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Sobolev spaces of vector-valued functions on compact groups

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Abstract. This paper deals with a class of Sobolev spaces of vector-valued functions on a compact group. Using some results among which are the inversion formula and the Plancherel type theorem involving the Fourier transform of vector-valued functions, we define Sobolev spaces of Bessel potential type. Then, some continuous embedding results are proved.

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1. Introduction

Sobolev spaces have proven their effectiveness in mathematical sciences. They are well studied on certain classical spaces such as Euclidean spaces, and on more general differential manifolds. Some of them can be constructed via the Fourier transform and therefore they can be studied using techniques from abstract/classical harmonic analysis.

In the context of abstract harmonic analysis, some studies of Sobolev spaces can be found in [3, 4, 6, 8, 7]. Particularly in [7], the authors introduced Sobolev spaces of complex-valued functions over compact groups and studied their properties. They obtained, among other results, some continuous embedding and compact embedding theorems.

The aim of this paper is to study the vector-valued aspect of some results in [7]. More precisely, we introduce Sobolev spaces of vector-valued functions on a compact group and prove some continuous embedding theorems.

The rest of the paper is organized as follows. Section 2 is devoted to preliminaries on harmonic analysis of vector-valued functions on compact groups. Section 3 contains our main results.

2. Preliminaries

In this section, we recall very briefly some facts concerning harmonic analysis on compact groups, mainly their representation theory [5] and the Fourier transform of vector-valued functions defined on such groups [1].

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Let G be a compact Hausdorff group which may not be necessarily abelian. A concrete example of such a group is the special unitary group $SU(2)$ consisting of matrices $A = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$, where $a, b \in \mathbb{C}$ are such that $|a|^2 + |b|^2 = 1$. A unitary representation σ of G on a Hilbert space H_σ is a homomorphism $\sigma : G \longrightarrow \mathcal{U}(H_\sigma)$, where $\mathcal{U}(H_\sigma)$ denotes the group of unitary operators on H_σ. The Hilbert space H_σ is called the representation space of σ and the dimension of H_σ is called the dimension of σ and it is denoted by d_σ. A unitary representation σ is said to be continuous if the mapping $G \longrightarrow H_\sigma, x \longmapsto \sigma(x)\xi$ is continuous for every $\xi \in H_\sigma$. A representation σ of G on H_σ is called irreducible if there is no proper closed subspace M of H_σ which is invariant by σ, that is, $\forall x \in G, \forall \xi \in M, \sigma(x)\xi \in M$. It is well known that the dimension of any unitary irreducible representation of a compact group is of finite dimension [5]. Two unitary represention σ_i, $i = 1, 2$ of G on H_{σ_i}, $i = 1, 2$ are said to be unitary equivalent if there exists a unitary linear operator $T : H_{\sigma_1} \longrightarrow H_{\sigma_2}$ such that $\forall x \in G$, $\sigma_2(x)T = T\sigma_1(x)$.

Let us denote by $\widehat{G}$ the set of equivalent classes of unitary irreducible representations of G. It is called the unitary dual of G and it is discrete since G is compact.

For $\sigma \in \widehat{G}$, choose an orthonormal basis $\{\xi_1^\sigma, \cdots, \xi_{d_\sigma}^\sigma\}$ of H_σ. The coefficients of the representation σ are the functions $u_{i,j}^\sigma$ defined by

$$u_{i,j}^\sigma(x) = \langle \sigma(x)\xi_i^\sigma, \xi_j^\sigma \rangle_\sigma,\ x \in G,$$

where $\langle \cdot, \cdot \rangle_\sigma$ is the inner product of the Hilbert space H_σ.

Let E be a complex Banach space. Denote by $L^1(G, E)$ the set of E-valued Bochner integrable functions on G.

