

Estimates of eigenvalues of a boundary value problem with a parameter*

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Abstract. We study an eigenvalue problem for the Laplace operator with a boundary condition containing a parameter. We estimate the rate of convergence of the eigenvalues to the eigenvalues of the Dirichlet problem for large positive values of the parameter.

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1. Introduction

We consider the eigenvalue problem

$$\Delta u + \lambda u = 0 \quad \text{in } \Omega, \quad (1)$$

$$\frac{\partial u}{\partial \nu} + \alpha \sigma(x) u = 0 \quad \text{on } \Gamma, \quad (2)$$

where $\Omega \subset R^n$, $n \geq 2$, is a bounded domain with boundary $\Gamma = \partial\Omega \in C^2$. By ν we denote the outward unit normal vector to Γ , α is a real parameter. The function $\sigma(x) \in C^1(\Gamma)$ is positive:

$$0 < \sigma_0 \leq \sigma(x) \leq \sigma_1, \quad \sigma_0 = \inf_{x \in \Gamma} \sigma(x) \quad \text{and} \quad \sigma_1 = \sup_{x \in \Gamma} \sigma(x).$$

Problem (1), (2) with $\sigma(x) = 1$ is known as the Robin (Fourier) problem for $\alpha > 0$ (see [6, Ch. 7, Par. 7.2]), and the generalized Robin problem for all α ([5]).

There is a sequence of eigenvalues $\lambda_1(\alpha) < \lambda_2(\alpha) \leq \dots$ of problem (1) - (2) enumerated according to their multiplicities with

$$\lim_{k \rightarrow \infty} \lambda_k(\alpha) = +\infty.$$

We also consider the sequence of eigenvalues $0 < \lambda_1^D < \lambda_2^D \leq \dots$ of the Dirichlet eigenvalue problem

$$\Delta u + \lambda u = 0 \quad \text{in } \Omega, \quad (3)$$

$$u = 0 \quad \text{on } \Gamma, \quad (4)$$

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with

$$\lim_{k \rightarrow \infty} \lambda_k^D = +\infty.$$

Note that the eigenvalues $\lambda_1(\alpha)$ and λ_1^D are simple and the corresponding eigenfunctions $u_{1,\alpha}(x)$ and $u_1^D(x)$ are positive.

In this paper, we estimate $\lambda_k(\alpha)$ for large values of α . We now give some known results.

It is easy to see that $\lambda_k(\alpha) \leq \lambda_k^D$, $k = 1, 2, \dots$. These inequalities give the upper bound of $\lambda_k(\alpha)$ for all values of α . It was announced in ([2, Ch. 6, Par. 2, No. 1]) that for $n = 2$ and a smooth boundary $\lim_{\alpha \rightarrow +\infty} \lambda_k(\alpha) = \lambda_k^D$.

Later the properties of the first eigenvalue $\lambda_1(\alpha)$ were studied more precisely. Consider the case $\sigma(x) = 1$. The following two-sided estimates:

$$\lambda_1^D \left(1 + \frac{\lambda_1^D}{\alpha q_1}\right)^{-1} \leq \lambda_1(\alpha) \leq \lambda_1^D \left(1 + \frac{4\pi}{\alpha |\Gamma|}\right)^{-1}, \quad \alpha > 0,$$

were obtained in [12] for $n = 2$. Here $|\Gamma|$ is the length of Γ and q_1 is the first eigenvalue of the Steklov problem

$$\begin{aligned} \Delta^2 u &= 0 \quad \text{in } \Omega, \\ u &= 0, \quad \Delta u - q \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma. \end{aligned}$$

In [4], for any $n \geq 2$ we establish the following asymptotic expansion:

$$\lambda_1(\alpha) = \lambda_1^D - \frac{\int_{\Gamma} \left(\frac{\partial u_1^D}{\partial \nu}\right)^2 ds}{\int_{\Omega} (u_1^D)^2 dx} \alpha^{-1} + o(\alpha^{-1}), \quad \alpha \rightarrow +\infty.$$

The case $\alpha < 0$ has recently attracted attention (see, for instance, [9]). It was shown in [9] that for a piecewise- C^1 boundary

$$\liminf_{\alpha \rightarrow -\infty} \lambda_1(\alpha)/(-\alpha^2) \geq 1.$$

For C^1 boundaries it was proved ([10]) that

$$\lim_{\alpha \rightarrow -\infty} \lambda_1(\alpha)/(-\alpha^2) = 1.$$

The C^1 -condition is optimal. In [9], the authors constructed plane triangle domains for which

$$\lim_{\alpha \rightarrow -\infty} \lambda_1(\alpha)/(-\alpha^2) > 1.$$

In [3], the authors proved that for C^1 boundaries

$$\lim_{\alpha \rightarrow -\infty} \frac{\lambda_k(\alpha)}{-\alpha^2} = 1 \tag{5}$$

for all $k = 1, 2, \dots$

2. Main results

The main result of this paper reads as follows.

