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A tiling involution for the Sury's identity

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Abstract. We study integer sequences defined by the recurrence $U_{n+2} = pU_{n+1} + U_n$ and the initial values $U_0 = a$, $U_1 = 1$, for $n \geq 0$. We find families of identities of these sequences, some of which Sury's identities are a special case. We prove these identities by using a combinatorial interpretation by means of tiling. In particular, we present a tiling involution of the alternating sign dual of the first Sury's identity.

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1. Introduction

Recently, Kuhapatanakul and Thongsing [6] have defined a refined two parametric integer sequence denoted as $U_n(a,p)$, briefly U_n, by the following relation:

$$U_{n+2}(a,p) = p\,U_{n+1}(a,p) + U_n(a,p) \;\; (n \geq 0), \tag{1}$$

with the initial values $U_0(a,p) = a$ and $U_1(a,p) = 1$, where a is a nonnegative and p is a positive integer. This family of sequences encounters some well-known integer sequences. Note that $U_n(0,1) = F_n$ and $U_n(2,1) = L_n$ are the Fibonacci numbers and the Lucas numbers, respectively, while for $U_n(2,3)$ we get a generalized Pellian sequence.

Martinjak [7] gave an algebraic proof for an interesting relation involving Fibonacci and Lucas numbers:

$$\sum_{k=0}^{n} (-1)^k\, 2^{n-k}\, L_{k+1} = (-1)^n\, F_{n+1}, \tag{2}$$

for which Kuhapatanakul and Thongsing found a generalized version using U_n. This Fibonacci–Lucas identity came in a couple with the identity

$$\sum_{k=0}^{n} 2^k\, L_k = 2^{n+1} F_{n+1} \tag{3}$$

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that can be found in the paper by Sury [9]. Relations (2) and (3) are called Sury's identities (and we shall call (3) the first Sury's identity, and its alternating companion the second Sury's identity).

Let us recall that an n-board consists of n cells (a 1×1 square), and when all cells of the board are covered by tiles, we have an n-tiling. A well-known combinatorial interpretation of Fibonacci number F_n is that it enumerates the number of tilings of an $(n-1)$-board with 1×1 tiles (monomios) and 1×2 tiles (dominoes). Furthemore, the number of tilings with m colors of monomios and m^2 colors of dominoes is equal to $m^n F_n$. For details, see the book by Benjamin and Quinn [3], and the paper by Brigham et al. [4].

It is worth mentioning integer sequence identities that are proved by means of tiling found by Benjamin et al. [2] and K. Edwards and M. A. Allen [5]. Furthermore, there are a few results involving generalized Fibonacci numbers. Let us mention that Ait-Amrane and Behloul extended Cassini's formula to generalized Fibonacci numbers [1].

Motivated by these results, here we aim at finding a combinatorial interpretation for the numbers U_n. Once we have such an interpretation, we use it to prove a family of identities involving these numbers. In addition, since a combinatorial proof for Sury's identity (3) is already known [8], we were curious to find a combinatorial proof for Sury's identity with alternating sign (2).

2. Identities for the sequence $(U_n)_{n\geq 0}$

We begin with a few relations for the sequence $(U_n)_{n\geq 0}$, which we prove by strong induction. After that, we use one of these results for proving an identity that is a generalization of relation (2).

Proposition 1. *Let n be a nonnegative integer. Then for the sequence $(U_n)_{n\geq 0}$ we have*

$$U_{n+1}(a,p) = U_{n+2}(0,p) - (p-1)U_{n+1}(0,p) + (a-1)U_n(0,p). \tag{4}$$

Proof. We argue by strong induction on n and use relation (1) to complete the inductive step.

Relation (4) holds true for $n = 0$, since the right-hand side of the equation gives $p - (p-1) = 1$, which is equal to $U_1(a,p)\,(=1)$. Moreover, for $n = 1$, the left-hand side of the equation gives $U_2(a,p) = p + a$ by relation (1), and the right-hand side of the equation gives $p^2 + 1 - (p-1)p + a - 1 = p + a$ since $U_3(0,p) = p^2 + 1$. Thus, this relation holds true for $n = 1$.

