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Nonnegative integer solutions of the equation $L_n^{(k)} - L_m^{(k)} = 2 \cdot 3^a$

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Abstract. For an integer $k \geq 2$, let $(L_n^{(k)})_{n\geq -(k-2)}$ be the k-generalized Lucas sequence, which starts with $0, ..., 0, 2, 1$ (k terms) and each term afterwards is the sum of the k preceding terms. In 2019, Bitim found all the solutions of the Diophantine equation $L_n - L_m = 2 \cdot 3^a$. In this paper, we generalize this result by considering the k-generalized Lucas sequence, i.e., we solve the Diophantine equation $L_n^{(k)} - L_m^{(k)} = 2 \cdot 3^a$ in positive integers n, m, a with $k \geq 3$. To obtain our main result, we use Baker's method and the Baker-Davenport reduction method.

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1. Introduction

Let $k \geq 2$ be an integer and $(L_n^{(k)})_{n\geq -(k-2)}$ a linear recurrence sequence of order k given by:

$$L_{-(k-2)}^{(k)} = \cdots = L_{-1}^{(k)} = 0,\ L_0^{(k)} = 2,\ L_1^{(k)} = 1$$

and

$$L_n^{(k)} = L_{n-1}^{(k)} + \cdots + L_{n-k}^{(k)},$$

for $n \geq 2$. The sequence $(L_n^{(k)})_{n\geq -(k-2)}$ is called the k-generalized Lucas sequence or, for simplicity, the k-Lucas sequence. This sequence is a generalization of the classical Lucas sequence.

Next, we will review some facts and properties of the k-Lucas sequence that will be relevant later. The characteristic polynomial of this sequence is given by

$$\Psi_k(x) = x^k - x^{k-1} - \cdots - x - 1.$$

The polynomial $\Psi_k(x)$ is irreducible over $\mathbb{Q}[x]$ and has exactly one root, $\alpha(k)$, located outside the unit circle (see, for example, [12, 13], and [18]). This root has a real

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value satisfying $\alpha(k) > 1$, while all other roots are strictly within the unit circle. More precisely, in [18], Wolfram demonstrated that

$$2(1-2^{-k}) < \alpha(k) < 2, \quad k \geq 2.$$

For simplicity, we generally omit the dependence of α on k. Let $\alpha := \alpha^{(1)}, \ldots, \alpha^{(k)}$ be the zeros of $\Psi_k(x)$ and $f_k(x) := \dfrac{x-1}{2+(k+1)(x-2)}$, for $x > 2(1-2^{-k})$. In [5], Bravo et al. proved that the inequalities

$$1/2 < f_k(\alpha) < 3/4 \quad \text{and} \quad \left|f_k(\alpha^{(i)})\right| < 1,\ 2 \leq i \leq k$$

hold. These facts imply that the number $f_k(\alpha)$ is not an algebraic integer. The Binet-like formula was established by Bravo and Luca in [5]. Specifically, they demonstrated that

$$L_n^{(k)} = \sum_{i=1}^{k}(2\alpha^{(i)}-1)f_k(\alpha^{(i)}){\alpha^{(i)}}^{n-1} \quad \text{and} \quad \left|L_n^{(k)} - (2\alpha-1)f_k(\alpha)\alpha^{n-1}\right| < \frac{3}{2} \tag{1}$$

hold, for $n \geq 1$ and $k \geq 2$. Additionally, in the same paper, they proved that for $n \geq 1$ and $k \geq 2$,

$$\alpha^{n-1} \leq L_n^{(k)} \leq 2\alpha^n. \tag{2}$$

Recently, many Diophantine equations involving k-Lucas sequences have been studied [1, 6, 3, 14, 17, 15, 16]. In [8], Bitim studied the Diophantine equation

$$L_n - L_m = 2 \cdot 3^a.$$

The aim of the present paper is to extend his result by proving the following theorem.

Theorem 1. *All the solutions of the Diophantine equation*

$$L_n^{(k)} - L_m^{(k)} = 2 \cdot 3^a \tag{3}$$

in nonegative integers (a, m, n, k) *are* $(2,1,5,3), (3,5,7,3), (3,7,8,3)$.

