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Covering numbers with involutions in decomposing infinite matrices

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Abstract. Let D be a division ring. The aim of this paper is to explore the problem of decomposing an infinite matrix over D into a product of involutions and a product of commutators of involutions within the context of covering numbers. Specifically, we focus on decomposing matrices in the commutator subgroup $\mathrm{SL}_{VK,\infty}(D)$ of the Vershik–Kerov group and in the subgroup $\mathrm{SL}_\infty(D)$ of the stable general linear group $\mathrm{GL}_\infty(D)$.

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1. Introduction

Let G be a group with the identity element 1 and let X be a set of generators of G such that $x^{-1} \in X$ for every $x \in X$. Then, for every $g \in G$, there exist $x_1, \dots, x_k \in X$ such that $g = x_1 x_2 \cdots x_k$, where k is a positive integer. We denote $X^k = \{x_1 x_2 \cdots x_k \mid x_i \in X, i = 1, \dots, k\}$. The *covering number of* G *by* X, denoted by $\mathrm{cn}_X(G)$, is defined to be the smallest integer k such that $X^k = G$ or ∞ if no such k exists. For instance, if $\mathcal{C}$ is the set of all commutators $aba^{-1}b^{-1}$, where a, b range over G and $G' = [G, G]$ is the commutator subgroup of G, then the covering number $\mathrm{cn}_\mathcal{C}(G')$ is called the *commutator width* of G.

An element x in the group G is called an *involution* if $x^2 = 1$. If a and b are involutions in G, then $[a, b] = aba^{-1}b^{-1}$ is called the *commutator of involutions*. We denote $\mathcal{I}$ and $\mathcal{CI}$ as the sets of involutions and the commutators of involutions in the group G, respectively. Clearly, $\mathcal{I}$ and $\mathcal{CI}$ are closed under taking the inverse. In this paper, we will evaluate the covering numbers of the subgroup $\mathrm{SL}_\infty(D)$ of the stable general linear group $\mathrm{GL}_\infty(D)$ and the commutator subgroup $\mathrm{SL}_{VK,\infty}(D)$ of the Vershik–Kerov group by these sets. To evaluate such covering numbers, we will decompose infinite matrices within these groups into a product of involutions and a product of commutators of involutions.

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The decomposition of elements in a group, especially linear groups, into products of involutions has received significant attention from the mathematical community, e.g. see [1, 2, 3, 7, 8, 13, 15]. Assume that R is a unitary associative ring. The notation $\mathrm{GL}_n(R)$ denotes the group of invertible $n\times n$ matrices over R, and $\mathrm{SL}_n(R)$ is the commutator subgroup of $\mathrm{GL}_n(R)$. A matrix $A \in \mathrm{GL}_n(R)$ is called an *involution* if $A^2 = \mathbf{1}_n$, where $\mathbf{1}_n$ is the identity matrix. Over an arbitrary field, every matrix with determinant ± 1 can be expressed as a product of at most four involutions [7, Theorem]. Note that if A is an involution, then both A^{-1} and $B^{-1}AB$ are also involutions. Therefore, a commutator of involutions is essentially the product of two involutions. In connection with this topic, X. Hou in [10] and T. N. Son et al. in [13] proved that a matrix over a field is a product of at most two commutators of involutions. These results have been extended to division rings in [2, Theorem 4.5 and Theorem 6.3].

Let D be a division ring. We define the notation $\mathrm{GL}_{c,\infty}(D)$ to denote the group consisting of all countable-dimensional column-finite invertible matrices. Moreover, we introduce $\mathbf{1}_\infty \in \mathrm{GL}_{c,\infty}(D)$ to denote the diagonal matrix with 1 entries along its diagonal. If we consider a matrix $A \in \mathrm{GL}_n(D)$ as the matrix $\begin{pmatrix} A & 0 \\ 0 & \mathbf{1}_\infty \end{pmatrix} \in \mathrm{GL}_{c,\infty}(D)$, then $\mathrm{GL}_n(D)$ becomes a subgroup of $\mathrm{GL}_{c,\infty}(D)$. A matrix in $\mathrm{GL}_{c,\infty}(D)$ is considered *unitriangular* if it is upper triangular and has diagonal entries equal to 1. The subgroup $\mathrm{T}_\infty(D)$ of $\mathrm{GL}_{c,\infty}(D)$ consists of all upper triangular matrices, while $\mathbb{UT}_\infty(D)$ denotes the subgroup of upper unitriangular matrices. Recall that the *Vershik–Kerov group*, denoted as $\mathrm{GL}_{VK,\infty}(D)$, is a subgroup of $\mathrm{GL}_{c,\infty}(D)$. This subgroup consists of matrices in the form $A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$, where $A_1 \in \mathrm{GL}_n(D), A_3 \in \mathrm{T}_\infty(D)$ and A_2 has the size $n \times \mathbb{N}$. We denote $\mathrm{SL}_{\infty,n}(D)$ as a subset of $\mathrm{GL}_{VK,\infty}(D)$, where $A_1 \in \mathrm{SL}_n(D)$ and $A_3 \in \mathrm{T}_\infty(D)$ with the main diagonal entries represented by elements $s_i \in D'$. Let $\mathrm{SL}_{VK,\infty}(D) = \bigcup_{n\geq 1} \mathrm{SL}_{\infty,n}(D)$. According to [1, Corollary 1.3], if D is a centrally finite division ring with more than three elements, then the commutator subgroup of $\mathrm{GL}_{VK,\infty}(D)$ is equal to $\mathrm{SL}_{VK,\infty}(D)$.