Theorem 1. *The eigenvalues $\lambda_k(\alpha)$, $k = 1, 2, \dots$, satisfy the estimates*

$$0 \leq \lambda_k^D - \lambda_k(\alpha) \leq C_1 \alpha^{-1/2} (\lambda_k^D)^2, \quad \alpha > 0, \quad (6)$$

where the constant C_1 does not depend on k .

In the following theorem we gather the qualitative properties of eigenvalues of problem (1) - (2) (see also [2, Ch. 6] for i) and [9] for ii) and iii) for $\sigma(x) = 1$)

Theorem 2. *The eigenvalues have the following properties:*

i) $\lambda_k(\alpha)$, $k = 1, 2, \dots$, are continuous functions of α and

$$\lambda_k(\alpha_1) \leq \lambda_k(\alpha_2), \quad \alpha_1 < \alpha_2; \quad (7)$$

ii) $\lambda_1(\alpha)$ is a concave function of α :

$$\lambda_1(\beta\alpha_1 + (1 - \beta)\alpha_2) \geq \beta\lambda_1(\alpha_1) + (1 - \beta)\lambda_1(\alpha_2), \quad 0 < \beta < 1; \quad (8)$$

iii) $\lambda_1(\alpha)$ is differentiable and

$$\lambda_1'(\alpha) = \frac{\int_{\Gamma} \sigma u_{1,\alpha}^2 ds}{\int_{\Omega} u_{1,\alpha}^2 dx} > 0; \quad (9)$$

iv) the following estimate

$$\liminf_{\alpha \rightarrow -\infty} \frac{\lambda_1'(\alpha)}{-\alpha} \geq \sigma_1^2 \quad (10)$$

holds.

3. Operator treatment

In this section, we introduce two linear operators associated with problems (1) - (2) and (3) - (4) to derive the eigenvalue estimates (6).

Consider problem (1) - (2) in the space $H^1(\Omega)$ ([1, 11]). We define an eigenvalue of problem (1), (2) as a value λ for which there exists the non-zero function $u \in H^1(\Omega)$ satisfying the integral identity

$$\int_{\Omega} (\nabla u, \nabla v) dx + \alpha \int_{\Gamma} \sigma uv ds = \lambda \int_{\Omega} uv dx \quad (11)$$

for any $v \in H^1(\Omega)$. Relation (11) can be rewritten as

$$\int_{\Omega} ((\nabla u, \nabla v) + Muv) dx + \alpha \int_{\Gamma} \sigma uv ds = (\lambda + M) \int_{\Omega} uv dx, \quad M > 0. \quad (12)$$

Let us define an equivalent scalar product in the space $H^1(\Omega)$ by the formula

$$[u, v]_M = \int_{\Omega} ((\nabla u, \nabla v) + Muv) \, dx, \quad \|u\|_M^2 = [u, u]_M. \tag{13}$$

Now (12) transforms to

$$[u, v]_M + \alpha[Tu, v]_M = (\lambda + M)[Bu, v]_M,$$

where the linear self-adjoint non-negative operators $T : H^1(\Omega) \rightarrow H^1(\Omega)$ and $B : H^1(\Omega) \rightarrow H^1(\Omega)$ were defined by the bilinear forms

$$[Tu, v]_M = \int_{\Gamma} \sigma uv \, ds, \quad [Bu, v]_M = \int_{\Omega} uv \, dx, \quad u, v \in H^1(\Omega). \tag{14}$$

Hence we have an equation in the space $H^1(\Omega)$ with the norm $\|\cdot\|_M$:

$$(I + \alpha T)u = (\lambda + M)Bu. \tag{15}$$

Now we use the inequality ([11, Ch. 3, Par. 5, Formula 19])

$$\|v\|_{L_2(\Gamma)}^2 \leq \varepsilon \|\nabla v\|_{L_2(\Omega)}^2 + C_{\varepsilon} \|v\|_{L_2(\Omega)}^2, \tag{16}$$

which is valid for $v \in H^1(\Omega)$ with an arbitrary $\varepsilon > 0$. Using (14), (16), we obtain

$$\begin{aligned} \|Tu\|_M^2 &= [Tu, Tu]_M = \int_{\Gamma} \sigma u Tu \, ds \leq \sigma_1 \|u\|_{L_2(\Gamma)} \|Tu\|_{L_2(\Gamma)} \\ &\leq \sigma_1 \varepsilon \left(\int_{\Omega} \left(|\nabla Tu|^2 + \frac{C_{\varepsilon}}{\varepsilon} (Tu)^2 \right) dx \right)^{1/2} \\ &\quad \times \left(\int_{\Omega} \left(|\nabla u|^2 + \frac{C_{\varepsilon}}{\varepsilon} u^2 \right) dx \right)^{1/2} \leq C_2 \varepsilon \|Tu\|_M \|u\|_M, \end{aligned} \tag{17}$$

where $\varepsilon > 0$, $M = M_{\varepsilon}$. It follows from (17) that

$$\|Tu\|_{M_{\varepsilon}} \leq C_2 \varepsilon \|u\|_{M_{\varepsilon}},$$

and for any arbitrary small ε we have $\|\alpha T\|_{H^1(\Omega) \rightarrow H^1(\Omega)} < 1$ for $|\alpha| < 1/C_2\varepsilon$. Therefore, the inverse operator $(I + \alpha T)^{-1}$ is bounded and