Let $f \in L^1(G, E)$. Following [1], the Fourier transform $\widehat{f}$ of f is the collection $\left(\widehat{f}(\sigma)\right)_{\sigma \in \widehat{G}}$ of sesquilinear maps, where for each σ, the sesquilinear map $\widehat{f}(\sigma)$ is defined from $H_\sigma \times H_\sigma$ into E by

$$\widehat{f}(\sigma)(\xi, \eta) = \int_G \langle \sigma(x)^*\xi, \eta \rangle_\sigma f(x)dx,\ \xi, \eta \in H_\sigma. \tag{1}$$

Since G is compact, the space of E-valued Bochner square integrable functions on G, denoted by $L^2(G, E)$, is a subspace of $L^1(G, E)$. Clearly, the Fourier transform of functions in $L^2(G, E)$ is well-defined by formula (1). For $f \in L^2(G, E)$, the inversion formula is given by

$$f(x) = \sum_{\sigma \in \widehat{G}} d_\sigma \sum_{i=1}^{d_\sigma} \sum_{j=1}^{d_\sigma} \widehat{f}(\sigma)(\xi_j^\sigma, \xi_i^\sigma) u_{i,j}^\sigma(x),\ x \in G.$$

Denote by $\mathscr{S}(H_\sigma \times H_\sigma, E)$ the set of E-valued sesquilinear maps on $H_\sigma \times H_\sigma$. Set

$$\mathscr{S}(\widehat{G}, E) = \prod_{\sigma \in \widehat{G}} \mathscr{S}(H_\sigma \times H_\sigma, E).$$

Define $\mathscr{S}_p(\widehat{G},E), p \geqslant 1$, to be the set of elements ϕ of $\mathscr{S}(\widehat{G},E)$ such that

$$\sum_{\sigma\in\widehat{G}} d_\sigma \sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma} \|\phi(\xi_j^\sigma,\xi_i^\sigma)\|_E^p < \infty.$$

Also, on $\mathscr{S}_p(\widehat{G},E)$, consider the norm

$$\|\phi\|_{\mathscr{S}_p} = \left(\sum_{\sigma\in\widehat{G}} d_\sigma \sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma} \|\phi(\xi_j^\sigma,\xi_i^\sigma)\|_E^p\right)^{\frac{1}{p}}.$$

The proof of completeness and other properties of the spaces $\mathscr{S}_p(\widehat{G},E)$ can be found in [9].

The space $\mathscr{S}_2(\widehat{G},E)$ is of particular interest: the Fourier transformation is an isometry from $L^2(G,E)$ onto $\mathscr{S}_2(\widehat{G},E)$ [1].

Finally, let us recall the following well-known fact which will be used later. For $x = (x_1,\cdots,x_n)$, where $x_1,\cdots,x_n$ are real (or complex) numbers, and for $p \geqslant 1$, set

$$\|x\|_p = (|x_1|^p + \cdots + |x_n|^p)^{\frac{1}{p}}.$$

It is known that if $1 \leqslant p \leqslant q \leqslant \infty$, then

$$\|x\|_q \leqslant \|x\|_p \leqslant n^{\frac{1}{p}-\frac{1}{q}}\|x\|_q. \tag{2}$$

3. Main results

In this section, we introduce the Sobolev spaces of E-valued functions on the compact Hausdorff group G and prove some continuous embedding results. Throughout the rest of the paper, the symbol $X \hookrightarrow Y$ means that the space X is continuously embedded in the space Y.

Let $(\gamma(\sigma))_{\sigma\in\widehat{G}}$ be a sequence of nonnegative real numbers. Pick s in $[0,\infty)$. The Sobolev space $H^s_\gamma(G,E)$ is defined to be the subspace of $L^2(G,E)$ consisting of functions f such that

$$\sum_{\sigma\in\widehat{G}} d_\sigma (1+\gamma(\sigma)^2)^s \sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma} \|\widehat{f}(\sigma)(\xi_j^\sigma,\xi_i^\sigma)\|_E^2 < \infty.$$

The following norm is defined on $H^s_\gamma(G,E)$:

$$\|f\|_{H^s_\gamma} = \left(\sum_{\sigma\in\widehat{G}} d_\sigma (1+\gamma(\sigma)^2)^s \sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma} \|\widehat{f}(\sigma)(\xi_j^\sigma,\xi_i^\sigma)\|_E^2\right)^{\frac{1}{2}}.$$

Theorem 1. *The space $H^s_\gamma(G,E)$ is a Banach space.*

Proof. The map $f \longmapsto \left(1+\gamma(\sigma)^2\right)^{\frac{s}{2}} \widehat{f}$ is an isometric bijection from $H^s_\gamma(G,E)$ onto $\mathscr{S}_2(\widehat{G},E)$. Since $\mathscr{S}_2(\widehat{G},E)$ is a Banach space, then so is $H^s_\gamma(G,E)$. $\square$