In Section 2, we prove that every matrix in $\mathrm{SL}_{VK,\infty}(D)$ can be expressed as a product of at most $8s+4$ involutions in $\mathrm{GL}_{VK,\infty}(D)$ provided that $\mathrm{cn}_{\mathcal{C}}(D') = s$, where D is a centrally finite division ring with more than three elements. We also prove that $\mathrm{cn}_{\mathcal{CI}}(\mathrm{SL}_{VK,\infty}(D)) \leq 9s+2$ if D is a noncommutative centrally finite division ring of characteristic different from 2 and $\mathrm{cn}_{\mathcal{C}}(D') = s$.

Recall that if D is a division ring, then the direct limit $\mathrm{GL}_\infty(D) = \varinjlim \mathrm{GL}_n(D)$ with respect to the transition homomorphisms $\mathrm{GL}_n(D) \longrightarrow \mathrm{GL}_{n+1}(D)$ given by $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$ is called the *stable general linear group* over D. The subgroup $\mathrm{SL}_\infty(D)$ of $\mathrm{GL}_\infty(D)$ is defined as $\mathrm{SL}_\infty(D) = \varinjlim \mathrm{SL}_n(D)$.

In Section 3, we prove that $\mathrm{cn}_{\mathcal{I}}(\mathrm{SL}_\infty(D)) \leq 4$ and $\mathrm{cn}_{\mathcal{CI}}(\mathrm{SL}_\infty(D)) \leq 5$ when D is a noncommutative centrally finite division ring such that $\mathrm{cn}_{\mathcal{C}}(D') < \infty$.

We present some remarks frequently utilized in this paper. The proofs of these claims are simple and for convenience we provide them here.

Remark 1. *Assume that D is a division ring. Then,*

(i) *The matrix* $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ *can be decomposed into a product of at most two involutions, where* $a \in D^*$.

(ii) *If* s *is a commutator, then* $\mathrm{diag}(1, s)$ *can be decomposed into a product of at most four involutions.*

(iii) *If* B_i *is an involution, then* $\oplus_{i\in\mathbb{N}} B_i$ *is also an involution for every* $i \in \mathbb{N}$.

(iv) *If* $A \in \mathrm{GL}_n(D)$ *and* $B \in \mathrm{GL}_m(D)$ *are products of* k *and* ℓ *involution matrices, respectively, then* $A \oplus B$ *is the product of at most* $\max\{k, \ell\}$ *involution matrices in* $\mathrm{GL}_{n+m}(D)$.

Proof. (i) It is demonstrated in [2, Lemma 5.2].

(ii) It follows from [2, Lemma 4.1].

(iii) We have $B_i^2 = \mathbf{1}_n$ for every $i \in \mathbb{N}$, leading to $(\oplus_{\lambda\in\mathbb{N}} B_i)^2 = \oplus_{\lambda\in\mathbb{N}} B_i^2 = \mathbf{1}_\infty$, which is also an involution.

(iv) We can assume that $k \leq \ell$. For every $i = 1, \ldots, k; j = 1, \ldots, \ell$ assume A_i, B_j are involution matrices such that $A = A_1 \ldots A_k$ and $B = B_1 \ldots B_\ell$. Then, $A \oplus B = C_1 \oplus \ldots \oplus C_\ell$ is a product of ℓ involutions, in which $C_i = A_i \oplus B_i$ for $i = 1, \ldots, k$ and $C_i = \mathbf{1}_n \oplus B_i$ for $i = k + 1, \ldots, \ell$. □

Remark 2. *Assume that* D *is a noncommutative division ring.*

(i) *Suppose* D *is a centrally finite division ring such that* $\mathrm{cn}_\mathcal{C}(D') < \infty$. *In this case,* $\mathrm{diag}(1, \ldots, 1, s) \in \mathrm{GL}_n(D)$ *is a product of at most* $3\mathrm{cn}_\mathcal{C}(D')$ *commutators of involutions if* $n \geq 3$ *or* $\mathrm{char} D \neq 2$. *Particularly, if* s *is a commutator, then the matrix* $\mathrm{diag}(1, \ldots, 1, s)$ *is a product of at most three commutators of involutions.*