$$\|(I + \alpha T)^{-1}\| \leq (1 - |\alpha| \|T\|)^{-1}.$$

Hence, equation (15) is equivalent to

$$(I - (\lambda + M)(I + \alpha T)^{-1}B)u = 0.$$

The operator B is compact ([11, Ch. 3, Par. 5, Th. 3]) and the operator $(I + \alpha T)^{-1}B : H^1(\Omega) \rightarrow H^1(\Omega)$ is also compact. Hence the spectrum of problem (15) consists of real eigenvalues $\lambda_j(\alpha)$, $j = 1, 2, \dots$, of finite multiplicity with the only limit point at the infinity. From (14), (15) we obtain the inequality

$$\lambda_j(\alpha) \geq -M_{\varepsilon} + (1 - |\alpha| \|T\|) \frac{\|u_{j,\alpha}\|_{M_{\varepsilon}}^2}{\|u_{j,\alpha}\|_{L_2(\Omega)}^2} \geq -M_{\varepsilon}$$

with the corresponding eigenfunction $u_{j,\alpha}$. Thus, $\lambda_j(\alpha) \rightarrow +\infty, j \rightarrow \infty$.

By the variational principle ([11, Ch. 4, Par. 1, No. 4]) we have

$$\lambda_k(\alpha) = \sup_{v_1, \dots, v_{k-1} \in L_2(\Omega)} \inf_{\substack{v \in H^1(\Omega) \\ (v, v_j)_{L_2(\Omega)} = 0 \\ j = 1, \dots, k-1}} \frac{\int_{\Omega} |\nabla v|^2 dx + \alpha \int_{\Gamma} \sigma v^2 ds}{\int_{\Omega} v^2 dx}, \tag{18}$$

$$\lambda_k^D = \sup_{v_1, \dots, v_{k-1} \in L_2(\Omega)} \inf_{\substack{v \in \mathring{H}^1(\Omega) \\ (v, v_j)_{L_2(\Omega)} = 0 \\ j = 1, \dots, k-1}} \frac{\int_{\Omega} |\nabla v|^2 dx}{\int_{\Omega} v^2 dx}, \quad k = 1, 2, \dots \tag{19}$$

To prove inequalities (6) we apply the following statement (see [6, Ch. 2, Th. 2.3.1]).

Theorem 3. *Let T_1 and T_2 be two linear self-adjoint, compact and positive operators on a separable Hilbert space H . Assume also that $\mu_k(T_1)$ and $\mu_k(T_2)$ are their k -th respective eigenvalues. Then*

$$|\mu_k(T_1) - \mu_k(T_2)| \leq \|T_1 - T_2\|. \tag{20}$$

Now we give the proof of Theorem 1.

Proof. Consider the boundary value problem

$$-\Delta u + u = h \quad \text{in } \Omega, \tag{21}$$

$$\frac{\partial u}{\partial \nu} + \alpha \sigma(x)u = 0 \quad \text{on } \Gamma, \quad \alpha > 0, \tag{22}$$

with $h \in L_2(\Omega)$. A weak solution $u \in H^1(\Omega)$ of problem (21), (22) satisfy the integral identity

$$\int_{\Omega} ((\nabla u, \nabla v) + uv) dx + \alpha \int_{\Gamma} \sigma uv ds = \int_{\Omega} hv dx \tag{23}$$

for all $v \in H^1(\Omega)$. Let us define the scalar product in the space $H^1(\Omega)$ as

$$(u, v)_{H^1(\Omega), \alpha} = \int_{\Omega} ((\nabla u, \nabla v) + uv) dx + \alpha \int_{\Gamma} \sigma uv ds \tag{24}$$

and the corresponding norm by

$$\|u\|_{H^1(\Omega), \alpha}^2 = (u, u)_{H^1(\Omega), \alpha}.$$

Due to (16), scalar product (24) is equivalent to the standard one

$$(u, v)_{H^1(\Omega)} = \int_{\Omega} ((\nabla u, \nabla v) + uv) dx. \tag{25}$$

Using (23), (24), we obtain the relation

$$(u, v)_{H^1(\Omega), \alpha} = (h, v)_{L_2(\Omega)}. \tag{26}$$

Hence, consider the linear functional

$$l_h(v) = (h, v)_{L_2(\Omega)}.$$

This functional is bounded on the space $H^1(\Omega)$:

$$|l_h(v)| \leq \|h\|_{L_2(\Omega)} \|v\|_{L_2(\Omega)}. \quad (27)$$

Now, by the Riesz lemma there exists the unique function $u \in H^1(\Omega)$ satisfying integral identity (23). Applying (26) with $v = u$, we obtain

$$\|u\|_{H^1(\Omega), \alpha}^2 \leq \|h\|_{L_2(\Omega)} \|u\|_{H^1(\Omega), \alpha}.$$