Our strategy for proving Theorem 1 is the following: first, we rewrite equation (3) in suitable ways in order to obtain two different linear forms in logarithms of algebraic numbers which are both nonzero and small. Next, we use twice lower bounds on such nonzero linear forms in logarithms of algebraic numbers, due to Matveev, to bound n polynomially in terms of k. When $k \leq 730$, we use the reduction algorithm due to Baker-Davenport to reduce the upper bounds to a size that can be more easily handled. When $k > 730$, we use some estimates from [4, 17] based on the fact that the dominant root of the k-Lucas sequence is exponentially close to 2. Therefore, one can replace this root by 2 in future calculations with linear forms in logarithms and end up with absolute upper bounds for all the variables, which we then reduce using the Baker-Davenport reduction method again.

2. Preliminaries and known results

This section is devoted to collecting a few definitions, notations, proprieties and results which will be used in the remaining part of this paper.

2.1. Linear forms in logarithms

For any nonzero algebraic number η of degree d over $\mathbb{Q}$, whose minimal polynomial over $\mathbb{Z}$ is $a\prod_{j=1}^{d}\left(X-\eta^{(j)}\right)$, we denote by

$$h(\eta)=\frac{1}{d}\left(\log|a|+\sum_{j=1}^{d}\log\max\left(1,|\eta^{(j)}|\right)\right)$$

the usual absolute logarithmic height of η. In particular, if $\eta = p/q$ is a rational number with $\gcd(p,q)=1$ and $q>0$, then $h(\eta)=\log\max\{|p|,q\}$. The following properties of the function logarithmic height h, which will be used in the next sections without any special reference, are also known:

$$h(\eta\pm\gamma)\le h(\eta)+h(\gamma)+\log 2, \tag{4}$$
$$h(\eta\gamma^{\pm 1})\le h(\eta)+h(\gamma), \tag{5}$$
$$h(\eta^{s})=|s|\,h(\eta)\quad (s\in\mathbb{Z}). \tag{6}$$

In [2] (see also p. 73 of [4]), Bravo et al. proved that the logarithmic height of $f_k(\alpha)$ satisfies

$$h(f_k(\alpha))<\log(k+1)+\log 4,\quad k\ge 2. \tag{7}$$

With this notation, Matveev proved the following theorem (see [10]).

Theorem 2. *Let $\eta_1,\dots,\eta_s$ be real algebraic numbers and let $b_1,\dots,b_s$ be nonzero rational integers. Let $d_{\mathbb{K}}$ be the degree of the number field $\mathbb{Q}(\eta_1,\dots,\eta_s)$ over $\mathbb{Q}$ and let A_j be a positive real number satisfying*

$$A_j=\max\{d_{\mathbb{K}}h(\eta_j),|\log\eta_j|,0.16\},\quad \text{for}\quad j=1,\dots,s.$$

Assume that

$$B\ge\max\{|b_1|,\dots,|b_s|\}.$$

If $\eta_1^{b_1}\cdots\eta_s^{b_s}-1\neq 0$, then

$$|\eta_1^{b_1}\cdots\eta_s^{b_s}-1|\ge\exp(-1.4\cdot 30^{s+3}\cdot s^{4.5}\cdot d_{\mathbb{K}}^{2}(1+\log d_{\mathbb{K}})(1+\log B)A_1\cdots A_s).$$

2.2. Reduction algorithms

Here, we present a variant of the reduction method of Baker and Davenport due to Dujella and Pethő [9].

Lemma 1. *Let M be a positive integer, p/q a convergent of the continued fraction of the irrational γ such that $q>6M$, and let A,B,μ be some real numbers with $A>0$ and $B>1$. Let*

$$\varepsilon=||\mu q||-M\cdot||\gamma q||,$$

where $||\cdot||$ denotes the distance from the nearest integer. If $\varepsilon>0$, then there is no solution of the inequality

$$0<|u\gamma-v+\mu|<AB^{-w}$$

in positive integers u, v *and* w *with*

$$u \leq M \quad \text{and} \quad w \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

Note that Lemma 1 cannot be applied when $\mu = 0$ (since then $\varepsilon < 0$) or when μ is a multiple of γ. For this case, we use the following well-known technical result from Diophantine approximations, known as Legendre's criterion.

Lemma 2 (see [7])**.** *Let* η *be an irrational number.*

(*i*) *If* n, m *are integers such that* $m \geq 1$ *and*

$$\left|\eta - \frac{n}{m}\right| < \frac{1}{2m^2},$$

then n/m *is a convergent of* η.

(*ii*) *Let* M *be a positive real number and* $p_0/q_0, p_1/q_1, \ldots$ *all the convergents of the continued fraction* $[a_0, a_1, a_2, \ldots]$ *of* η. *Let* N *be the smallest positive integer such that* $q_N > M$. *Put* $a(M) := \max\{a_k : k = 0, 1, \ldots, N\}$. *Then, the inequality*

$$\left|\eta - \frac{n}{m}\right| > \frac{1}{(a(M)+2)m^2}$$

holds for all pairs (n, m) *of integers with* $0 < m < M$.