Theorem 2. *If $t>s$, then $H^t_\gamma(G,E) \hookrightarrow H^s_\gamma(G,E)$ with $\|f\|_{H^s_\gamma} \leqslant \|f\|_{H^t_\gamma}$.*

Proof. The result comes from the fact that if $t>s$, then $(1+\gamma(\sigma)^2)^t > (1+\gamma(\sigma)^2)^s$ since $1+\gamma(\sigma)^2 > 1$. $\square$

Theorem 3. *We have $H^s_\gamma(G,E) \hookrightarrow L^2(G,E)$ with $\|f\|_{L^2} \leqslant \|f\|_{H^s_\gamma}$.*

Proof. Let $f \in H^s_\gamma(G,E)$. Then,

$$\begin{aligned}
\|f\|^2_{L^2} &= \|\widehat{f}\|^2_{\mathscr{S}_2}\\
&= \sum_{\sigma\in\widehat{G}} d_\sigma \sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma} \|\widehat{f}(\sigma)(\xi^\sigma_j,\xi^\sigma_i)\|^2_E\\
&\leqslant \sum_{\sigma\in\widehat{G}} d_\sigma (1+\gamma(\sigma)^2)^s \sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma} \|\widehat{f}(\sigma)(\xi^\sigma_j,\xi^\sigma_i)\|^2_E\\
&= \|f\|^2_{H^s_\gamma}.
\end{aligned}$$

$\square$

Lemma 1. *Let σ be a continuous representation of G. Let $a \in G$ and let $\varepsilon > 0$. Then, there exists a neighborhood U of a such that*

$$\forall x \in U,\ |u^\sigma_{i,j}(x) - u^\sigma_{i,j}(a)| < \varepsilon.$$

Proof. Since the representation σ is continuous, there exists a neighbourhood U of a such that $\|\sigma(x)-\sigma(a)\| < \varepsilon$ whenever $x \in U$. Then,

$$\begin{aligned}
|u^\sigma_{i,j}(x) - u^\sigma_{i,j}(a)| &= |\langle \sigma(x)\xi^\sigma_i, \xi^\sigma_j\rangle_\sigma - \langle \sigma(a)\xi^\sigma_i, \xi^\sigma_j\rangle_\sigma|\\
&= |\langle (\sigma(x)-\sigma(a))\,\xi^\sigma_i, \xi^\sigma_j\rangle_\sigma|\\
&\leqslant \|\sigma(x)-\sigma(a)\|\|\xi^\sigma_i\|\|\xi^\sigma_j\|\\
&\leqslant \|\sigma(x)-\sigma(a)\| < \varepsilon,
\end{aligned}$$

where meanwhile we have used the Cauchy-Schwarz inequality, the boundedness of the operator $\sigma(x)-\sigma(a)$ and the fact that ξ^σ_i's are unit vectors. $\square$

Lemma 2. *Assume that $\sum_{\sigma\in\widehat{G}} \dfrac{d^3_\sigma}{(1+\gamma(\sigma)^2)^s} < \infty$. If $f \in H^s_\gamma(G,E)$, then f is continuous on G.*

Proof. Let $a \in G$ and let U be as in Lemma 1. Let $f \in H^s_\gamma(G,E)$. If $x \in U$, then

$$\begin{aligned}\|f(x)-f(a)\|_E &= \left\| \sum_{\sigma\in\widehat{G}} d_\sigma \sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma} \widehat{f}(\sigma)(\xi_j^\sigma,\xi_i^\sigma)(u^\sigma_{i,j}(x)-u^\sigma_{i,j}(a))\right\|_E \\ &\leqslant \sum_{\sigma\in\widehat{G}} d_\sigma \sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma} \left\|\widehat{f}(\sigma)(\xi_j^\sigma,\xi_i^\sigma)\right\|_E |u^\sigma_{i,j}(x)-u^\sigma_{i,j}(a)| \\ &\leqslant \varepsilon \sum_{\sigma\in\widehat{G}} d_\sigma \sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma} \left\|\widehat{f}(\sigma)(\xi_j^\sigma,\xi_i^\sigma)\right\|_E \text{ (by Lemma 1)} \\ &= \varepsilon \sum_{\sigma\in\widehat{G}} d_\sigma \sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma} (1+\gamma(\sigma)^2)^{\frac{s}{2}} \left\|\widehat{f}(\sigma)(\xi_j^\sigma,\xi_i^\sigma)\right\|_E (1+\gamma(\sigma)^2)^{-\frac{s}{2}}.\end{aligned}$$