(ii) *If* A_i *is a commutator of involutions for each* $i \in \mathbb{N}$, *then* $\oplus_{i\in\mathbb{N}} A_i$ *is also a commutator of involutions.*

(iii) *If* $A \in \mathrm{GL}_n(D)$ *and* $B \in \mathrm{GL}_m(D)$ *are each expressed as products of* k *and* ℓ *commutators of involutions, respectively, then* $A \oplus B$ *can be decomposed into a product of at most* $\max\{k, \ell\}$ *commutators of involutions in* $\mathrm{GL}_{n+m}(D)$.

Proof. The first statement is established in [3, Lemma 4.3 and Lemma 4.4]. The last two statements can be proven similarly to (iii) and (iv) of Remark 1. □

In this paper, we define a centrally finite division ring as one that has finite dimensionality over its center. We use the following notations: Let D be a division ring, and denote $D' = [D^*, D^*]$, where $D^* = D \setminus \{0\}$. We represent the diagonal matrix with elements $a_1, \ldots, a_n \in D$ on the main diagonal as $\mathrm{diag}(a_1, \ldots, a_n)$.

2. Decompositions of matrices in $\mathrm{SL}_{VK,\infty}(D)$

In this section, we decompose matrices in the subgroup $\mathrm{SL}_{VK,\infty}(D)$ of the Vershik–Kerov group, where D is a centrally finite division ring containing more than three elements, and evaluate the covering numbers of the subgroup $\mathrm{SL}_{VK,\infty}(D)$ by the set of commutators of involutions $\mathcal{CI}$.

Assume that $(N_\lambda)_{\lambda\in\Lambda}$, where $\Lambda \subseteq \mathbb{N}$ is a partition of $\mathbb{N}$. Then, a *finite or infinite Jordan block* is denoted as

$$J_{|N_\lambda|}(1,1) = \begin{pmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & \ddots & \ddots & \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix}.$$

Lemma 1. *Assume that D is a division ring, and $A \in \mathbb{UT}_\infty(D)$. Then, A is similar to an infinite Jordan block $\oplus_{\lambda\in\Lambda} J_{|N_\lambda|}(1,1)$, where $(N_\lambda)_{\lambda\in\Lambda}$ is a partition of $\mathbb{N}$, and $\Lambda \subseteq \mathbb{N}$ is a subset of the natural numbers.*

Proof. The lemma is established in [4, Corollary 3.4]. □

Suppose that R is a unitary associative ring. According to [11, Theorem 1.1], if 2 is invertible in R, then every matrix in the groups $\mathbb{UT}_n(R)$ and $\mathbb{UT}_\infty(R)$ can be written as a product of at most two commutators of involutions in $\mathrm{T}_\infty(R)$. Recently, we have shown that if R is a division ring, then every matrix in the group $\mathbb{UT}_n(R)$ can be expressed as a product of two involutions, which is a special case of [2, Lemma 4.3].

In the following lemma, we continue considering the group $\mathbb{UT}_\infty(R)$, where R is a division ring. Our goal is to reduce the number of involutions in the decomposition to 2, and the number of commutators of involutions to 1.

Lemma 2. *Assume that D is a division ring and $A \in \mathbb{UT}_\infty(D)$. Then,*

(i) *Every matrix in $\mathbb{UT}_\infty(D)$ can be expressed as a product of at most two involutions.*

(ii) *Every matrix in $\mathbb{UT}_\infty(D)$ can be written as a commutator of involutions, provided that* $\mathrm{char} D \neq 2$.

Proof. Assume that $A \in \mathbb{UT}_\infty(D)$. According to Lemma 1, the matrix A is similar to $\oplus_{\lambda\in\Lambda} J_{|N_\lambda|}(1,1)$.

(i) According to [9, Theorem 2.3], the matrix $J_{|N_\lambda|}(1,1)$ is a product of two involutions in $\mathbb{UT}_\infty(D)$. Therefore, A can be decomposed into a product of two involutions according to Remark 1.

(ii) According to [15, Lemma 7] and [11, Corollary 2.7], the matrix $J_{|N_\lambda|}(1,1)$ is a commutator of involutions if $\mathrm{char} D \neq 2$, so A is also a commutator of involutions. □

It is known that if D is a field with characteristic different from 2, then every matrix in $\mathrm{SL}_{VK,\infty}(D)$ can be expressed as a product of at most two commutators of involutions according to [11, Theorem 1.3]. Since matrices similar to involutions are also involutions, every matrix in $\mathrm{SL}_{VK,\infty}(D)$ can be expressed as a product of at most four involutions. The results presented below address this problem for division rings.

