}}{\partial x^3} = 0;$$

for $x \in [0, x_{1,2}^{(3/2)}]$, Lemma 6 implies that

$$|\frac{\partial^3 w_{i,1}}{\partial x^3}(x,t)| \leq |\frac{\partial^3 w_i^L}{\partial x^3}(x,t)| + \sum_{q=2}^{r+1}|\frac{\partial^3 w_{i,q}}{\partial x^3}(x,t)| \leq C\sum_{q=1}^{m}\frac{B_q^L(x)}{\varepsilon_q^{3/2}} \leq C\frac{B_1^L(x)}{\varepsilon_1^{3/2}}.$$

The bounds for $\frac{\partial^3 w_{i,l}}{\partial x^3}(x,t)$, for $i = k+1,...,n$ and $l = r+1,...,1$ are obtained using the above steps with an appropriate bound of $\frac{\partial^3 w_i^L}{\partial x^3}(x,t), i = k+1,...,n$, from Lemma 6. For the bounds on the second derivatives note that for each $i = 1,...,k$ and l, $1 \leq l \leq r$: for $x \in [x_{l,l+1}^{(3/2)}, 1]$, $\frac{\partial^2 w_{i,l}}{\partial x^2} = 0$;
for $x \in [0, x_{l,l+1}^{(3/2)}]$,

$$\int_x^{x_{l,l+1}^{(3/2)}}\frac{\partial^3 w_{i,l}}{\partial x^3}(s,t)ds = \frac{\partial^2 w_{i,l}}{\partial x^2}(x_{l,l+1}^{(3/2)},t) - \frac{\partial^2 w_{i,l}}{\partial x^2}(x,t) = -\frac{\partial^2 w_{i,l}}{\partial x^2}(x,t),$$

and so

$$|\frac{\partial^2 w_{i,l}}{\partial x^2}(x,t)| \leq \int_x^{x_{l,l+1}^{(3/2)}}|\frac{\partial^3 w_{i,l}}{\partial x^3}(s,t)|ds \leq \frac{C}{\varepsilon_l^{3/2}}\int_x^{x_{l,l+1}^{(3/2)}}B_l^L(s)ds \leq C\frac{B_l^L(x)}{\varepsilon_l}.$$

Similarly, for $i = k+1,...,n$ and each l, $1 \leq l \leq r$, $|\frac{\partial^2 w_{i,l}}{\partial x^2}(x,t)| \leq CB_l^L(x)$. This completes the proof of the estimates for $s = 3/2$.
Secondly, consider decomposition (56) corresponding to the polynomials $p_i^{(1)}$. From the above definitions it follows that for each i, $i = 1,...,k$ and l, $1 \leq l \leq r$, $w_{i,l} = 0$ on $[x_{l,l+1}^{(1)}, 1]$.
To establish the bounds on the fourth derivatives it is obvious that: for $x \in [x_{r,r+1}^{(1)}, 1]$, Lemma 6, (14) and $x \geq x_{r,r+1}^{(1)}$ imply that

$$|\varepsilon_i\frac{\partial^4 w_{i,r+1}}{\partial x^4}(x,t)| = |\varepsilon_i\frac{\partial^4 w_i^L}{\partial x^4}(x,t)| \leq C\sum_{q=1}^{m}\frac{B_q^L(x)}{\varepsilon_q} \leq C\sum_{q=r+1}^{m}\frac{B_q^L(x)}{\varepsilon_q};$$

for $x \in [0, x^{(1)}_{r,r+1}]$, Lemma 6, (14) and $x \leq x^{(1)}_{r,r+1}$ imply that

$$|\varepsilon_i \frac{\partial^4 w_{i,r+1}}{\partial x^4}(x,t)| = |\varepsilon_i \frac{\partial^4 w^L_i}{\partial x^4}(x^{(1)}_{r,r+1},t)| \leq \sum_{q=1}^{m} \frac{B^L_q(x^{(1)}_{r,r+1})}{\varepsilon_q}$$

$$\leq C \sum_{q=r+1}^{m} \frac{B^L_q(x^{(1)}_{r,r+1})}{\varepsilon_q} \leq C \sum_{q=r+1}^{m} \frac{B^L_q(x)}{\varepsilon_q};$$

and for each $l = r, \ \ldots \ , 2$, it follows that
for $x \in [x^{(1)}_{l,l+1}, 1]$,