Now, applying the Hölder inequality, we have

$$\begin{aligned}\|f(x)-f(a)\|_E \leqslant& \varepsilon \left(\sum_{\sigma\in\widehat{G}} d_\sigma \sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma} (1+\gamma(\sigma)^2)^{s} \|\widehat{f}(\sigma)(\xi_j^\sigma,\xi_i^\sigma)\|_E^2 \right)^{\frac{1}{2}} \\ &\times \left(\sum_{\sigma\in\widehat{G}} d_\sigma \sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma} (1+\gamma(\sigma)^2)^{-s} \right)^{\frac{1}{2}} \\ =& \varepsilon \left(\sum_{\sigma\in\widehat{G}} d_\sigma \sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma} (1+\gamma(\sigma)^2)^{s} \|\widehat{f}(\sigma)(\xi_j^\sigma,\xi_i^\sigma)\|_E^2 \right)^{\frac{1}{2}} \left(\sum_{\sigma\in\widehat{G}} d_\sigma^3 (1+\gamma(\sigma)^2)^{-s}\right)^{\frac{1}{2}} \\ =& \varepsilon \|f\|_{H^s_\gamma} \left(\sum_{\sigma\in\widehat{G}} d_\sigma^3 (1+\gamma(\sigma)^2)^{-s}\right)^{\frac{1}{2}}.\end{aligned}$$

Thus, f is continuous at a. Since a is an arbitrary element of G, then f is continuous on G. □

Lemma 3. *Assume that* $\sum_{\sigma\in\widehat{G}} \frac{d_\sigma^3}{(1+\gamma(\sigma)^2)^s} < \infty$. *If* $f \in H^s_\gamma(G,E)$, *then there exists a constant* $C(\gamma,s)$, *depending only on* γ *and* s, *such that*

$$\|f\|_\infty := \sup\{\|f(x)\|_E : x \in G\} \leqslant C(\gamma,s)\|f\|_{H^s_\gamma}.$$

Proof. Let $x \in G$. Then,

$$\begin{aligned}\|f(x)\|_E &= \left\| \sum_{\sigma\in\widehat{G}} d_\sigma \sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma} \widehat{f}(\sigma)(\xi_j^\sigma,\xi_i^\sigma)u^\sigma_{i,j}(x)\right\|_E \\ &\leqslant \sum_{\sigma\in\widehat{G}} d_\sigma \sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma} |u^\sigma_{i,j}(x)| \|\widehat{f}(\sigma)(\xi_j^\sigma,\xi_i^\sigma)\|_E.\end{aligned}$$

Since σ is a unitary representation, then by the Cauchy-Schwarz inequality,

$$|u_{i,j}^{\sigma}(x)| = |\langle \sigma(x)\xi_i, \xi_j\rangle_\sigma| \leqslant \|\sigma(x)\|\|\xi_i^\sigma\|\|\xi_j^\sigma\| = 1.$$

Then,

$$\begin{aligned}\|f(x)\|_E &\leqslant \sum_{\sigma\in\widehat{G}} d_\sigma \sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma} \|\widehat{f}(\sigma)(\xi_j^\sigma,\xi_i^\sigma)\|_E \\ &= \sum_{\sigma\in\widehat{G}} d_\sigma \sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma}(1+\gamma(\sigma)^2)^{\frac{s}{2}} \left\|\widehat{f}(\sigma)(\xi_j^\sigma,\xi_i^\sigma)\right\|_E (1+\gamma(\sigma)^2)^{-\frac{s}{2}}.\end{aligned}$$