Theorem 1. *Let D be a centrally finite division ring with more than three elements. If $\mathrm{cn}_{\mathcal{C}}(D') = s$, then every element in $\mathrm{SL}_{VK,\infty}(D)$ can be decomposed into a product of at most $8s+4$ involutions in $\mathrm{GL}_{VK,\infty}(D)$.*

Proof. Let $A \in \mathrm{SL}_{VK,\infty}(D)$ and $F = Z(D)$. If A is central in $\mathrm{SL}_{VK,\infty}(D)$, then according to [1, Lemma 2.6], $A = \lambda \mathbf{1}_\infty$ for $\lambda \in F \cap D'$. Furthermore,

$$A = \mathrm{diag}(\lambda, 1, \lambda, 1, \dots)\mathrm{diag}(1, \lambda, 1, \lambda, \dots).$$

Since $\lambda \in D'$, there exist commutators $\lambda_1, \dots, \lambda_s$ such that $\lambda = \lambda_1 \dots \lambda_s$. Then,

$$\mathrm{diag}(\lambda, 1) = \mathrm{diag}(\lambda_1, 1)\mathrm{diag}(\lambda_2, 1) \dots \mathrm{diag}(\lambda_s, 1),$$

where each λ_i is a commutator for $i = 1, \dots, s$. By Remark 1, $\mathrm{diag}(\lambda, 1)$ can be expressed as a product of at most $4s$ involutions, and the same holds for

$$\mathrm{diag}(\lambda, 1, \lambda, 1 \dots).$$

Therefore, A is a product of at most $8s$ involutions.

Now, assume A is noncentral in $\mathrm{SL}_{VK,\infty}(D)$. In this case, $A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$, where A_1 is a matrix in $\mathrm{SL}_n(D)$, A_2 is an $n \times \mathbb{N}$ matrix, and $A_3 \in \mathrm{T}(\infty, D)$ with diagonal entries $\alpha_i \in D'$. We can express $A = MN$, where

$$M = \begin{pmatrix} \mathbf{1}_n & A_2A_3^{-1} \\ 0 & \mathbf{1}_\infty \end{pmatrix} \text{ and } N = \begin{pmatrix} A_1 & 0 \\ 0 & A_3 \end{pmatrix}.$$

We have $A_3 = UD$, where $D = \mathrm{diag}(\alpha_1, \alpha_2, \dots)$ and $U \in \mathbb{UT}_\infty(D)$. According to Lemma 2, the matrix U can be decomposed into a product of at most two involutions. Moreover,

$$D = \mathrm{diag}(\alpha_1, 1, \alpha_3, 1, \dots)\mathrm{diag}(1, \alpha_2, 1, \alpha_4, \dots).$$

Since $\alpha_i \in D'$, there exist commutators $a_1^i, \dots, a_s^i$ such that $\alpha_i = a_1^i \dots a_s^i$. Thus,

$$\mathrm{diag}(1, \alpha_i) = \mathrm{diag}(1, a_1^i) \dots \mathrm{diag}(1, a_s^i).$$

For each $k = 1, \dots, s$, the matrix $\mathrm{diag}(1, a_k^i)$ is a product of at most four involutions according to Remark 1. Therefore, $\mathrm{diag}(1, \alpha_i)$ is a product of $4s$ involutions. Similarly, according to Remark 1, the matrix $\mathrm{diag}(\alpha_1, 1, \alpha_3, 1, \ldots)$ is a product of at most $4s$ involutions. Hence, the matrix D is a product of at most $8s$ involutions. It follows that A_3 can be decomposed into a product of at most $8s+2$ involutions. Since $A_1 \in \mathrm{SL}_n(D)$, by [2, Theorem 4.5], the matrix A_1 is a product of at most $4s+4$ involutions. Thus, N is a product of at most $8s+2$ involutions. Therefore, A can be decomposed into a product of at most $8s+4$ involutions, as M can be expressed as a product of at most two involutions according to Lemma 2. □

Below is an alternative version of [11, Theorem 1.3] for a noncommutative division ring of characteristic different from 2.