3. The proof of Theorem 1

This section is devoted to showing Theorem 1. This will be done in four main steps.

3.1. Preliminary considerations

We start our study of (3) for $2 \leq m < n \leq k$. In this case, we have $L_n^{(k)} = 3 \cdot 2^{n-2}$ and $L_m^{(k)} = 3 \cdot 2^{m-2}$, so that equation (3) becomes

$$2^{m-2}(2^{n-m} - 1) = 2 \cdot 3^{a-1}. \tag{8}$$

If $m \geq 4$, the left-hand side of equation (8) is divisible by 4, while the right-hand side is not. Therefore, equation (3) has no solutions for $4 \leq m < n \leq k$. When $m = 2$, the left-hand side of equation (8) is odd, but the right-hand side is even; hence the equation has no solutions for $2 = m < n \leq k$. For $m = 3$, Catalan's result (see [11]) shows that equation (8) has no solutions, and thus there are no solutions of (8) for $3 = m < n \leq k$. Hence, we can assume that $n \geq k + 1$, which implies that $n \geq 4$. Furthermore, we have $a < n$. Indeed, using inequality (2), one has

$$2 \cdot 3^a = L_n^{(k)} - L_m^{(k)} < L_n^{(k)} \leq 2\alpha^n,$$

which yields $a < n$.

3.2. An inequality for n versus k

Now, we show the following lemma.

Lemma 3. *If (n, m, k, a) is a solution in integers of equation (3) with $k \geq 3$ and $n \geq k+1$, then the inequality*

$$n < 2.4 \cdot 10^{28} k^7 \log^5 k \tag{9}$$

holds.

Proof. Equation (3) can be reorganized as

$$(2\alpha - 1) f_k(\alpha) \alpha^{n-1} - 2 \cdot 3^a = L_m^{(k)} - e_k(n),$$

where

$$e_k(n) = \sum_{i=2}^{k} (2\alpha^{(i)} - 1) f_k(\alpha^{(i)}) {\alpha^{(i)}}^{n-1}.$$

Hence, estimate (1) and the above equation give us

$$\left| (2\alpha - 1) f_k(\alpha) \alpha^{n-1} - 2 \cdot 3^a \right| \leq 2\alpha^m + \frac{3}{2}.$$

Dividing both sides by $(2\alpha - 1) f_k(\alpha) \alpha^{n-1}$, we obtain

$$|\Gamma_1| \leq \frac{2\alpha}{(2\alpha - 1) f_k(\alpha) \alpha^{n-m}} + \frac{3\alpha}{2(2\alpha - 1) f_k(\alpha) \alpha^n} < \frac{3.2}{\alpha^{n-m}} + \frac{2.4}{\alpha^n} < \frac{5.6}{\alpha^{n-m}}, \tag{10}$$

where

$$\Gamma_1 := 1 - \left(\frac{(2\alpha - 1) f_k(\alpha)}{2} \right)^{-1} \cdot \alpha^{-(n-1)} \cdot 3^a. \tag{11}$$

Now, we need to show that $\Gamma_1 \neq 0$. Indeed, if $\Gamma_1 = 0$, then we would get

$$2 \cdot 3^a = (2\alpha - 1) f_k(\alpha) \alpha^{n-1}.$$

By conjugating with a Galois automorphism which sends α to α_i, where $i \geq 2$, and taking the absolute value of both sides of the resulting equation, we obtain

$$6 \leq 2 \cdot 3^a \leq (2|\alpha_i| + 1) \cdot |f_k(\alpha_i)| \cdot |\alpha_i|^{n-1} \leq 3,$$

which leads to a contradiction. Thus, $\Gamma_1 \neq 0$. To apply Theorem 2 to Γ_1, given by (11), we choose the parameters as

$$(\eta_1, b_1) := (((2\alpha - 1) f_k(\alpha))/2, -1), \quad (\eta_2, b_2) := (\alpha, -(n-1)), \quad (\eta_3, b_3) := (3, a).$$

We have $\eta_1, \eta_2, \eta_3 \in \mathbb{K} := \mathbb{Q}(\alpha)$ and $d_{\mathbb{K}} = k$. Since $h(\eta_2) = (\log \alpha)/k < (\log 2)/k$ and $h(\eta_3) = \log 3$, then we can take $A_2 := \log 2$ and $A_3 := k \log 3$. Next, we estimate A_1. By using estimate (7) and the proprieties (5) and (6), for $k \geq 3$, it results that

$$\begin{aligned} h(\eta_1) &\leq 2h(2) + h(\alpha) + h(f_k(\alpha)) + \log 2 \\ &< 6 \log 2 + \log(k+1) \\ &< 5.2 \log k. \end{aligned}$$