By the Hölder inequality, we obtain

$$\begin{aligned}\|f(x)\|_E &\leqslant \left(\sum_{\sigma\in\widehat{G}} d_\sigma \sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma}(1+\gamma(\sigma)^2)^s\|\widehat{f}(\sigma)(\xi_j^\sigma,\xi_i^\sigma)\|_E^2\right)^{\frac{1}{2}}\left(\sum_{\sigma\in\widehat{G}} d_\sigma \sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma}(1+\gamma(\sigma)^2)^{-s}\right)^{\frac{1}{2}} \\ &= \left(\sum_{\sigma\in\widehat{G}} d_\sigma \sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma}(1+\gamma(\sigma)^2)^s\|\widehat{f}(\sigma)(\xi_j^\sigma,\xi_i^\sigma)\|_E^2\right)^{\frac{1}{2}}\left(\sum_{\sigma\in\widehat{G}} d_\sigma^3(1+\gamma(\sigma)^2)^{-s}\right)^{\frac{1}{2}} \\ &= \|f\|_{H_\gamma^s}\left(\sum_{\sigma\in\widehat{G}} d_\sigma^3(1+\gamma(\sigma)^2)^{-s}\right)^{\frac{1}{2}} \\ &= C(\gamma,s)\|f\|_{H_\gamma^s} < \infty,\end{aligned}$$

where $C(\gamma,s) = \left(\sum_{\sigma\in\widehat{G}} d_\sigma^3(1+\gamma(\sigma)^2)^{-s}\right)^{\frac{1}{2}}$. Hence, $\|f\|_\infty \leqslant C(\gamma,s)\|f\|_{H_\gamma^s}$. □

Let us denote by $\mathcal{C}(G,E)$ the space of E-valued continuous functions on G.

Theorem 4. *If* $\sum_{\sigma\in\widehat{G}} \frac{d_\sigma^3}{(1+\gamma(\sigma)^2)^s} < \infty$, *then* $H_\gamma^s(G,E) \hookrightarrow \mathcal{C}(G,E)$.

Proof. This theorem is the conjunction of Lemma 1 by which f is continuous, and Lemma 3 by which the continuous embedding inequality $\|f\|_\infty \leqslant C(\gamma,s)\|f\|_{H_\gamma^s}$ holds. □

Lemma 4. *Let* $\phi \in \prod_{\sigma\in\widehat{G}} \mathscr{S}(H_\sigma\times H_\sigma, E)$. *If* $1 \leqslant p \leqslant q$, *then*

$$\left(\sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma}\|\phi(\sigma)(\xi_j^\sigma,\xi_i^\sigma)\|_E^p\right)^{\frac{1}{p}} \leq (d_\sigma^2)^{\frac{1}{p}-\frac{1}{q}}\left(\sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma}\|\phi(\sigma)(\xi_j^\sigma,\xi_i^\sigma)\|_E^q\right)^{\frac{1}{q}}.$$

Proof. Use the right-hand side inequality in (2). □

Theorem 5. *Let $t > s > 0$. Set $\alpha' = \dfrac{2t}{t-s}$. If $\sum_{\sigma\in\widehat{G}} \dfrac{d_\sigma^3}{(1+\gamma(\sigma)^2)^t} < \infty$,*

then $H^s_\gamma(G,E) \hookrightarrow L^{\alpha'}(G,E)$ with $\|f\|_{L^{\alpha'}} \leqslant \left(\sum_{\sigma\in\widehat{G}} \frac{d_\sigma^3}{(1+\gamma(\sigma)^2)^t}\right)^{\frac{s}{2t}} \|f\|_{H^s_\gamma}$.

Proof. Let α be the Hölder conjugate of α'. That is, $\dfrac{1}{\alpha} + \dfrac{1}{\alpha'} = 1$. Then, $\alpha = \dfrac{2t}{s+t}$ and $\dfrac{s}{t} = \dfrac{2-\alpha}{\alpha}$. It follows that $1 < \alpha < 2$. By the inverse Hausdorff-Young inequality [2, Lemma 5.1], we have

$$\|f\|_{L^{\alpha'}} \leqslant \|\widehat{f}\|_{\mathscr{S}_\alpha}.$$

We have $\|\widehat{f}\|^\alpha_{\mathscr{S}_\alpha} = \sum_{\sigma\in\widehat{G}} d_\sigma \sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma} \|\widehat{f}(\sigma)(\xi_j^\sigma,\xi_i^\sigma)\|^\alpha_E$. Using Lemma 4 with the fact that $1 < \alpha < 2$, we attain

$$\left(\sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma} \|\widehat{f}(\sigma)(\xi_j^\sigma,\xi_i^\sigma)\|^\alpha_E\right)^{\frac{1}{\alpha}} \leqslant (d_\sigma^2)^{\frac{1}{\alpha}-\frac{1}{2}} \left(\sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma} \|\widehat{f}(\sigma)(\xi_j^\sigma,\xi_i^\sigma)\|^2_E\right)^{\frac{1}{2}}.$$