Furthermore, we have

$$\begin{aligned}
U_{k+1}(a,p) &= p\,U_k(a,p) + U_{k-1}(a,p)\\
&= p\,(U_{k+1}(0,p) - (p-1)U_k(0,p) + (a-1)U_{k-1}(0,p))\\
&\quad + U_k(0,p) - (p-1)U_{k-1}(0,p) + (a-1)U_{k-2}(0,p)\\
&= U_{k+2}(0,p) - (p-1)U_{k+1}(0,p) + (a-1)U_k(0,p).
\end{aligned}$$

□

In particular, when $a = 2$ and $p = 1$, Proposition 1 gives the basic Fibonacci–Lucas relation

$$L_{n+1} = F_{n+2} + F_n.$$

Lemma 1. *Let n be a nonnegative integer. Then for the sequence $(U_n)_{n\geq 0}$ we have*

$$U_{n+1}(a,p) = U_{n+1}(0,p) + a\,U_n(0,p). \tag{5}$$

Proof. Again, we use relation (1) to complete the inductive step. Relation (5) obviously holds true for $n = 0$ since $U_1(a,p) = 1$, $U_1(0,p) = 1$ and $U_0(0,p) = 0$. This relation holds true for $n = 1$ since $U_2(a,p) = p + a$, $U_2(0,p) = p$ and $U_1(0,p) = 1$.

Furthermore, we have

$$\begin{aligned} U_{k+1}(a,p) &= pU_k(a,p) + U_{k-1}(a,p) \\ &= p\left(U_k(0,p) + aU_{k-1}(0,p)\right) + U_{k-1}(0,p) + aU_{k-2}(0,p) \\ &= U_{k+1}(0,p) + aU_k(0,p). \end{aligned}$$

□

Now, we are ready to prove the announced generalization of the second Sury's identity.

Theorem 1. *Let n be a nonnegative integer. Then for the sequence $(U_n)_{n\geq 0}$ we have*

$$\sum_{k=0}^{n}(-1)^k\,a^{n-k}\,U_{k+1}(a,p) = (-1)^n U_{n+1}(0,p).$$

Proof. For purpose of proving it, we use Lemma 1. For an even n we have

$$\begin{aligned} \sum_{k=0}^{n}(-1)^k a^{n-k} U_{k+1}(a,p) &= \sum_{k=0}^{n}(-1)^k a^{n-k}\left(U_{k+1}(0,p) + aU_k(0,p)\right) \\ &= a^n\left(U_1(0,p) + aU_0(0,p)\right) - a^{n-1}\left(U_2(0,p) + aU_1(0,p)\right) \\ &\quad + \cdots - a\left(U_n(0,p) + aU_{n-1}(0,p)\right) + U_{n+1}(0,p) + aU_n(0,p) \\ &= U_{n+1}(0,p). \end{aligned}$$

We deal with an odd n in the same fashion, which completes the proof. □

In particular, when $p = 1$ and $a = 2$, Theorem 1 gives Sury's identity with alternating sign (2).

3. A combinatorial interpretation of the sequence $(U_n)_{n\geq 0}$

Definition 1. *A tiling with monomios and dominoes such that*

(*i*) *there are p colors for monomios, except the ones covering the first cell of a board which remain uncolored,* (*ii*) *dominoes that cover the first two cells in a tiling come in a phases, while the other dominoes are in one color,*

is called a phased tiling*. In particular, for $a = 0$, the number of domino phases is zero and there are no tilings starting with a domino.*

We let u_n be the number of distinct phased tilings of an n–board. Note that a tiling can start either with a monomio or a domino (in a phases). These tiles can be followed by a monomio in p colors or by a domino. Thus, there are four possibilities for the first two tiles in a tiling, as illustrated in Figure 1.