Theorem 2. *Let D be a noncommutative centrally finite division ring of characteristic different from 2 and $\mathrm{cn}_{\mathcal{C}}(D') = s$. Then, $\mathrm{cn}_{\mathcal{CI}}(\mathrm{SL}_{VK,\infty}(D)) \leq 9s+2$.*

Proof. Assume $A \in \mathrm{SL}_{VK,\infty}(D)$ and $F = Z(D)$. If A is central in $\mathrm{SL}_{VK,\infty}(D)$, then $A = \lambda \mathbf{1}_\infty$ for $\lambda \in F \cap D'$. We have

$$A = \mathrm{diag}(\lambda, 1, 1, \lambda, 1, 1, \dots)\mathrm{diag}(1, \lambda, 1, 1, \lambda, 1, \dots)\mathrm{diag}(1, 1, \lambda, 1, 1, \lambda, \dots).$$

According to Remark 2, we observe that $\mathrm{diag}(\lambda, 1, 1)$ can be decomposed into a product of at most $3s$ commutators of involutions, and the same holds for

$$\mathrm{diag}(\lambda, 1, 1, \lambda, 1, 1, \dots).$$

Similarly, $\mathrm{diag}(1, \lambda, 1, 1, \lambda, 1, \dots)$ and $\mathrm{diag}(1, 1, \lambda, 1, 1, \lambda, \dots)$ are also a product of $3s$ commutators of involutions. Therefore, A is a product of at most $9s$ commutators of involutions.

Now consider A to be noncentral in $\mathrm{SL}_{VK,\infty}(D)$. Similarly, we have $A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$, where A_1 is a matrix in $\mathrm{SL}_n(D)$, and $A_3 \in \mathrm{T}_\infty(D)$ with entries on the main diagonal being $s_i \in D'$. Again, we have $A = MN$ with

$$M = \begin{pmatrix} \mathbf{1}_n & A_2 A_3^{-1} \\ 0 & \mathbf{1}_\infty \end{pmatrix}; \quad N = \begin{pmatrix} A_1 & 0 \\ 0 & A_3 \end{pmatrix}.$$

Since $M \in \mathbb{UT}_\infty(D)$, according to Lemma 2, M can be written as a commutator of involutions. Consider the matrix $A_3 = UD$ with $D = \mathrm{diag}(s_1, s_2, \dots)$ and $U \in \mathbb{UT}_\infty(D)$. Again, according to Lemma 2, the matrix U is a commutator of involutions. Moreover,

$$D = \mathrm{diag}(s_1, 1, 1, s_4, 1, 1, \dots)\mathrm{diag}(1, s_2, 1, 1, s_5, 1, \dots)\mathrm{diag}(1, 1, s_3, 1, 1, s_6, \dots).$$

Using an argument similar to the above, D can be decomposed into a product of at most $9s$ commutators of involutions. Therefore, A_3 can be represented as a product of at most $9s+1$ commutators of involutions. Furthermore, according to [2, Theorem 6.3], the matrix A_1 can be expressed as a product of at most $3s + 2$ commutators of involutions. Thus, N can be decomposed into a product of at most $9s + 1$ commutators of involutions. Therefore, A is a product of at most $9s + 2$ commutators of involutions. Furthermore, according to [1, Corollary 1.3] a product of commutators of involutions belongs to $\mathrm{SL}_{VK,\infty}(D)$. Therefore, $\mathrm{cn}_{\mathcal{CI}}(\mathrm{SL}_{VK,\infty}(D)) \le 9s + 2$. □

3. Decompositions of matrices in $\mathrm{SL}_\infty(D)$

In this section, we will evaluate the covering numbers of the subgroup $\mathrm{SL}_\infty(D)$ of the stable general linear group $\mathrm{GL}_\infty(D)$ by the set of involutions $\mathcal{I}$ and the set of commutators of involutions $\mathcal{CI}$. We need several lemmas for this purpose. To do this, we introduce the notation $\mathbb{LT}_n(D)$ (respectively, $\mathbb{UT}_n(D)$) to denote the group of lower (respectively, upper) unitriangular matrices in $\mathrm{GL}_n(D)$, where each matrix has elements on the main diagonal equal to 1 and (x) is a square matrix of size 1.

Lemma 3 (see [8], Lemma 10)**.** *Let $c = d_1 + \cdots + d_n$ be a partition of the number c, where $d_n \in \mathbb{N}$. If $g \in \mathrm{SL}_n(D) \setminus Z(\mathrm{SL}_n(D))$, then there is $\gamma \in \mathrm{SL}_n(D)$ such that $\gamma g \gamma^{-1} = vhu$, where $v \in \mathbb{LT}_n(D), u \in \mathbb{UT}_n(D)$ and $h = \mathrm{diag}(\epsilon_1, \cdots, \epsilon_n)$, where $\epsilon_i \in D'$ and each ϵ_i is a product of at most d_i commutators for all $1 \le i \le n$.*

Lemma 4. *Let D be a division ring and $n \geq 1$. If A is a noncentral matrix in $\mathrm{SL}_n(D)$, then there exists $P \in \mathrm{GL}_n(D)$ and $s \in D'$ such that*

$$P^{-1}AP = XHY,$$

where $X \in \mathbb{LT}_n(D), Y \in \mathbb{UT}_n(D)$ and $H = \mathrm{diag}(1, 1, \ldots, s)$. In specific cases, when D is finite dimensional over its center and A represents a lower or upper triangular matrix with pairwise nonconjugate diagonal entries $a_{11}, \ldots, a_{nn} \in D$, it follows that A is similar to the diagonal matrix $\mathrm{diag}(a_{11}, \ldots, a_{nn})$.