$$\frac{\partial^4 w_{i,l}}{\partial x^4} = 0;$$

for $x \in [x^{(1)}_{l-1,l}, x^{(1)}_{l,l+1}]$, Lemma 6 implies that

$$|\varepsilon_i \frac{\partial^4 w_{i,l}}{\partial x^4}(x,t)| \leq |\varepsilon_i \frac{\partial^4 w^L_i}{\partial x^4}(x,t)| + \sum_{q=l+1}^{r+1} |\varepsilon_i \frac{\partial^4 w_{i,q}}{\partial x^4}(x,t)|$$

$$\leq C \sum_{q=1}^{m} \frac{B^L_q(x)}{\varepsilon_q} \leq C \frac{B^L_l(x)}{\varepsilon_l}, \ \text{ using (14);}$$

for $x \in [0, x^{(1)}_{l-1,l}]$, Lemma 6 and $x \leq x^{(1)}_{l-1,l}$ imply that

$$|\varepsilon_i \frac{\partial^4 w_{i,l}}{\partial x^4}(x,t)| = |\varepsilon_i \frac{\partial^4 w^L_i}{\partial x^4}(x^{(1)}_{l-1,l},t)| \leq C \sum_{q=1}^{m} \frac{B^L_q(x^{(1)}_{l-1,l})}{\varepsilon_q}$$

$$\leq C \frac{B^L_l(x^{(1)}_{l-1,l})}{\varepsilon_l} \leq C \frac{B^L_l(x)}{\varepsilon_l}, \ \text{ using (12) and (14);}$$

for $x \in [x^{(1)}_{1,2}, 1]$,

$$\frac{\partial^4 w_{i,1}}{\partial x^4} = 0;$$

for $x \in [0, x^{(1)}_{1,2}]$, Lemma 6 implies that

$$|\varepsilon_i \frac{\partial^4 w_{i,1}}{\partial x^4}(x,t)| \leq |\varepsilon_i \frac{\partial^4 w^L_i}{\partial x^4}(x,t)| + \sum_{q=2}^{r+1} |\varepsilon_i \frac{\partial^4 w_{i,q}}{\partial x^4}(x,t)|$$

$$\leq C \sum_{q=1}^{m} \frac{B^L_q(x)}{\varepsilon_q} \leq C \frac{B^L_1(x)}{\varepsilon_1}.$$

For the bounds on the second and third derivatives note that for each l, $1 \leq l \leq r$:
for $x \in [x^{(1)}_{l,l+1}, 1]$,

$$\frac{\partial^2 w_{i,l}}{\partial x^2} = 0 = \frac{\partial^3 w_{i,l}}{\partial x^3};$$

for $x \in [0, x^{(1)}_{l,l+1}]$,

$$\int_x^{x^{(1)}_{l,l+1}} \varepsilon_i \frac{\partial^4 w_{i,l}}{\partial x^4}(s,t)ds = \varepsilon_i \frac{\partial^3 w_{i,l}}{\partial x^3}(x^{(1)}_{l,l+1},t) - \varepsilon_i \frac{\partial^3 w_{i,l}}{\partial x^3}(x,t) = -\varepsilon_i \frac{\partial^3 w_{i,l}}{\partial x^3}(x,t),$$

and so

$$|\varepsilon_i \frac{\partial^3 w_{i,l}}{\partial x^3}(x,t)| \leq \int_x^{x^{(1)}_{l,l+1}} |\varepsilon_i \frac{\partial^4 w_{i,1}}{\partial x^4}(s,t)|ds \leq \frac{C}{\varepsilon_l} \int_x^{x^{(1)}_{l,l+1}} B^L_l(s)ds \leq C\frac{B^L_l(x)}{\sqrt{\varepsilon_l}}.$$

In a similar way, it can be shown that

$$|\varepsilon_i \frac{\partial^2 w_{i,l}}{\partial x^2}(x,t)| \leq CB^L_l(x).$$

Using similar arguments it is easy to get the bounds of $\frac{\partial^s w_{i,l}}{\partial x^s}(x,t)$, for $s = 2,3$, $k+1 \leq i \leq n$ and $1 \leq l \leq r+1$.
The proof for the w^R_i and their derivatives is similar. □

Lemma 14. *Assume that $d_r > 0$ for some r, $1 \leq r \leq m$. Let assumptions* (3) - (6) *hold. Then, if $x_j \notin J$,*

$$|(\vec{L}^{M,N}(\vec{W}^L - \vec{w}^L))_i(x_j,t_k)| \leq C[M^{-1} + B^L_r(x_{j-1}) + \frac{(x_{j+1}-x_{j-1})^2}{\varepsilon_{r+1}}] \tag{57}$$

and if $x_j \in J$,

$$|(\vec{L}^{M,N}(\vec{W}^L - \vec{w}^L))_i(x_j,t_k)| \leq C[M^{-1} + N^{-2} + \frac{\varepsilon_i}{\sqrt{\varepsilon_r \varepsilon_{r+1}}}N^{-1}]. \tag{58}$$

Analogous results hold for the $\vec{W}^R - \vec{w}^R$.