Therefore,

$$\|u\|_{L_2(\Omega)} \leq \|u\|_{H^1(\Omega), \alpha} \leq \|h\|_{L_2(\Omega)}, \quad (28)$$

and we can define the bounded linear operator $A_\alpha : L_2(\Omega) \rightarrow L_2(\Omega)$ such that $u = A_\alpha h$ and $\|A_\alpha\| \leq 1$. Moreover, if the domain Ω with C^2 boundary is bounded, then the space $H^1(\Omega)$ embeds compactly into the space $L_2(\Omega)$ ([6, Ch. 1, Th. 1.1.1]). It means that the operator A_α is compact. Note that

$$\begin{aligned} (h, A_\alpha g)_{L_2(\Omega)} &= \int_{\Omega} h A_\alpha g \, dx = \int_{\Omega} h v \, dx \\ &= \int_{\Omega} ((\nabla u, \nabla v) + uv) \, dx + \alpha \int_{\Gamma} \sigma uv \, ds \\ &= \int_{\Omega} u g \, dx = (A_\alpha h, g)_{L_2(\Omega)}, \quad f, g \in L_2(\Omega), \end{aligned} \quad (29)$$

with $u = A_\alpha h$, $v = A_\alpha g$, $u, v \in H^1(\Omega)$. Relation (29) means that A_α is a self-adjoint operator. Now, by relation (29) we have

$$\begin{aligned} (h, A_\alpha h)_{L_2(\Omega)} &= \int_{\Omega} u h \, dx \\ &= \int_{\Omega} (|\nabla u|^2 + u^2) \, dx + \alpha \int_{\Gamma} \sigma u^2 \, ds = \|u\|_{H^1(\Omega), \alpha}^2 > 0, \quad h \neq 0. \end{aligned}$$

Hence, the operator A_α is positive. Finally, A_α is a self-adjoint positive compact operator in the Hilbert space $H = L_2(\Omega)$. By the well-known theorem ([6, Ch. 1, Th. 1.2.1]), A_α has a sequence of eigenvalues $\{\mu_k(\alpha)\}$, $k = 1, 2, \dots$ with finite multiplicities such that $\mu_k(\alpha) > 0$, $\mu_k(\alpha) \searrow 0$, $k \rightarrow \infty$. Let us denote by $u_{k,\alpha}(x) \in L_2(\Omega)$ the eigenfunction satisfying $A_\alpha u_{k,\alpha} = \mu_k(\alpha) u_{k,\alpha}$. Thus,

$$\mu_k(\alpha) (u_{k,\alpha}, v)_{H^1(\Omega), \alpha} = (u_{k,\alpha}, v)_{L_2(\Omega)}$$

and

$$\mu_k(\alpha) \left(\int_{\Omega} ((\nabla u_{k,\alpha}, \nabla v) + u_{k,\alpha} v) \, dx + \alpha \int_{\Gamma} \sigma u_{k,\alpha} v \, ds \right) = \int_{\Omega} u_{k,\alpha} v \, dx.$$

It can be seen that

$$\mu_k(\alpha) = \frac{1}{\lambda_k(\alpha) + 1}.$$

Let us note that for $\alpha > 0$ we have

$$\mu_k(\alpha) \leq \frac{1}{\lambda_1(\alpha) + 1} < 1,$$

so $\|A_\alpha\| < 1$.

Furthermore, consider a Dirichlet problem

$$-\Delta y + y = h \quad \text{in } \Omega, \quad (30)$$

$$y = 0 \quad \text{on } \Gamma. \quad (31)$$

For $h \in L_2(\Omega)$ a weak solution $y \in \overset{\circ}{H}^1(\Omega)$ of problem (30), (31) satisfies the integral identity

$$\int_{\Omega} ((\nabla y, \nabla v) + yv) dx = \int_{\Omega} hv dx \quad (32)$$

for all $v \in \overset{\circ}{H}^1(\Omega)$. Define the scalar product in the space $\overset{\circ}{H}^1(\Omega)$ by (25). Using (25), (32), we obtain the relation

$$(y, v)_{\overset{\circ}{H}^1(\Omega)} = l_h(v). \quad (33)$$

Now, by (27) and the Riesz lemma there exists the unique function $y \in \overset{\circ}{H}^1(\Omega)$ satisfying integral identity (32). Using (32) with $v = y$, we obtain

$$\|y\|_{\overset{\circ}{H}^1(\Omega)}^2 \leq \|h\|_{L_2(\Omega)} \|y\|_{\overset{\circ}{H}^1(\Omega)}. \quad (34)$$