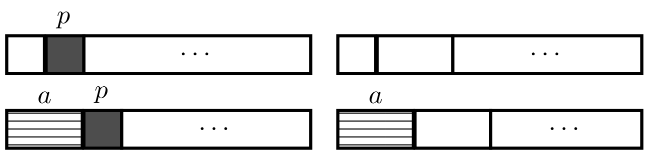

Figure 1: *Possibilities of the first two tiles*

The total number of phased tilings of an $(n+2)$-board beginning with monomio–monomio and those beginning with domino–monomio is equal to pu_{n+1}. In order to prove this, we establish a $1 : p$ correspondence between the set of such tilings (both beginning with monomio–monomio and domino–monomio) of an $(n + 2)$-board and the set of all tilings of an $(n + 1)$-board. The correspondence is done by adding (removing) a monomio after the first tile in a tiling.

Similarly, dealing with a domino instead of a monomio, one can see that the total number of phased tilings of the remaining two possibilities (monomio-domino and domino-domino) is equal to u_n. Thus, for the number of phased tilings of an $(n + 2)$-board we obtain the recurrence

$$u_{n+2} = pu_{n+1} + u_n.$$

Since the initial values $u_0 = a$ and $u_1 = 1$ correspond with those of the sequence $(U_n)_{n\geq 0}$, we get

$$u_n = U_n(a, p).$$

This proves the following

Lemma 2. *There are U_n distinct phased tilings of an n–board.*

As an example, one can easily check that there are 7 phased tilings of a 4–board for $p = 1$ and $a = 2$ (see Figure 2). Furthermore, there are 11 such tilings of a 5–board, 18 tilings of a 6–board, etc. In general, there are $U_n(2, 1) = L_n$ such tilings of an n–board.

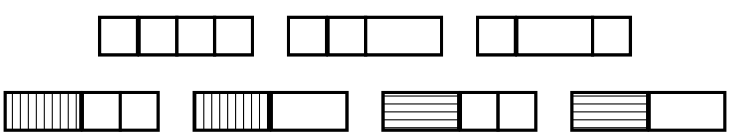

Figure 2: *Phased tilings of the 4–board*

Now, we are able to prove Proposition 1 in another way this time, using phased tilings. Let us write relation (4) as

$$U_n(a,p) + (p-1)U_n(0,p) = U_{n+1}(0,p) + (a-1)U_{n-1}(0,p),$$

and let us consider phased tilings of p n–boards beginning with a monomio and phased tilings of an n–board beginning with a domino in a phases; for the purpose of counting them in two ways. Take tilings of a board beginning with a monomio and all those tilings beginning with a domino. By Lemma 2, the number of such tilings is equal to $U_n(a,p)$ and the number of remaining tilings is equal to $(p-1)U_n(0,p)$, which is left-hand side of the identity above.

On the other hand, take tilings beginning with a domino in one of a phases and all those tilings beginning with a monomio. By adding a monomio at the beginning of each tiling we establish correspondence with the set of phased tilings of an $(n+1)$–board for $a = 0$. So, the number of those tilings is equal to $U_{n+1}(0,p)$. As for the rest of tilings beginning with a domino (in $a-1$ phases), we replace the first domino with a monomio and in this way we obtain phased tilings of $a-1$ $(n-1)$–boards beginning with a monomio. Thus, in total, we have $U_{n+1}(0,p) + (a-1)U_{n-1}(0,p)$ tilings as desired.

4. A tiling involution

Here, we shall present a combinatorial proof for identity (2) using phased tilings. Since identity (2) is a special case of Theorem 1, we shall also see how to modify our combinatorial approach to Theorem 1.

For n a nonnegative even integer, we consider phased tilings of boards of consecutive lengths for parameters $a = 2$ and $p = 1$. Additionally, let us assign a color to the first tile: if we have a $(k+1)$–board, we choose among 2^{n-k} colors for the first tile. We let $\mathcal{P}_{k+1}$ be the set consisting of such tilings of a $(k+1)$–board, $k = 0, \ldots, n$. The number of tilings of a $(k+1)$–board for parameters $a = 2$ and $p = 1$ is $U_{k+1}(2,1)$, which is equal to L_{k+1}. The total number of tilings in $\mathcal{P}_{k+1}$ is $2^{n-k}L_{k+1}$.