Proof. The first part of this lemma follows from [6, Theorem 2.1]. The remaining part is derived from [3, Lemma 3.2]. □

We know that if D is a centrally finite division ring, then every matrix in the group $\mathrm{SL}_\infty(D)$ is a commutator in $\mathrm{SL}_\infty(D)$, as stated in [8, Corollary 5]. Assume that D is a field; it has been shown in [12, Theorem 1.1] that every $A \in \mathrm{GL}_\infty(D)$ can be expressed as a product of three involutions if and only if $\det(A) = \pm 1$. The following theorem is an example showing that the result of [12, Theorem 1.1] does not hold when D is a noncommutative division ring.

For each element a in the division ring D, we denote $N(a)$ as the norm of the element a. For further details on the norm of elements in finite dimensional division rings, please refer to [5, p. 143]. Before presenting the main result, we state the following lemma.

Lemma 5. *Let D be a noncommutative centrally finite division ring and $n \geq 1$ an integer. Assume $g_1, g_2, \ldots, g_n \in D \setminus F^*$ are pairwise non-conjugate elements. Then there exist $\alpha \in F^*$ such that $\alpha g_1, \alpha g_2, \ldots, \alpha g_n, (\alpha g_1)^{-1}, \ldots, (\alpha g_n)^{-1}$ are pairwise non-conjugate.*

Proof. Let F be the center of D and $S = \{t \in F : t^{2m} \in T\}$, in which $m^2 = \dim_F D$ and $T = \left\{N(g_i^{-1})N(g_j^{-1}) : 1 \leq i, j \leq n\right\}$. For each pair $1 \leq i, j \leq n$, the equation $t^{2m} = N(g_i^{-1})N(g_j^{-1})$ represents a polynomial of degree $2m$ over the field F. It is well known that this equation can have at most $2m$ roots in F, thus the set S is finite. Because D is a noncommutative division ring and $\dim_F D$ is finite, F is infinite (see [14, Theorems 13.11 and 15.13]). Let $\alpha \in F^* \setminus S$. We shall show that α satisfies the required condition. Indeed, for every $i \neq j$, since g_i and g_j are non-conjugate, αg_i and αg_j are also non-conjugate, and similarly, $(\alpha g_i)^{-1}$ and $(\alpha g_j)^{-1}$ are non-conjugate as well. Next, we will prove that αg_i and $(\alpha g_j)^{-1}$ are non-conjugate by using the method of contradiction. Assume αg_i and $(\alpha g_j)^{-1}$ are conjugate for every $1 \leq i; j \leq n$. Then, there exists $h \in D^*$ such that $\alpha g_i = h^{-1}(\alpha g_j)^{-1}h$, implying $\alpha^2 = h^{-1}g_j^{-1}hg_i^{-1}$. Consequently, $N(\alpha^2) = N(h^{-1})N(g_j^{-1})N(h)N(g_i^{-1}) = N(g_j^{-1})N(g_i^{-1})$. By [5, p. 143], we have $\alpha^{2m} = N(g_j^{-1})N(g_i^{-1})$. This implies that $\alpha \in S$, which contradicts our initial choice of α. Therefore, αg_i and $(\alpha g_j)^{-1}$ are non-conjugate for every $1 \leq i; j \leq n$. Hence, $\alpha g_1, \alpha g_2, \ldots, \alpha g_n, (\alpha g_1)^{-1}, \ldots, (\alpha g_n)^{-1}$ are pairwise non-conjugate. □

By applying the above lemma, we obtain the following result.

Theorem 3. *Let D be a noncommutative centrally finite division ring such that $\mathrm{cn}_{\mathcal{C}}(D') < \infty$. Then,*

(i) $\mathrm{cn}_{\mathcal{I}}(\mathrm{SL}_\infty(D)) \leq 4$.

(ii) $\mathrm{cn}_{\mathcal{CI}}(\mathrm{SL}_\infty(D)) \leq 5$.

Proof. (i) Assume $A \in \mathrm{SL}_\infty(D)$. If $A = \mathbf{1}_\infty$, then A is always an involution. If $A \neq \mathbf{1}_\infty$, then there exists $n \geq 2$ such that $A \in \mathrm{SL}_n(D)$ and A is a noncentral element in $\mathrm{SL}_n(D)$. By Lemma 4, there exists $P \in \mathrm{GL}_n(D)$ such that $P^{-1}AP = UHV$, where $U \in \mathbb{LT}_n(D)$, $V \in \mathbb{UT}_n(D)$, and $H = \mathrm{diag}(1, 1, \dots, 1, x)$ for some $x \in D'$.