Therefore,

$$\|y\|_{L_2(\Omega)} \leq \|y\|_{\overset{\circ}{H}^1(\Omega)} \leq \|h\|_{L_2(\Omega)}, \quad (35)$$

and we can define the bounded linear operator $A^D : L_2(\Omega) \rightarrow L_2(\Omega)$ such that $u = A^D h$ and $\|A^D\| \leq 1$. If the domain Ω is bounded, then the space $\overset{\circ}{H}^1(\Omega)$ embeds compactly into the space $L_2(\Omega)$ ([6, Ch. 1, Th. 1.1.1]). Hence, the operator A^D is compact. Note that

$$\begin{aligned} (h, A^D g)_{L_2(\Omega)} &= \int_{\Omega} h A^D g dx = \int_{\Omega} hv dx = \int_{\Omega} ((\nabla y, \nabla v) + yv) dx \\ &= \int_{\Omega} yg dx = (A^D h, g)_{L_2(\Omega)}, \quad f, g \in L_2(\Omega), \end{aligned} \quad (36)$$

with $y = A^D h$, $v = A^D g$, $y, v \in \overset{\circ}{H}^1(\Omega)$. Relation (36) means that A^D is a self-adjoint operator. Now, by (36) we have

$$(h, A^D h)_{L_2(\Omega)} = \int_{\Omega} yh dx = \int_{\Omega} (|\nabla y|^2 + y^2) dx = \|y\|_{\overset{\circ}{H}^1(\Omega)}^2 > 0, \quad h \neq 0.$$

Hence, the operator A^D is positive. Finally, A^D is a self-adjoint positive compact operator in the Hilbert space $H = L_2(\Omega)$. By ([6, Ch. 1, Th. 1.2.1]), there exists a

sequence of eigenvalues $\{\mu_k^D\}$, $k = 1, 2, \dots$, of the operator A^D with finite multiplicities such that $\mu_k^D > 0$, $\mu_k^D \searrow 0$, $k \rightarrow \infty$. Denote by $y_k(x) \in L_2(\Omega)$ the respective eigenfunction satisfying $A^D y_k = \mu_k^D y_k$. Thus, $\mu_k^D (y_k, v)_{H^1(\Omega)} = (y_k, v)_{L_2(\Omega)}$ and

$$\mu_k^D \int_{\Omega} ((\nabla y_k, \nabla v) + y_k v) dx = \int_{\Omega} y_k v dx.$$

Then,

$$\mu_k^D = \frac{1}{\lambda_k^D + 1}.$$

Note that

$$\mu_k^D \leq \frac{1}{\lambda_1^D + 1} < 1,$$

so $\|A^D\| < 1$.

Now we estimate the norm $\|A_\alpha - A^D\|_{L_2(\Omega) \rightarrow L_2(\Omega)}$ for large positive values of α .

Let us remind that in domains with C^2 -class boundaries and positive $\sigma(x) \in C^1(\Gamma)$ the functions $u = A_\alpha h$ and $y = A^D h$ are strong solutions and belong to $H^2(\Omega)$ ([11, Ch. 4, Par. 2, Th. 4]). Moreover, the following estimate

$$\|y\|_{H^2(\Omega)} \leq C_2 \|h\|_{L_2(\Omega)} \tag{37}$$

holds. Now we use estimate (16) with $\varepsilon = 1$:

$$\|y\|_{L_2(\Gamma)} \leq C_3 \|y\|_{H^1(\Omega)}. \tag{38}$$

Combining (37) and (38) we derive the inequality

$$\|\nabla y\|_{L_2(\Gamma)} \leq C_4 \|y\|_{H^2(\Omega)}. \tag{39}$$

Since $\left| \frac{\partial y}{\partial \nu} \right| \leq |\nabla y|$ on Γ , from (37), (39) we obtain the estimate

$$\left\| \frac{\partial y}{\partial \nu} \right\|_{L_2(\Gamma)} \leq C_5 \|h\|_{L_2(\Omega)}. \tag{40}$$

Suppose that $w = (A^D - A_\alpha) h$. By (21), (22), (30), (31) the function w is a solution of the boundary value problem

$$-\Delta w + w = 0 \quad \text{in } \Omega, \tag{41}$$

$$\frac{\partial w}{\partial \nu} + \alpha \sigma w = \frac{\partial y}{\partial \nu} \quad \text{on } \Gamma. \tag{42}$$

Multiplying equation (41) by w and integrating it over Ω with respect to boundary condition (42), we get the relation

$$\int_{\Omega} (|\nabla w|^2 + w^2) dx + \frac{1}{\alpha} \int_{\Gamma} \left(\frac{\partial w}{\partial \nu} \right)^2 \frac{ds}{\sigma} = \frac{1}{\alpha} \int_{\Gamma} \frac{\partial w}{\partial \nu} \frac{\partial y}{\partial \nu} \frac{ds}{\sigma}, \quad \alpha > 0. \tag{43}$$

Then we obtain the inequality

$$\|w\|_{L_2(\Omega)}^2 + \frac{1}{\alpha} \left\| \frac{\partial w}{\partial \nu} \right\|_{L_2(\Gamma)}^2 \leq \frac{C_6}{\alpha} \left\| \frac{\partial w}{\partial \nu} \right\|_{L_2(\Gamma)} \left\| \frac{\partial y}{\partial \nu} \right\|_{L_2(\Gamma)}$$

and, consequently,

$$\|w\|_{L_2(\Omega)}^2 + \frac{1}{\alpha} \left\| \frac{\partial w}{\partial \nu} \right\|_{L_2(\Gamma)}^2 \leq \frac{1}{2\alpha} \left\| \frac{\partial w}{\partial \nu} \right\|_{L_2(\Gamma)}^2 + \frac{C_6^2}{2\alpha} \left\| \frac{\partial y}{\partial \nu} \right\|_{L_2(\Gamma)}^2.$$