Moreover, since $7/4 < \alpha < 2$ and $1/2 < f_k(\alpha) < 3/4$, then we have

$$\eta_1 = \frac{(2\alpha-1)f_k(\alpha)}{2} < \frac{9}{8} \quad \text{and} \quad \eta_1^{-1} = \frac{2}{(2\alpha-1)f_k(\alpha)} < \frac{8}{5}.$$

Hence, we can choose $A_1 := 5.2k\log k$. Finally, the fact that $a < n$ implies that we can take $B := n$. Therefore, applying Theorem 2 and using the facts that $1+\log k < 2\log k$ and $1+\log n < 1.8\log n$, which hold for $k \geq 3$ and $n \geq 4$, we deduce that

$$|\Gamma_1| > \exp\left(-2.1\cdot 10^{12}\cdot k^4\log^2 k\log n\right). \tag{12}$$

Comparing the lower bound (12) and the upper bound (10) of $|\Gamma_1|$ yields

$$(n-m)\log\alpha < 2.2\cdot 10^{12}k^4\log^2 k\log n. \tag{13}$$

We go back to equation (3) and rewrite it as

$$(2\alpha-1)f_k(\alpha)(1-\alpha^{m-n})\alpha^{n-1} - 2\cdot 3^a = -e_k(n) + e_k(m),$$

where

$$e_k(m) = \sum_{i=2}^{k}(2\alpha^{(i)}-1)f_k(\alpha^{(i)}){\alpha^{(i)}}^{m-1}.$$

Hence, we obtain

$$\left|(2\alpha-1)f_k(\alpha)(1-\alpha^{m-n})\alpha^{n-1} - 2\cdot 3^a\right| < 3.$$

Dividing through by $(2\alpha-1)f_k(\alpha)(1-\alpha^{m-n})\alpha^{n-1}$, we get

$$|\Gamma_2| \leq \frac{3}{(2\alpha-1)f_k(\alpha)(1-\alpha^{-1})\alpha^n} < \frac{9.6}{\alpha^n}, \tag{14}$$

where

$$\Gamma_2 := 1 - \left(\frac{(2\alpha-1)f_k(\alpha)(1-\alpha^{m-n})}{2}\right)^{-1}\cdot\alpha^{-(n-1)}\cdot 3^a.$$

We can show that $\Gamma_2 \neq 0$ by a similar method used to show that $\Gamma_1 \neq 0$. Now, we will apply Theorem 2 to Γ_2 by putting

$$\begin{aligned}(\eta_1, b_1) &:= (((2\alpha-1)f_k(\alpha)(1-\alpha^{m-n}))/2, -1),\\ (\eta_2, b_2) &:= \quad (\alpha, -(n-1)), \quad (\eta_3, b_3) := (3, a).\end{aligned}$$

Obviously, η_1, η_2, η_3 belong to $\mathbb{K} := \mathbb{Q}(\alpha)$ and $d_{\mathbb{K}} = k$. As calculated before, we take

$$A_2 := \log 2, \quad A_3 := k\log 3, \quad \text{and} \quad B := n.$$

We need to compute A_1. For $k \geq 3$, estimates (7), (13) and the proprieties (4) - (6) give

$$\begin{aligned}h(\eta_1) &\leq 2h(2) + h(\alpha) + h(f_k(\alpha)) + (n-m)h(\alpha) + 2\log 2\\ &< 7\log 2 + \log(k+1) + 2.2\cdot 10^{12}k^3\log^2 k\log n\\ &< 2.3\cdot 10^{12}k^3\log^2 k\log n.\end{aligned}$$

So, we can take

$$\max\{kh(\eta_1), |\log \eta_1|, 0.16\} < 2.3 \cdot 10^{12} k^4 \log^2 k \log n := A_1.$$

Applying Theorem 2 and comparing the resulting inequality with (14), we conclude that

$$\frac{n}{\log^2 n} < 1.7 \cdot 10^{24} k^7 \log^3 k,$$

where we have used the facts $1 + \log k < 2\log k$ and $1 + \log n < 1.8 \log n$, which hold for $k \geq 3$ and $n \geq 4$.