We let $\mathcal{O}_n$ and $\mathcal{E}_n$ denote the sets of such tilings of odd and even lengths, respectively. More precisely,

$$\mathcal{O}_n \coloneqq \bigcup_{\substack{0 \le k \le n \\ 1 \equiv (k+1) \pmod 2}} \mathcal{P}_{k+1}$$

and

$$\mathcal{E}_n \coloneqq \bigcup_{\substack{0 \le k \le n \\ 0 \equiv (k+1) \pmod 2}} \mathcal{P}_{k+1}.$$

Then we have

$$|\mathcal{O}_n| = 2^n L_1 + 2^{n-2} L_3 + \cdots + 2^2 L_{n-1} + L_{n+1} = \sum_{k \text{ even}}^{n} 2^{n-k} L_{k+1}$$

and

$$|\mathcal{E}_n| = 2^{n-1}L_2 + 2^{n-3}L_4 + \cdots + 2L_n = \sum_{k \text{ odd}}^{n} 2^{n-k} L_{k+1}.$$

Thus,

$$|\mathcal{O}_n| - |\mathcal{E}_n| = \sum_{k=0}^{n} (-1)^k \, 2^{n-k} \, L_{k+1}.$$

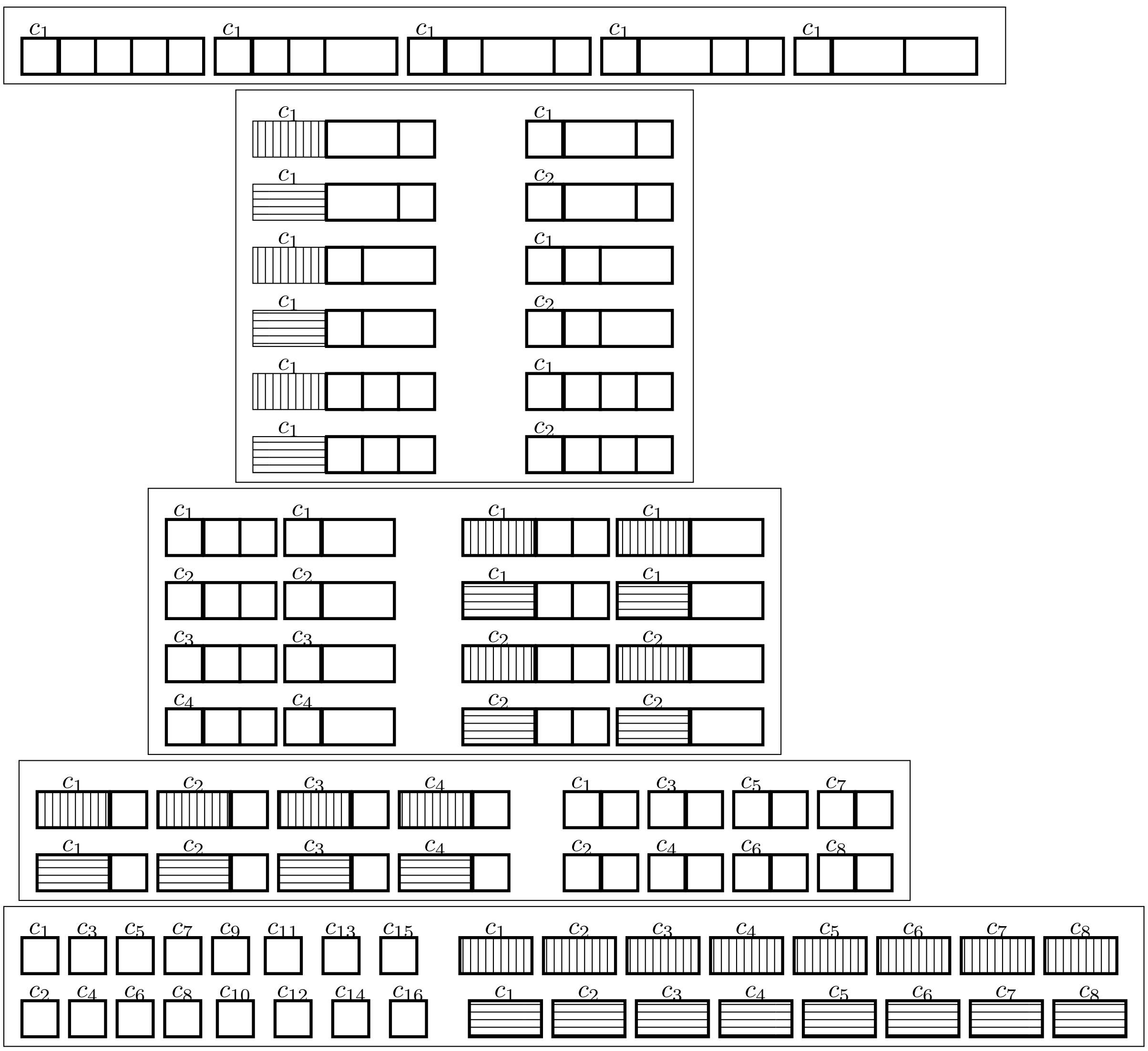

Figure 3: *The sets $\mathcal{O}_4$ and $\mathcal{E}_4$*

The first domino comes in two phases, p_1 and p_2. Colors for the first tile of a $(k+1)$–board are denoted by c_i, $1 \le i \le 2^{n-k}$. We establish correspondence between tilings described above of even and odd boards in the following way. We replace the first tiles, a monomio with a domino and vice versa, as follows:

- a monomio in color c_{2i-1} and a domino in color c_i and in phase p_1,
- a monomio in color c_{2i} and a domino in color c_i and in phase p_2.

When replacing the first tile, we change the parity of the length of a tiling.

Since we consider tilings of lengths $1, 2, \ldots, n+1$, it is not possible to perform such replacement among all tilings. The only exceptions are tilings with maximum length $n+1$ starting with a monomio. For these unpaired tilings $a = 0$ and the total number of such tilings is $U_{n+1}(0, 1)$, which is equal to F_{n+1}. Thus, for an even n, we have

$$|\mathcal{O}_n| - |\mathcal{E}_n| = F_{n+1}.$$

We deal with an odd n in the same fashion. Thus, for the odd n we have

$$|\mathcal{E}_n| - |\mathcal{O}_n| = F_{n+1},$$

which completes the proof.