Therefore, we have the estimate

$$\|w\|_{L_2(\Omega)} \leq \frac{C_6}{\sqrt{2\alpha}} \left\| \frac{\partial y}{\partial \nu} \right\|_{L_2(\Gamma)}, \quad \alpha > 0. \quad (44)$$

Combining (44) with (40), we get

$$\|w\|_{L_2(\Omega)} \leq C_7 \alpha^{-1/2} \|h\|_{L_2(\Omega)}, \quad \alpha > 0,$$

with the constant C_6 independent of α . Thus, for all $h \in L_2(\Omega)$ we have the estimate

$$\|(A^D - A_\alpha)h\|_{L_2(\Omega)} \leq C_7 \alpha^{-1/2} \|h\|_{L_2(\Omega)}$$

and

$$\|A^D - A_\alpha\| \leq C_7 \alpha^{-1/2}, \quad \alpha > 0. \quad (45)$$

Now we apply (20) to the operators $T_1 = A_\alpha$, $T_2 = A^D$. Then, by the relations

$$\mu_k(\alpha) = \frac{1}{\lambda_k(\alpha) + 1}, \quad \mu_k^D = \frac{1}{\lambda_k^D + 1},$$

and inequalities (20), (45) we get the estimate

$$\left| \frac{1}{\lambda_k(\alpha) + 1} - \frac{1}{\lambda_k^D + 1} \right| \leq C_7 \alpha^{-1/2}. \quad (46)$$

Therefore,

$$|\lambda_k^D - \lambda_k(\alpha)| \leq C_7 \alpha^{-1/2} (\lambda_k^D + 1) (\lambda_k(\alpha) + 1). \quad (47)$$

and taking into account inequalities (49) (see Section 4), we obtain the estimate

$$0 \leq \lambda_k^D - \lambda_k(\alpha) \leq C_7 \alpha^{-1/2} (\lambda_k^D + 1)^2 \leq C_1 \alpha^{-1/2} (\lambda_k^D)^2. \quad (48)$$

The proof of Theorem 1 is completed. \square

4. General properties of eigenvalues

In this Section, we give the proof of Theorem 2.

Proof. Due to (18), $\lambda_k(\cdot)$ is an increasing function. Using (19) and the inclusion $\overset{\circ}{H}^1(\Omega) \subset H^1(\Omega)$, we have

$$\begin{aligned} \lambda_k(\alpha) &= \sup_{v_1, \dots, v_{k-1} \in L_2(\Omega)} \inf_{\substack{v \in H^1(\Omega) \\ (v, v_j)_{L_2(\Omega)} = 0 \\ j = 1, \dots, k-1}} \frac{\int_{\Omega} |\nabla v|^2 dx + \alpha \int_{\Gamma} \sigma v^2 ds}{\int_{\Omega} v^2 dx} \\ &\leq \sup_{v_1, \dots, v_{k-1} \in L_2(\Omega)} \inf_{\substack{v \in \overset{\circ}{H}^1(\Omega) \\ (v, v_j)_{L_2(\Omega)} = 0 \\ j = 1, \dots, k-1}} \frac{\int_{\Omega} |\nabla v|^2 dx + \alpha \int_{\Gamma} \sigma v^2 ds}{\int_{\Omega} v^2 dx} \\ &= \sup_{v_1, \dots, v_{k-1} \in L_2(\Omega)} \inf_{\substack{v \in \overset{\circ}{H}^1(\Omega) \\ (v, v_j)_{L_2(\Omega)} = 0 \\ j = 1, \dots, k-1}} \frac{\int_{\Omega} |\nabla v|^2 dx}{\int_{\Omega} v^2 dx} = \lambda_k^D. \end{aligned} \tag{49}$$

The continuity of $\lambda_k(\alpha)$ was proved in ([2, Ch. 6, Par. 2, No. 6]).

Inequality (8) can be proved by the following:

$$\begin{aligned} \lambda_1(\beta\alpha_1 + (1 - \beta)\alpha_2) &= \inf_{v \in H^1(\Omega)} \frac{\int_{\Omega} |\nabla v|^2 dx + (\beta\alpha_1 + (1 - \beta)\alpha_2) \int_{\Gamma} \sigma v^2 ds}{\int_{\Omega} v^2 dx} \\ &\geq \beta \inf_{v \in H^1(\Omega)} \frac{\int_{\Omega} |\nabla v|^2 dx + \alpha_1 \int_{\Gamma} \sigma v^2 ds}{\int_{\Omega} v^2 dx} \\ &\quad + (1 - \beta) \inf_{v \in H^1(\Omega)} \frac{\int_{\Omega} |\nabla v|^2 dx + \alpha_2 \int_{\Gamma} \sigma v^2 ds}{\int_{\Omega} v^2 dx} \\ &= \beta\lambda_1(\alpha_1) + (1 - \beta)\lambda_1(\alpha_2), \quad 0 < \beta < 1. \end{aligned}$$