Proof. The proof is as in Lemma 8.4 of [2]. It is not hard to check that the proof of Lemma 8.4 of [2] follows for all $\varepsilon_i,\ i = 1,...,n$, given in (1) for (2). □

Lemma 15. *Let assumptions* (3) - (6) *hold. Then, on $\Omega^{M,N}$, for each $1 \leq i \leq n$, the following estimates hold*

$$|(\vec{L}^{M,N}(\vec{W}^L - \vec{w}^L))_i(x_j,t_k)| \leq C(M^{-1} + B^L_m(x_{j-1})). \tag{59}$$

An analogous result holds for $\vec{W}^R - \vec{w}^R$.

Proof. From (40) and Lemma 6, for each $i = 1,\ldots,m$, it follows that on $\Omega^{M,N}$,

$$\begin{aligned}|(\vec{L}^{M,N}(\vec{W}^L - \vec{w}^L))_i(x_j,t_k)| &= |((\tfrac{\partial}{\partial t} - D^-_t) - \varepsilon_i(\tfrac{\partial^2}{\partial x^2} - \delta^2_x))w^L_i(x_j,t_k)| \\ &\leq C(M^{-1} + \varepsilon_i \sum_{q=i}^{m} \frac{B^L_q(x_{j-1})}{\varepsilon_q}) \\ &\leq C(M^{-1} + B^L_m(x_{j-1})),\end{aligned}$$

for each $i = m+1, \dots, k$,

$$\begin{aligned}|(\vec{L}^{M,N}(\vec{W}^L - \vec{w}^L))_i(x_j, t_k)| &= |((\tfrac{\partial}{\partial t} - D_t^-) - \varepsilon_m(\tfrac{\partial^2}{\partial x^2} - \delta_x^2))w_i^L(x_j, t_k)| \\ &\le C(M^{-1} + \varepsilon_m \frac{B_m^L(x_{j-1})}{\varepsilon_m}) \\ &\le C(M^{-1} + B_m^L(x_{j-1})),\end{aligned}$$

and for $i = k+1, \dots, n$, it follows that on $\Omega^{M,N}$,

$$\begin{aligned}|(\vec{L}^{M,N}(\vec{W}^L - \vec{w}^L))_i(x_j, t_k)| &= |((\tfrac{\partial}{\partial t} - D_t^-) - (\tfrac{\partial^2}{\partial x^2} - \delta_x^2))w_i^L(x_j, t_k)| \\ &\le C(M^{-1} + B_m^L(x_{j-1})),\end{aligned}$$

The proof for $\vec{W}^R - \vec{w}^R$ is similar. □

The following theorem gives the estimate of the singular component of the error.

Theorem 2. *Let assumptions* (3)–(6) *hold. Let* $\vec{w}$ *denote the singular component of the exact solution from (1) and* $\vec{W}$ *the singular component of the discrete solution from (34). Then*

$$||\vec{W} - \vec{w}|| \le C(M^{-1} + (N^{-1}\ln N)^2). \tag{60}$$

Proof. Since $\vec{w} = \vec{w}^L + \vec{w}^R$, it suffices to prove the result for $\vec{w}^L$ and $\vec{w}^R$ separately. Here it is proved for $\vec{w}^L$; a similar proof holds for $\vec{w}^R$.
By the comparison principle in Lemma 11 it suffices to show that for all i, j, k, and some constant C,

$$|(\vec{L}^{M,N}(\vec{W}^L - \vec{w}^L))_i(x_j, t_k)| \le (\vec{L}^{M,N}\vec{\Phi})_i(x_j, t_k). \tag{61}$$

This is proved for each mesh point $x_j \in (0,1)$ by considering separately the 4 kinds of sub-interval
(a) $(0, \sigma_1)$, (b) $[\sigma_1, \sigma_2)$, (c) $[\sigma_l, \sigma_{l+1})$ for some l, $2 \le l \le m-1$ and (d) $[\sigma_m, 1)$.

(a) Clearly $x_j \notin J$ and $x_{j+1} - x_{j-1} \le C\sqrt{\varepsilon_1}N^{-1}\ln N$. Then, Lemma 12 and (46) give (61).

(b) There are 2 possibilities:

(b1) $d_1 = 0$ and (b2) $d_1 > 0$.