Since $\mathrm{cn}_{\mathcal{C}}(D') = s < \infty$, there exist elements $a_1, \dots, a_s$ and $b_1, \dots, b_s$ in D^* such that $x = [a_1, b_1] \dots [a_s, b_s]$. Now, we shall show $\mathrm{SL}_\infty(D) \subseteq \mathcal{I}^4$, where $\mathcal{I}$ is the set of involutions of $\mathrm{SL}_\infty(D)$.

Assume $n \leq 2s$. By [1, Lemma 2.3], we can choose elements $\lambda_1, \lambda_2, \dots, \lambda_s$ in F such that $\lambda_1 b_1, \lambda_2 b_2, \dots, \lambda_s b_s$ are pairwise non-conjugate. Since $[a_i, b_i] = [a_i, \lambda_i b_i]$, without loss of generality, we assume that $b_1, \dots, b_s$ are pairwise non-conjugate. By Lemma 5, there exists $\alpha \in F^*$ such that $\alpha b_1, \alpha b_2, \dots, \alpha b_s, (\alpha b_1)^{-1}, (\alpha b_2)^{-1}, \dots, (\alpha b_s)^{-1}$ are pairwise non-conjugate. By appropriately adjusting n, specifically by inserting $2s - n$ elements 1 into the main diagonal of A, we then consider A as a matrix of size $n = 2s$. By Lemma 3, the matrix A is similar to $U_1 H_1 V_1$, where $U_1 \in \mathbb{LT}_n(D)$, $V_1 \in \mathbb{UT}_n(D)$, and $H_1 = \mathrm{diag}(1, [a_1, b_1], 1, [a_2, b_2], \dots, 1, [a_s, b_s])$.

Put $S = \mathrm{diag}(1, a_1, 1, a_2, \dots, 1, a_s)$. Then

$$U_1 H_1 V_1 = SXS^{-1}Y,$$

where

$$X = \begin{pmatrix} (\alpha b_1)^{-1} & & & & \\ & \alpha b_1 & & & \\ & & \ddots & & \\ & * & & (\alpha b_s)^{-1} & \\ & & & & \alpha b_s \end{pmatrix}; Y = \begin{pmatrix} \alpha b_1 & & & & \\ & (\alpha b_1)^{-1} & & & \\ & & \ddots & & * \\ & & & \alpha b_s & \\ & & & & (\alpha b_s)^{-1} \end{pmatrix}.$$

Furthermore, by Lemma 4, X is similar to $\mathrm{diag}((\alpha b_1)^{-1}, \alpha b_1, \dots, (\alpha b_s)^{-1}, \alpha b_s)$. Then, according to Remark 1, X is a product of two involutions. Similarly, Y is also a product of two involutions. Therefore, A can be expressed as a product of at most four involutions. This leads to $\mathrm{SL}_\infty(D) \subseteq \mathcal{I}^4$.

Assume $n > 2s$ and n is even. By similar reasoning, we have $x = [a_1, b_1] \dots [a_s, b_s]$, in which $b_1, \dots, b_s$ in $D^* \setminus F$. We choose $b_{s+1}, b_{s+2}, \dots, b_{\frac{n}{2}}$ in $D^* \setminus F$. By Lemma 5, there exists $\lambda \in F^*$ such that $\lambda b_1, \lambda b_2, \dots, \lambda b_{\frac{n}{2}}, (\lambda b_1)^{-1}, \lambda b_1 \dots, (\lambda b_{\frac{n}{2}})^{-1}$ are non-conjugate. By Lemma 3, the matrix A is similar to $U_1 H_1 V_1$, where $U_1 \in \mathbb{LT}_n(D)$, $V_1 \in \mathbb{UT}_n(D)$ and $H_1 = \mathrm{diag}(1, \dots, 1, [a_1, b_1], 1, [a_2, b_2], \dots, 1, [a_s, b_s])$. Similarly, put $S = \mathrm{diag}(1, \dots, 1, a_1, 1, a_2, \dots, 1, a_s)$. Then

$$U_1 H_1 V_1 = SXS^{-1}Y,$$

where

$$X=\begin{pmatrix} (\lambda b_{s+1})^{-1} & & & & & & & & & \\ & \lambda b_{s+1} & & & & & & & & \\ & & \ddots & & & & & & & \\ & & & (\lambda b_{\frac{n}{2}})^{-1} & & & & & & \\ & & & & \lambda b_{\frac{n}{2}} & & & & & \\ & & & & & (\lambda b_1)^{-1} & & & & \\ & & & & & & \lambda b_1 & & & \\ & & \ast & & & & & \ddots & & \\ & & & & & & & & (\lambda b_s)^{-1} & \\ & & & & & & & & & \lambda b_s \end{pmatrix};$$