The eigenvalue $\lambda_1(\alpha)$ is simple for all $-\infty < \alpha < \infty$. The family of self-adjoint operators $(I + \alpha T)^{-1}B$ in the space $H^1(\Omega)$ with norm (13) satisfies the conditions of the asymptotic perturbation theorem ([7, Ch. 8, Par. 4, Th. 2.9]). It means that the eigenvalue $\lambda_1(\alpha)$ is a differentiable function of α . So

$$\lim_{j \rightarrow \infty} \frac{\lambda_1(\alpha_j) - \lambda_1(\alpha)}{\alpha_j - \alpha} = \lambda_1'(\alpha) \tag{50}$$

for an arbitrary sequence $\alpha_j \rightarrow \alpha, j \rightarrow \infty, \alpha_j \neq \alpha$. Let $\alpha_j \rightarrow \alpha, j \rightarrow \infty$, and $\|u_{1,\alpha_j}\|_{L_2(\Omega)} = 1, u_{1,\alpha_j} \geq 0$. Therefore, $\|u_{1,\alpha_j}\|_{H^1(\Omega)} \leq C_8$. By (11), the functions u_{1,α_j} satisfy

$$\int_{\Omega} (\nabla u_{1,\alpha_j}, \nabla v) dx + \alpha_j \int_{\Gamma} \sigma u_{1,\alpha_j} v ds = \lambda_1(\alpha_j) \int_{\Omega} u_{1,\alpha_j} v dx. \tag{51}$$

Now, we can choose a subsequence $u_{1,\alpha_j} \rightharpoonup u$ weakly in $H^1(\Omega)$ and $\|u_{1,\alpha_j} - u\|_{L_2(\Omega)} \rightarrow 0$, $\|u_{1,\alpha_j} - u\|_{L_2(\Gamma)} \rightarrow 0$. It means that $u \geq 0$ and $\|u\|_{L_2(\Omega)} = 1$. Due to (51), u satisfies the integral identity

$$\int_{\Omega} (\nabla u, \nabla v) dx + \alpha \int_{\Gamma} \sigma uv ds = \lambda_1(\alpha) \int_{\Omega} uv dx. \tag{52}$$

Hence, by the uniqueness of the first positive normalized eigenfunction $u = u_{1,\alpha}$ and

$$\|u_{1,\alpha_j} - u_{1,\alpha}\|_{L_2(\Omega)} \rightarrow 0, \quad j \rightarrow \infty. \tag{53}$$

Now, we have

$$\begin{aligned} & \int_{\Omega} |\nabla(u_{1,\alpha_j} - u_{1,\alpha})|^2 dx + \alpha \int_{\Gamma} \sigma(u_{1,\alpha_j} - u_{1,\alpha})^2 ds \\ &= \lambda_1(\alpha) \int_{\Omega} (u_{1,\alpha_j} - u_{1,\alpha})^2 dx \\ & \quad + (\lambda_1(\alpha_j) - \lambda_1(\alpha)) \int_{\Omega} u_{1,\alpha_j} (u_{1,\alpha_j} - u_{1,\alpha}) dx \\ & \quad - (\alpha_j - \alpha) \int_{\Gamma} \sigma u_{1,\alpha_j} (u_{1,\alpha_j} - u_{1,\alpha}) ds. \end{aligned} \tag{54}$$

It follows from (54) that

$$\begin{aligned} \|u_{1,\alpha_j} - u_{1,\alpha}\|_{H^1(\Omega)}^2 &\leq C_9 \left(|\alpha| \|u_{1,\alpha_j} - u_{1,\alpha}\|_{L_2(\Gamma)}^2 \right. \\ & \quad + (|\lambda_1(\alpha)| + 1) \|u_{1,\alpha_j} - u_{1,\alpha}\|_{L_2(\Omega)}^2 \\ & \quad + |\lambda_1(\alpha_j) - \lambda_1(\alpha)| \|u_{1,\alpha_j} - u_{1,\alpha}\|_{L_2(\Omega)} \|u_{1,\alpha_j}\|_{L_2(\Omega)} \\ & \quad \left. + |\alpha_j - \alpha| \|u_{1,\alpha_j} - u_{1,\alpha}\|_{L_2(\Gamma)} \|u_{1,\alpha_j}\|_{L_2(\Gamma)} \right). \end{aligned} \tag{55}$$

Applying (50) and (16) with sufficiently small ε we obtain

$$\|u_{1,\alpha_j} - u_{1,\alpha}\|_{H^1(\Omega)}^2 \leq C_{10} \left(\|u_{1,\alpha_j} - u_{1,\alpha}\|_{L_2(\Omega)}^2 + (\alpha_j - \alpha)^2 \|u_{1,\alpha_j}\|_{H^1(\Omega)}^2 \right). \tag{56}$$