(b1) Since $\sigma_1 = \frac{\sigma_2}{2}$ and the mesh is uniform in $(0, \sigma_2)$, it follows that $x_j \notin J$, and $x_{j+1} - x_{j-1} \le C\sqrt{\varepsilon_1}N^{-1}\ln N$. Then Lemma 12 and (46) give (61).

(b2) Either $x_j \notin J$ or $x_j \in J$.
If $x_j \notin J$, then $x_{j+1} - x_{j-1} \le C\sqrt{\varepsilon_2}N^{-1}\ln N$ and by Lemma 8 $B_1^L(x_{j-1}) \le B_1^L(\sigma_1 - h_1^-) \le CN^{-2}$, so Lemma 14 (57) with $r = 1$ and (46) give (61).
On the other hand, if $x_j \in J$, then Lemma 14 (58) with $r = 1$ and (47) give (61).

(c) There are 3 possibilities:

(c1) $d_1 = d_2 = \dots = d_l = 0$, (c2) $d_r > 0$ and $d_{r+1} = \dots = d_l = 0$ for some r, $1 \le r \le l-1$ and (c3) $d_l > 0$.

(c1)Since $\sigma_1 = C\sigma_{l+1}$ and the mesh is uniform in $(0, \sigma_{l+1})$, it follows that $x_j \notin J$ and $x_{j+1} - x_{j-1} \le C\sqrt{\varepsilon_1}N^{-1}\ln N$. Then Lemma 12 and (46) give (61).

(c2) Either $x_j \notin J$ or $x_j \in J$.
If $x_j \notin J$, then $\sigma_{r+1} = C\sigma_{l+1}$, $x_{j+1} - x_{j-1} \leq C\sqrt{\varepsilon_{l+1}}N^{-1}\ln N$ and by Lemma 8 $B_r^L(x_{j-1}) \leq B_r^L(\sigma_l - h_l^-) \leq B_r^L(\sigma_r - h_r^-) \leq CN^{-2}$. Thus Lemma 14 (57) and (46) give (61).
On the other hand, if $x_j \in J$, then $x_j = \sigma_l$, so Lemma 14 (58) with $r = l$ and (47) give (61).

(c3) Either $x_j \notin J$ or $x_j \in J$.
If $x_j \notin J$, then $x_{j+1} - x_{j-1} \leq C\sqrt{\varepsilon_{l+1}}N^{-1}\ln N$ and by Lemma 8 $B_l^L(x_{j-1}) \leq B_l^L(\sigma_l - h_l^-) \leq CN^{-2}$, so Lemma 14 (57) with $r = l$ and (46) give (61).
On the other hand, if $x_j = \sigma_l$, so Lemma 14 (58) with $r = l$ and (47) give (61).

(d) There are 3 possibilities:
(d1) $d_1 = \ \dots\ = d_m = 0$, (d2) $d_r > 0$ and $d_{r+1} = \ \dots\ = d_m = 0$ for some r, $1 \leq r \leq m-1$ and (d3) $d_m > 0$.

(d1) Since $\sigma_1 = C$ and the mesh is uniform in $(0,1)$, it follows that $x_j \notin J$, $\frac{1}{\sqrt{\varepsilon_1}} \leq C\ln N$ and $x_{j+1} - x_{j-1} \leq CN^{-1}$. Then Lemma 12 and (46) give (61).

(d2) Either $x_j \notin J$ or $x_j \in J$.
If $x_j \notin J$, then $\sigma_{r+1} = C$, $\frac{1}{\sqrt{\varepsilon_{r+1}}} \leq C\ln N$, $x_{j+1} - x_{j-1} \leq CN^{-1}$ and by Lemma 8, $B_r^L(x_{j-1}) \leq B_r^L(\sigma_m - h_m^-) \leq B_r^L(\sigma_r - h_r^-) \leq CN^{-2}$. Thus Lemma 14 (57) and (46) give (61).
On the other hand, if $x_j \in J$, then $x_j \in \{\sigma_m, 1-\sigma_m, \ \dots\ , 1-\sigma_1\}$. Thus, Lemma 14 (58) and (47) give (61).

(d3) By Lemma 8 with $r = m$, $B_m^L(x_{j-1}) \leq B_m^L(\sigma_m - h_m^-) \leq CN^{-2}$. Then Lemma 15 and (46) give (61). □

The following theorem gives the first order in time and essentially the second order in space parameter-uniform error estimate.