and

$$Y=\begin{pmatrix} \lambda b_{s+1} & & & & & & & & & \\ & (\lambda b_{s+1})^{-1} & & & & & & & & \\ & & \ddots & & & & & & & \\ & & & \lambda b_{\frac{n}{2}} & & & & & & \\ & & & & (\lambda b_{\frac{n}{2}})^{-1} & & & \ast & & \\ & & & & & \lambda b_1 & & & & \\ & & & & & & (\lambda b_1)^{-1} & & & \\ & & & & & & & \ddots & & \\ & & & & & & & & \lambda b_s & \\ & & & & & & & & & (\lambda b_s)^{-1} \end{pmatrix};$$

By Lemma 4, X is similar to

$$\operatorname{diag}\big((\lambda b_{s+1})^{-1}, \lambda b_{s+1}, \dots, (\lambda b_{\frac{n}{2}})^{-1}, \lambda b_{\frac{n}{2}}, (\lambda b_1)^{-1}, \dots, \lambda b_s\big).$$

Therefore, X is a product of two involutions and Y as well. When n is odd, by a similar argument, we obtain that X is similar to

$$\operatorname{diag}\big(1, (\lambda b_{s+1})^{-1}, \lambda b_{s+1}, \dots, (\lambda b_{\frac{n}{2}})^{-1}, \lambda b_{\frac{n}{2}}, (\lambda b_1)^{-1}, \dots, \lambda b_s\big).$$

This leads to $\mathrm{SL}_\infty(D) \subseteq \mathcal{I}^4$.

Next, we shall show that $\mathcal{I}^4 \subseteq \mathrm{SL}_\infty(D)$. Let $A \in \mathrm{SL}_\infty(D)$, and for each $i = 1, \dots, 4$ suppose X_i is an involution matrix such that $A = X_1X_2X_3X_4$. We shall show $X_i \in \mathrm{SL}_\infty(D)$. Indeed, we see that $A \in \mathrm{SL}_n(D) \setminus Z(\mathrm{SL}_n(D))$, where $n \geq 3$. According to [2, Corollary 2.4], a matrix is a product of involutions if and only if its Dieudonné determinant equals $\overline{\pm 1}$. Therefore, we have the following cases:

Case 1. $\det(X_i) = \overline{1}$ *for all* $i = 1, \dots, 4$. Then, it is evident that $X_i \in \mathrm{SL}_\infty(D)$.

Case 2. There exist two matrices with determinants equal to $\overline{-1}$. Without loss of generality, we can assume that $X_1, X_2 \in \mathrm{GL}_n(D)$ and $\det(X_1) = \det(X_2) = \overline{-1}$. Increasing the number n, and we rewrite A as

$$A = [X_1 \oplus (-1)][(X_2 \oplus (-1)][X_3 \oplus (1)][X_4 \oplus (1)].$$

Because $\det[X_1 \oplus (-1)] = \overline{1}$, we have $X_1 \oplus (-1) \in \mathrm{SL}_{n+1}(D)$. Similarly, $X_2 \oplus (-1), X_3 \oplus (1), X_4 \oplus (1)$ also belong to $\mathrm{SL}_{n+1}(D)$. Therefore, $X_i \in \mathrm{SL}_\infty(D)$ for each $i = 1, \dots, 4$.

Case 3. $\det(X_i) = \overline{-1}$ *for all* $i = 1, \dots, 4$. Using the same argument as in the proof of Case 2 and expressing A in the form

$$A = [X_1 \oplus (-1)][(X_2 \oplus (-1)][X_3 \oplus (-1)][X_4 \oplus (-1)],$$

we conclude that $X_i \in \mathrm{SL}_\infty(D)$.

Therefore, $\mathrm{SL}_\infty(D) = \mathcal{I}^4$.

(ii) According to the above, there exists $Q \in \mathrm{GL}_n(D)$ such that $Q^{-1}AQ = U_1H_1V_1$. Put $X = U_1, Y = H_1V_1H_1^{-1}$, and $Z = H_1$. Then

$$Q^{-1}AQ = XYZ.$$

Furthermore, according to Remark 2, Z is a product of at most three commutators of involutions. Since $X \in \mathbb{UT}_n(D)$, and $Y \in \mathbb{LT}_n(D)$, according to [3, Theorem 3.4], XY is a product of at most two commutators of involutions. Therefore, A can be expressed as a product of at most five commutators of involutions. Therefore, $\mathrm{SL}_\infty(D) \subseteq \mathcal{CI}^5$. Furthermore, every commutator of involutions belongs to $\mathrm{SL}_\infty(D)$, so $\mathrm{cn}_{\mathcal{CI}}(\mathrm{SL}_\infty(D)) = 5$. □

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