The latter inequality implies

$$\sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma} \|\widehat{f}(\sigma)(\xi_j^\sigma,\xi_i^\sigma)\|^\alpha_E \leqslant d_\sigma^{2-\alpha} \left(\sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma} \|\widehat{f}(\sigma)(\xi_j^\sigma,\xi_i^\sigma)\|^2_E\right)^{\frac{\alpha}{2}}.$$

Therefore,

$$\begin{aligned}\|\widehat{f}\|^\alpha_{\mathscr{S}_\alpha} &\leqslant \sum_{\sigma\in\widehat{G}} d_\sigma d_\sigma^{2-\alpha} \left(\sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma} \|\widehat{f}(\sigma)(\xi_j^\sigma,\xi_i^\sigma)\|^2_E\right)^{\frac{\alpha}{2}} \\ &= \sum_{\sigma\in\widehat{G}} d_\sigma (1+\gamma(\sigma)^2)^{\frac{s\alpha}{2}} \left(\sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma} \|\widehat{f}(\sigma)(\xi_j^\sigma,\xi_i^\sigma)\|^2_E\right)^{\frac{\alpha}{2}} d_\sigma^{2-\alpha}(1+\gamma(\sigma)^2)^{-\frac{s\alpha}{2}}.\end{aligned}$$

Observe that $\dfrac{1}{\frac{2}{\alpha}} + \dfrac{1}{\frac{2}{2-\alpha}} = 1$. Now, apply the Hölder inequality to obtain

$$\begin{aligned}\|\widehat{f}\|^\alpha_{\mathscr{S}_\alpha} &\leqslant \left(\sum_{\sigma\in\widehat{G}} d_\sigma (1+\gamma(\sigma)^2)^s \sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma} \|\widehat{f}(\sigma)(\xi_j^\sigma,\xi_i^\sigma)\|^2_E\right)^{\frac{\alpha}{2}} \left(\sum_{\sigma\in\widehat{G}} d_\sigma d_\sigma^2 (1+\gamma(\sigma)^2)^{-\frac{s\alpha}{2-\alpha}}\right)^{\frac{2-\alpha}{2}} \\ &= \left(\sum_{\sigma\in\widehat{G}} d_\sigma (1+\gamma(\sigma)^2)^s \sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma} \|\widehat{f}(\sigma)(\xi_j^\sigma,\xi_i^\sigma)\|^2_E\right)^{\frac{\alpha}{2}} \left(\sum_{\sigma\in\widehat{G}} \frac{d_\sigma^3}{(1+\gamma(\sigma)^2)^{\frac{s\alpha}{2-\alpha}}}\right)^{\frac{2-\alpha}{2}}.\end{aligned}$$

Thus,

$$\|\widehat{f}\|_{\mathscr{S}_\alpha} \leqslant \left(\sum_{\sigma\in\widehat{G}} d_\sigma(1+\gamma(\sigma)^2)^s \sum_{i=1}^{d_\sigma}\sum_{j=1}^{d_\sigma} \|\widehat{f}(\sigma)(\xi_j^\sigma,\xi_i^\sigma)\|_E^2\right)^{\frac{1}{2}} \left(\sum_{\sigma\in\widehat{G}} \frac{d_\sigma^3}{(1+\gamma(\sigma)^2)^{\frac{s\alpha}{2-\alpha}}}\right)^{\frac{2-\alpha}{2\alpha}}$$

$$=\|f\|_{H^s_\gamma}\left(\sum_{\sigma\in\widehat{G}} \frac{d_\sigma^3}{(1+\gamma(\sigma)^2)^t}\right)^{\frac{s}{2t}}.$$

Finally, $\|f\|_{L^{\alpha'}} \leqslant \|\widehat{f}\|_{\mathscr{S}_\alpha} \leqslant \|f\|_{H^s_\gamma}\left(\sum_{\sigma\in\widehat{G}} \frac{d_\sigma^3}{(1+\gamma(\sigma)^2)^t}\right)^{\frac{s}{2t}}$. □

4. Conclusion

This paper explored continuous embeddings of Sobolev-type spaces consisting of vector-valued functions on compact groups in other function spaces. The main results include continuous embeddings between Sobolev spaces, between Sobolev spaces and spaces of continuous functions, and between Sobolev spaces and Lebesgue spaces.

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