Theorem 3. *Let assumptions* (3)–(6) *hold. Let $\vec{u}$ denote the exact solution of (1) and $\vec{U}$ the discrete solution of (34). Then*

$$||\vec{U} - \vec{u}|| \leq C(M^{-1} + (N^{-1}\ln N)^2). \tag{62}$$

Proof. An application of the triangle inequality and the results of Theorems 1 and 2 lead immediately to the required result. □

9. Numerical illustrations

In this section, a numerical illustration is presented. To get the order of convergence in the variable t exclusively, a fine Shishkin mesh is considered for x and the resulting problem is solved for various uniform meshes with respect to t. The two mesh algorithms, see [1] for more details, are applied to get the parameter-uniform order of convergence and the error constant. Similarly, a fine mesh for t is considered, the resulting problem is solved and the $x-$ order of convergence of the method is also found.
The notations D^N, p^N and C_p^N used in the tables are ε-uniform maximum pointwise

h!

η	Number of mesh points N				
	4	8	16	32	64
2^0	0.323E-01	0.224E-01	0.145E-01	0.839E-02	0.457E-02
2^{-1}	0.330E-01	0.230E-01	0.148E-01	0.856E-02	0.465E-02
2^{-2}	0.336E-01	0.233E-01	0.149E-01	0.863E-02	0.470E-02
2^{-3}	0.340E-01	0.235E-01	0.150E-01	0.866E-02	0.471E-02
2^{-4}	0.343E-01	0.237E-01	0.150E-01	0.862E-02	0.467E-02
2^{-5}	0.346E-01	0.238E-01	0.151E-01	0.864E-02	0.466E-02
D^N	0.346E-01	0.238E-01	0.151E-01	0.866E-02	0.471E-02
p^N	0.540E+00	0.659E+00	0.797E+00	0.877E+00	
C_p^N	0.234E+00	0.234E+00	0.215E+00	0.180E+00	0.143E+00
t-order of convergence= 0.5					
The error constant= 0.2					

Table 1: *t-convergence*

two-mesh differences, ε- uniform order of local convergence and ε- uniform error constant respectively.

η	Number of mesh points N				
	32	64	128	256	512
2^0	0.285E-02	0.755E-03	0.190E-03	0.478E-04	0.120E-04
2^{-1}	0.553E-02	0.148E-02	0.376E-03	0.949E-04	0.238E-04
2^{-2}	0.101E-01	0.282E-02	0.747E-03	0.188E-03	0.473E-04
2^{-3}	0.155E-01	0.549E-02	0.147E-02	0.373E-03	0.942E-04
2^{-4}	0.194E-01	0.101E-01	0.281E-02	0.744E-03	0.188E-03
2^{-5}	0.193E-01	0.155E-01	0.548E-02	0.146E-02	0.372E-03
D^N	0.194E-01	0.155E-01	0.548E-02	0.146E-02	0.372E-03
p^N	0.322E+00	0.150E+01	0.191E+01	0.197E+01	
C_p^N	0.295E+00	0.295E+00	0.131E+00	0.435E-01	0.139E-01
x- order of convergence= 0.3					
The error constant= 0.3					

Table 2: *x-convergence*

Example 1. *Consider the problem*

$$\frac{\partial \vec{u}}{\partial t} - E\frac{\partial^2 \vec{u}}{\partial x^2} + A\vec{u} = \vec{f} \text{ on } (0,1)\times(0,1], \quad \vec{u} = \vec{0} \text{ on } \Gamma,$$

where $E = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$, $A = \begin{pmatrix} 4(1+x+t) & -t & -x \\ -2(1-t) & 7+((2+t)x) & -(3-x) \\ -1 & -(x+t) & 4(1+\frac{x}{2}+\frac{t}{2}) \end{pmatrix}$,

$\vec{f} = (1+e^{x+t},\ 1+x+t^2,\ 1+e^t)^T$.

The variation in all the three parameters is given by considering $\varepsilon_3 = 1, \varepsilon_2 = \frac{\eta}{8}, \varepsilon_1 = \frac{\eta}{32}$, *where* η *is varied as shown in the tables.* α *is taken to be* 0.9.
Fixing a fine Shishkin mesh with 64 points horizontally, the problem is solved and the order of convergence in the variable t *is calculated. A fine uniform mesh on* t *with 32 points is considered and the order of convergence in the variable* x *is calculated. The order of convergence and the error constant for* t *and* x *are presented in Table 1 and Table 2 respectively. It is to be observed that the* $t-$ *order of convergence arrived at Table 1 well agrees with the theoretical result. On the other hand, the* x*-order of convergence proved theoretically is essentially two whereas numerically the* x*- order of convergence starts with* 0.3 *but it improves quickly to* 2 *for reasonable* N.

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