First, we use the fact that the inequality $n/\log^2 n < A$ implies $n < 4A\log^2 A$, whenever $A > 16^2$ and the fact that $55.8 + 7\log k + 3\log\log k < 58.5\log k$ holds, for $k \geq 3$. Then, we take $A := 1.7 \cdot 10^{24} k^7 \log^3 k$ to obtain

$$\begin{aligned} n &< 4(1.7 \cdot 10^{24} k^7 \log^3 k)(\log(1.7 \cdot 10^{24} k^7 \log^3 k))^2 \\ &< 6.8 \cdot 10^{24} k^7 \log^3 k \cdot (55.8 + 7\log k + 3\log\log k)^2 \\ &< 6.8 \cdot 10^{24} k^7 \log^5 k. \end{aligned}$$

This establishes (9) and finishes the proof of the lemma. □

3.3. The proof of Theorem 1 for $3 \leq k \leq 730$

In this subsection, we treat the cases where $k \in [3, 730]$. Setting

$$\Lambda_1 := \log(1 - \Gamma_1) = a \log 3 - (n-1)\log\alpha + \log\left(\frac{2}{(2\alpha - 1) f_k(\alpha)}\right).$$

Suppose that $n - m \geq 10$. Then, by (10), we have $|\Gamma_1| < 0.5$. Thus, we obtain

$$|\Lambda_1| < 2 \cdot |\Gamma_1| < 11.2 \cdot \alpha^{n-m},$$

which gives

$$0 < \left| a\left(\frac{\log 3}{\log \alpha}\right) - (n-1) + \frac{\log(2/(2\alpha-1) f_k(\alpha))}{\log \alpha} \right| < 20.1 \cdot \alpha^{-(n-m)}. \tag{15}$$

In order to apply Lemma 1 to (15), we set

$$\gamma := \frac{\log 3}{\log \alpha}, \quad \mu := \frac{\log(2/(2\alpha-1) f_k(\alpha))}{\log \alpha}, \quad A := 20.1, \quad B := \alpha.$$

For each $k \in [3, 730]$, we find a good approximation of α and a convergent p_ℓ/q_ℓ of the continued fraction of γ such that $q_\ell > 6M$, where $M = \lfloor 2.4 \cdot 10^{28} k^7 \log^5 k \rfloor$, which is an upper bound of $n - 1$ from Lemma 3. After doing this, we apply Lemma 1 to inequality (15). A computer search using Mathematica revealed that the maximum value of $\dfrac{\log(Aq/\varepsilon)}{\log B}$ over all $k \in [3, 730]$ is 727, which, according to Lemma 1, is an upper bound on $n - m$.

Now, we take $1 \leq n - m \leq 727$ and consider

$$\Lambda_2 := \log|\Gamma_2 - 1| = a\log 3 - (n-1)\log\alpha + \log(2/((2\alpha-1)f_k(\alpha)(1-\alpha)^{m-n})).$$

For $n \geq 10$, by (10), we have $|\Gamma_2| < 0.5$. Hence, one gets

$$|\Lambda_2| < 19.2 \cdot \alpha^{-n},$$

which gives

$$0 < \left| a\left(\frac{\log 3}{\log\alpha}\right) - (n-1) + \frac{\log(2/(2\alpha-1)f_k(\alpha)(1-\alpha)^{m-n})}{\log\alpha} \right| < 34.4 \cdot \alpha^{-(n-m)}. \quad (16)$$

We apply Lemma 1 to (16) with

$$\gamma := \frac{\log 3}{\log\alpha}, \quad \mu := \frac{\log(2/(2\alpha-1)f_k(\alpha)(1-\alpha)^{m-n})}{\log\alpha}, \quad A := 34.4, \quad B := \alpha.$$

Again, for each $(k, n-m) \in [3, 730] \cdot [1, 727]$, we find a good approximation of α and a convergent p_ℓ/q_ℓ of the continued fraction of γ such that $q_\ell > 6M$, where $M = \lfloor 2.4 \cdot 10^{28} k^7 \log^5 k \rfloor$, which is an upper bound of $n-1$ from Lemma 3. After doing this, again we apply Lemma 1 to inequality (16). A computer search using Mathematica revealed that the maximum value of $\dfrac{\log(Aq/\varepsilon)}{\log B}$ over all $k \in [3, 730]$ is 730, which, according to Lemma 1, is an upper bound on n.

Hence, we deduce that the possible solutions (a, m, n, k) of equation (3) for which $k \in [3, 730]$ have $0 \leq m < n \leq 730$ and $a < 730$.