Figure 3 illustrates this involution. Tilings in the first rectangle together with tilings on the left of the rest of rectangles form the set $\mathcal{O}_4$, while the rest of tilings that are on the right form the set $\mathcal{E}_4$. A color of the first tile in each tiling is indicated above it. Vertical lines are for phase p_1 and horizontal lines are for phase p_2. Tilings are put in an appropriate rectangle based on the values of i and on replacements we perform on the first tile. So, when we replace the first tile in some tiling on the left (right), we get a tiling on the right (left) of the same row of the same rectangle. The only tilings we could not match are those in the rectangle on the top. They begin with a monomio (all in the same color) and represent all phased tilings for $a = 0$ and $p = 1$ of length 5. Hence in the first rectangle we have $U_5(0, 1) = F_5$ tilings, and we can conclude,

$$|\mathcal{O}_4| - |\mathcal{E}_4| = F_5$$

in this particular case.

In the same manner as in the proof of Sury's identity one can expand this proof in order to prove Theorem 1. The changes in the proof above are as follows. The first domino comes in a phases, denoted by $p_1, p_2, \ldots, p_a$. For a $(k+1)$–board, one chooses among a^{n-k} colors for the first tile. We replace the first tiles, a monomio with a domino and vice versa, according to pairs:

- a monomio in color c_{ia+j} and a domino in color c_{i+1} and in phase p_j,
- a monomio in color c_{ia} and a domino in color c_i and in phase p_a.

The only exceptions that occur are those tilings of an $(n+1)$–board beginning with a monomio. Thus, for an even n we have

$$|\mathcal{O}_n| - |\mathcal{E}_n| = U_{n+1}(0, p),$$

and for an odd n we have

$$|\mathcal{E}_n| - |\mathcal{O}_n| = U_{n+1}(0, p),$$

which together gives

$$\sum_{k=0}^{n}(-1)^k\, a^{n-k}\, U_{k+1}(a,p) = (-1)^n U_{n+1}(0,p).$$

This completes the proof of Theorem 1.

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