Due to (16), (53) and (56) we get

$$\|u_{1,\alpha_j} - u_{1,\alpha}\|_{L_2(\Gamma)} \leq C_{11} \|u_{1,\alpha_j} - u_{1,\alpha}\|_{H^1(\Omega)} \rightarrow 0, \quad j \rightarrow \infty.$$

Therefore,

$$\int_{\Gamma} \sigma u_{1,\alpha_j}^2 ds \rightarrow \int_{\Gamma} \sigma u_{1,\alpha}^2 ds, \quad j \rightarrow \infty. \tag{57}$$

Now, to obtain (9) we use the inequalities

$$\begin{aligned} \lambda_1(\alpha_j) - \lambda_1(\alpha) &= \lambda_1(\alpha_j) - \inf_{v \in H^1(\Omega)} \frac{\int_{\Omega} |\nabla v|^2 dx + \alpha \int_{\Gamma} \sigma v^2 ds}{\int_{\Omega} v^2 dx} \\ &\geq \lambda_1(\alpha_j) - \frac{\int_{\Omega} |\nabla u_{1,\alpha_j}|^2 dx + \alpha \int_{\Gamma} \sigma u_{1,\alpha_j}^2 ds}{\int_{\Omega} u_{1,\alpha_j}^2 dx} = (\alpha_j - \alpha) \frac{\int_{\Gamma} \sigma u_{1,\alpha_j}^2 ds}{\int_{\Omega} u_{1,\alpha_j}^2 dx} \end{aligned}$$

and

$$\begin{aligned} \lambda_1(\alpha_j) - \lambda_1(\alpha) &= \inf_{v \in H^1(\Omega)} \frac{\int_{\Omega} |\nabla v|^2 dx + \alpha_j \int_{\Gamma} \sigma v^2 ds}{\int_{\Omega} v^2 dx} - \lambda_1(\alpha) \\ &\leq \frac{\int_{\Omega} |\nabla u_{1,\alpha}|^2 dx + \alpha_j \int_{\Gamma} \sigma u_{1,\alpha}^2 ds}{\int_{\Omega} u_{1,\alpha}^2 dx} - \lambda_1(\alpha) = (\alpha_j - \alpha) \frac{\int_{\Gamma} \sigma u_{1,\alpha}^2 ds}{\int_{\Omega} u_{1,\alpha}^2 dx}. \end{aligned}$$

Therefore, for $\alpha_j > \alpha$

$$\frac{\int_{\Gamma} \sigma u_{1,\alpha_j}^2 ds}{\int_{\Omega} u_{1,\alpha_j}^2 dx} \leq \frac{\lambda_1(\alpha_j) - \lambda_1(\alpha)}{\alpha_j - \alpha} \leq \frac{\int_{\Gamma} \sigma u_{1,\alpha}^2 ds}{\int_{\Omega} u_{1,\alpha}^2 dx}. \tag{58}$$

Finally, it follows from (50), (57) and (58) that

$$\lambda'_1(\alpha) = \frac{\int_{\Gamma} \sigma u_{1,\alpha}^2 ds}{\int_{\Omega} u_{1,\alpha}^2 dx}.$$

By ([11, Ch. 4, Par. 2, Th. 4]), $u_{1,\alpha} \in H^2(\Omega)$ and it satisfies equation (1) almost everywhere and the boundary condition in the sense of trace (the so-called strong solution). In the case $\int_{\Gamma} \sigma u_{1,\alpha}^2 ds = 0$, by (2) we have:

$$u_{1,\alpha} = \frac{\partial u_{1,\alpha}}{\partial \nu} = 0 \quad \text{on } \Gamma.$$

Applying the uniqueness theorem to the Cauchy problem for second-order elliptic equations ([8, Ch. 1, Par. 3, Th. 1.46]), we get $u_{1,\alpha} = 0$ in Ω . This contradiction proves that $\lambda'_1(\alpha) > 0$ for all α . Taking into account (9), we have the inequality $\lambda_1(\alpha) < \lambda_1^D$.

By combining the result from [10] with (9) we obtain the relations

$$\begin{aligned} \alpha \lambda'_1(\alpha) &= \frac{\alpha \int_{\Gamma} \sigma u_{1,\alpha}^2 ds}{\int_{\Omega} u_{1,\alpha}^2 dx} \leq \frac{\int_{\Omega} |\nabla u_{1,\alpha}|^2 dx + \alpha \int_{\Gamma} \sigma u_{1,\alpha}^2 ds}{\int_{\Omega} u_{1,\alpha}^2 dx} \\ &= \lambda_1(\alpha) = -\alpha^2 \sigma_1^2 (1 + \varrho(\alpha)), \quad \varrho(\alpha) \rightarrow 0, \quad \alpha \rightarrow -\infty. \end{aligned}$$

Hence,

$$\frac{\lambda'_1(\alpha)}{-\alpha} \geq \sigma_1^2 (1 + \varrho(\alpha)), \quad \alpha < 0,$$

and inequality (10) is proved.

This completes the proof of Theorem 2. □

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