Finally, we used Mathematica to compare $L_n^{(k)} - L_m^{(k)}$ and $2 \cdot 3^a$, for $k \in [3, 730]$, $0 \leq m < n \leq 730$ with $a < n$, and checked that the only solutions of equation (3) are $(a, m, n, k) \in \{(2, 1, 5, 3), (3, 5, 7, 3), (3, 7, 8, 3)\}$.

3.4. The proof of Theorem 1 for $k > 730$

The goal of this subsection is to prove the following proposition.

Proposition 1. *Equation* (3) *has no solution when* $k > 730$.

The proof of the above proposition will be done in two steps.

3.4.1. An absolute upper bound on k

The aim of this sub-subsection is to find the absolute bounds of k and n by proving the following lemma.

Lemma 4. *If* (n, m, k, a) *is a solution of Diophantine equation* (3) *with* $k > 730$ *and* $n \geq k+1$, *then* k *and* n *are bounded as*

$$k < 1.9 \cdot 10^{26} \quad \textit{and} \quad n < 1.8 \cdot 10^{221}.$$

Proof. For $k > 730$, it is easy to check that

$$n < 2.4 \cdot 10^{28} k^7 \log^5 k < 2^{k/2}.$$

In [17], it was proved that $L_n^{(k)}$ can be rewritten as

$$L_n^{(k)} = 3 \cdot 2^{n-2}(1+\zeta), \quad \text{where } |\zeta| < \frac{1}{2^{k/2}}. \tag{17}$$

Replacing (17) in (3), one gets

$$\left|3 \cdot 2^{n-2} - 2 \cdot 3^a\right| \leq 2\alpha^m + \frac{3 \cdot 2^{n-2}}{2^{k/2}}.$$

Consequently, we get

$$\left|1 - 2^{-(n-3)} \cdot 3^{a-1}\right| \leq \frac{8/3}{2^{n-m}} + \frac{1}{2^{k/2}} < \frac{3}{2^{\lambda}}, \tag{18}$$

where $\lambda := \min\{n-m, k/2\}$.

We will apply Theorem 2 to obtain a lower bound to the left-hand side of inequality (18). We take

$$s := 2, \quad (\eta_1, b_1) := (2, -(n-3)), \quad (\eta_2, b_2) := (3, a-1),$$

and

$$\Gamma_3 := 1 - 2^{-(n-3)} \cdot 3^{a-1}.$$

If $\Gamma_3 = 0$, then $3^{a-1} = 2^{n-3}$, which is false for $n \geq 4$. Therefore, we have $\Gamma_3 \neq 0$.

Since $\eta_1, \eta_2 \in \mathbb{K} := \mathbb{Q}$, then $d_{\mathbb{K}} = 1$. Furthermore, we can choose $B := n$ because $a \leq n$. On the other hand, since $h(\eta_1) = \log 2$ and $h(\eta_2) = \log 3$, we can take $A_1 := \log 2$ and $A_2 := \log 3$. Therefore, by applying Theorem 2, we obtain

$$|\Gamma_3| > \exp\left(-1.1 \cdot 10^9 \log n\right), \tag{19}$$

where we used the fact that $1 + \log n < 1.8 \log n$, for $n \geq 4$. Putting (18) together with (19) gives

$$\lambda < 1.6 \cdot 10^9 \log n. \tag{20}$$

Now, we distinguish two cases with respect to λ.

Case 1: $\lambda = k/2$**.** In this case, from (20) and Lemma 3, it follows that

$$k < 3.2 \cdot 10^9 \log\left(2.4 \cdot 10^{28} k^7 \log^5 k\right).$$

Solving the above equation gives

$$k < 8.8 \cdot 10^{11}.$$

Case 2: $\lambda = n - m$**.** In this case, from (20), we deduce that

$$n - m < 1.6 \cdot 10^9 \log n. \tag{21}$$

We go back to equation (3) and use properties of type (17) to obtain

$$3 \cdot 2^{n-2}(1 - 2^{m-n}) - 2 \cdot 3^a = 3 \cdot 2^{m-2}\zeta' - 3 \cdot 2^{n-2}\zeta;$$

thus, we get

$$\left|3 \cdot 2^{n-2}(1 - 2^{m-n}) - 2 \cdot 3^a\right| \le \frac{3 \cdot 2^{n-1}}{2^{k/2}}.$$

Hence, after dividing both sides by $3 \cdot 2^{n-2}(1 - 2^{m-n})$, we get

$$\left|1 - (1 - 2^{m-n})^{-1} \cdot 2^{-n+3} \cdot 3^{a-1}\right| \le \frac{2}{(1 - 2^{m-n}) \cdot 2^{k/2}} < \frac{4}{2^{k/2}}. \tag{22}$$

We apply Theorem 2 with the parameters

$$s := 3, \quad (\eta_1, b_1) := (1 - 2^{m-n}, -1), \quad (\eta_2, b_2) := (2, -n+3), \quad (\eta_3, b_3) := (3, a-1),$$

and

$$\Gamma_4 := 1 - (1 - 2^{m-n})^{-1} \cdot 2^{-n+3} \cdot 3^{a-1}.$$

If $\Gamma_4 = 0$, then $3^{a-1} = 2^{n-3} - 2^{m-3}$. According to the discussion in Subsection 3.1, this equation is impossible for any integers n, m, and $a \ge 4$. Therefore, $\Gamma_4 \neq 0$. As calculated before, we take

$$d_{\mathbb{K}} := 1, \quad A_2 := \log 2, \quad A_3 := \log 3, \quad \text{and} \quad B := n.$$

Furthermore, using (21), we get

$$h(\eta_1) \le (n - m)\log 2 + \log 2 \le 1.2 \cdot 10^9 \log n.$$

Hence, we take $A_1 := 1.2 \cdot 10^9 \log n$. By Theorem 2 and inequality (22), one gets

$$\exp(-2.4 \cdot 10^{20}(\log n)^2) < \frac{4}{2^{k/2}}.$$

Thus, we obtain

$$k < 7.3 \cdot 10^{20}(\log n)^2.$$

From this and Lemma 3, it follows that

$$k < 1.9 \cdot 10^{26}. \tag{23}$$

So, in both cases, inequality (23) holds. Thus, we obtain

$$n < 2.4 \cdot 10^{28}(1.9 \cdot 10^{26})^7(\log(1.9 \cdot 10^{26}))^5 < 1.8 \cdot 10^{221}.$$

Hence, the result is obtained. □

3.4.2. The proof of Proposition 1

To show Proposition 1, we will use the reduction algorithms described in Subsection 2.2. Let

$$\Lambda_3 := (a-1)\log 3 - (n-3)\log 2 = \log(1-\Gamma_3).$$

Assume that $n-m \geq 3$. Thus, from (18), we have $|\Gamma_3| < 1/2$. Hence, we get

$$|\Lambda_3| < 6 \cdot 2^{-\lambda}. \tag{24}$$

Dividing both sides of (24) by $(n-3)\log 3$, we obtain

$$\left|\frac{a-1}{n-3} - \frac{\log 2}{\log 3}\right| < \frac{6 \cdot 2^{-\lambda}}{(n-3)\log 3} < 5.5 \cdot 2^{-\lambda}. \tag{25}$$

Let $[a_0, a_1, \ldots]$ be the continued fraction expansion of $\dfrac{\log 2}{\log 3}$ and let p_k/q_k be its k-th convergent. We can see that $q_{425} > 1.8 \cdot 10^{221}$. Furthermore, one can see that

$$\max\{a_k : 0 \leq k \leq 425\} = a_{331} = 2436.$$

Thus, according to Lemma 2, we have

$$\left|\frac{a-1}{n-3} - \frac{\log 2}{\log 3}\right| > \frac{1}{2438(n-3)^2}. \tag{26}$$

Comparing (25) and (26), we deduce that

$$\lambda < \frac{\log(5.5 \cdot 2438 \cdot (1.8 \cdot 10^{221})^2)}{\log 2} < 1484.$$

Case 1 : $\lambda = k/2$**.** In this case, we get

$$k \leq 2968.$$

Case 2 : $\lambda = n - m$**.** Consider the linear form in logarithms

$$\Lambda_4 := (a-1)\log 3 - (n-3)\log 2 + \log((1-2^{m-n})^{-1}) = \log(\Gamma_4 + 1).$$

Since $k > 730$, from (22), we have $|\Gamma_4| < 1/2$. Hence, we obtain

$$|\Lambda_4| < 8 \cdot 2^{-k/2}.$$

Thus, we deduce that

$$0 < \left|(a-1)\frac{\log 3}{\log 2} - (n-3) + \frac{\log((1-2^{m-n})^{-1})}{\log 2}\right| < 11.6 \cdot 2^{-k/2}.$$

For $2 \leq n-m \leq 1484$, we take

$$\gamma := \frac{\log 3}{\log 2}, \quad \mu := \frac{\log((1-2^{m-n})^{-1})}{\log 2}, \quad A := 11.6, \quad B := 2, \quad M := 1.8 \cdot 10^{221},$$

in Lemma 1. Using Mathematica, we find that q_{640} satisfies the hypotheses of Lemma 1, and we obtain $k < 2974$. For $n-m=1$ or 2, Λ_4 turns into

$$\Lambda_4 = \begin{cases} (a-1)\log 3 - (n-4)\log 2, \text{ if } n-m=1; \\ (a-1)\log 3 - (n-5)\log 2, \text{ if } n-m=2. \end{cases}$$

Therefore, one can see that

$$\left|\frac{a-1}{n-t} - \frac{\log 2}{\log 3}\right| < \frac{8 \cdot 2^{-k/2}}{(n-3)\log 3} < 7.3 \cdot 2^{-k/2},$$

where $t = n-4$ if $n-m=1$ and $t=n-5$ if $n-m=2$. In this case, we apply Lemma 2 with the same parameters given above to have

$$k < \frac{2 \cdot \log(7.3 \cdot 2438 \cdot (1.8 \cdot 10^{221})^2)}{\log 2} < 1499.$$

So, $k < 2974$ always holds. With this new upper bound on k, we get

$$n < 1.7 \cdot 10^{57}.$$

Now, we repeat the reduction process but with this new bound on n, and obtain that $k \leq 782$. Hence, we get

$$n < 5.7 \cdot 10^{52}.$$

In the third application with the above bound on n, we get that $k < 724$, which contradicts our assumption $k > 730$. This completes the proof of Theorem 1 for $k > 730$ and the entire proof of the theorem.

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References

[1] E. F. Bravo, J. J. Bravo, C. A. Gómez, *Generalized Lucas numbers which are concatenations of two repdigits*, Results Math. **76**(2021), 139.

[2] J. J. Bravo, C. A. Gómez, F. Luca, *Powers of two as sums of two k-Fibonacci numbers*, Miskolc Math. Notes **17**(2016), 85–100.

[3] J. J. Bravo, J. L. Herrera, *Fermat k-Fibonacci and k-Lucas numbers*, Math. Bohem. **145**(2020), 19–32.

[4] J. J. Bravo, F. Luca, *Powers of two in generalized Fibonacci sequences*, Rev. Colombiana Mat. **46**(2012), 67–79.

[5] J. J. Bravo, F. Luca, *Repdigits in k-Lucas sequences*, Proc. Indian Acad. Sci. Math. Sci. **124**(2014), 141–154.

[6] E. F. Bravo, J. J. Bravo, F. Luca, *Coincidences in generalized Lucas sequences*, Fibonacci Quart. **52**(2014), 296–306.

[7] H. Cohen, *A Course in Computational Algebraic Number Theory*, Springer, New York, 1993.

[8] B. Demirtürk Bitim, *On the Diophantine equation $L_n - L_m = 2 \cdot 3^a$*, Period. Math. Hungar. **79**(2019), 210–217.

[9] A. Dujella, A. Pethő, *A generalization of a theorem of Baker and Davenport*, Quart. J. Math. Oxford Ser. (2) **49**(1998), 291–306.

[10] E. M. Matveev, *An explicit lower bound for a homogeneous rational linear form in the logarithms of algebraic numbers, II*, Izv. Math. **64**(2000) 1217–1269.

[11] P. Mihăilescu, *Primary cyclotomic units and a proof of Catalan's conjecture*, J. Reine Angew. Math. **572**(2004), 167–195.

[12] E. P. Jr. Miles, *Generalized Fibonacci numbers and associated matrices*, Amer. Math. Monthly **67**(1960), 745–752.

[13] M. D. Miller, *Mathematical notes: on generalized Fibonacci numbers*, Amer. Math. Monthly **78**(1971), 1108–1109.

[14] S. G. Rayaguru, J. J. Bravo, *Repdigits as sums of two generalized Lucas numbers*, Turkish J. Math. **45**(2021), 1166–1179.

[15] S. E. Rihane, *k-Fibonacci and k-Lucas numbers as product of two repdigits*, Results Math. **76**(2021), 208.

[16] S. E. Rihane, B. Faye, F. Luca, A. Togbé, *Powers of two in generalized Lucas sequences*, Fibonacci Quart. **58**(2020), 254–260.

[17] S. E. Rihane, A. Togbé, *On the intersection between k-Lucas sequences and some binary sequences*, Period. Math. Hungar. **84**(2022), 125–145.

[18] D. A. Wolfram, *Solving generalized Fibonacci recurrences*, Fibonacci Quart. **36**(1998), 129–145.