Math. Commun. **31** (2026), 61–77
DOI: 10.64785/mc.31.1.6

Mathematical Communications

On the dynamics of discrete-time Rayleigh-Duffing oscillators

Mehdi Fatehi Nia[1], Zehra Nurkanović[2,*], and Najmeh Khajoei[3]

[1] Department of Mathematics, Yazd University, P. O. Box 89195-741, Yazd, Iran
[2] Department of Mathematics, Faculty of Natural Sciences and Mathematics, University of Tuzla, Univerzitetska 4, 75 000 Tuzla, Bosnia and Herzegovina
[3] Department of Mathematics, Shahid Bahonar University of Kerman, 22 Bahman Boulevard, Kerman 76169-14111, Iran

Abstract. This paper investigates the dynamics of a discrete-time Rayleigh-Duffing oscillator exhibiting chaos in the Li-Yorke sense. We use Marotto's theorem to prove the existence of chaos by finding a snap-back repeller. It is shown that the system undergoes Neimark-Sacker and a period-doubling bifurcations. We compute the Lyapunov exponents numerically to show sensitive dependence on initial conditions and chaotic behavior. The results are illustrated using numerical simulations.

AMS subject classifications: 39A10, 39A28, 39A30
Keywords: period-doubling bifurcation; Neimark-Sacker bifurcation; Lyapunov exponent; Marotto's method; Li-Yorke chaos
Received October 30, 2024; accepted July 9, 2025

1. Introduction

Discretization of ordinary differential equations is an approach to approximate and analyze their dynamical behavior. Some discrete schemes are e.g. the trapezoidal rule method, midpoint, forward, and backward Euler's methods [20]. For an autonomous differential equation

$$y' = f(y), \tag{1}$$

where $y, f \in \mathbb{R}^n$ the flow map is defined by the function $\phi_\tau : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, where $\phi_\tau(y_0) = y(\tau)$ and $y(t)$ is the solution of the associated differential equation (1) with the initial condition $y(0) = y_0$ for $t \in [0, \tau]$. The flow map of a differential equation defines a discrete dynamical system (difference equation) when the vector field f is globally Lipschitz, since there is a unique solution to any initial condition $y_0 \in \mathbb{R}^n$ for all time. Among the discretization methods of ordinary differential equations, forward Euler's method approximates the flow map.

This paper considers the Rayleigh-Duffing oscillator, which is demonstrated as $\ddot{x} + \beta\dot{x} + \dot{x}^3 + \alpha x + x^3 = 0$. Duffing's function $\alpha x + x^3$ shows a nonlinear stiffness in this system, and Rayleigh's function $\beta\dot{x} + \dot{x}^3$ shows nonlinear damping. Also, α and β describe the linear damping coefficient and linear stiffness, respectively. This system can be shown equivalently in the form of the planar system

$$\dot{x} = y, \quad \dot{y} = -\alpha x - \beta y - x^3 - y^3, \tag{2}$$

where $\alpha, \beta \in \mathbb{R}$. This system is a mathematical model in many physical and mechanical problems; for instance, see [1, 11, 23].

It behaves similarly to the Van der Pol-Duffing and Van der Pol oscillator, see [28]. In [13], the authors studied qualitative analyses with small positive linear and cubic damping. In [4], Freire et al. studied the bifurcation diagram when $|\alpha|$ and $|\beta|$ are small. In [3], Chen and Zou studied the global dynamics of Rayleigh-Duffing oscillators. Also, the dynamic behavior of the general form of Rayleigh-Duffing oscillators is investigated by Chen et al. in [2].

Email addresses: fatehiniam@yazd.ac.ir (M. Fatehi Nia), zehra.nurkanovic@untz.ba (Z. Nurkanović), khajuee.najmeh@yahoo.com (N. Khajoei)

*Corresponding author.

https://www.mathos.unios.hr/mc
© 2026 School of Applied Mathematics and Informatics, University of Osijek

By choosing various values for parameters α and β, system (2) has different behaviors. So, the forward Euler method applied to system (2) is an approach to investigate the approximate dynamical behavior of its flow map. Hence, by applying the forward Euler scheme to system (2), we obtain the following discrete-time model:

$$x_{n+1} = x_n + hy_n, \quad y_{n+1} = y_n - h(\alpha x_n + \beta y_n + x_n^3 + y_n^3), \tag{3}$$

where the step size is $h > 0$.

The main goal of this paper is to study chaotic behaviors and bifurcation conditions of the given discrete system. This paper is organized as follows: The following section will briefly discuss the existence of fixed points and describe their local stability conditions. In Section 3, sufficient conditions for the existence of codimension-one bifurcations, including Neimark-Sacker and period-doubling bifurcations, are investigated. In Section 4, it is shown that system (3) is chaotic in the sense of Li-Yorke using Marotto's method. Finally, we used Maple and Mathematica to conduct numerical simulations, confirming our theoretical results.

2. Local stability of fixed points

In this section, we determine the existence of the fixed points of the discrete-time system and analyze their stability. It is clear that system (3) has $E_0 = (0,0)$ as a fixed point for all $\alpha, \beta \in \mathbb{R}$. Furthermore, it has the origin and $E_\pm = (\pm\sqrt{-\alpha}, 0)$ as fixed points for $\alpha < 0$.

Let T denote the map associated with system (3), i.e.

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + hy \\ y - h\left(\alpha x + \beta y + x^3 + y^3\right) \end{pmatrix},$$

the Jacobian matrix of T at the point (x, y) is

$$J_T(x,y) = \begin{bmatrix} 1 & h \\ -\alpha h - 3hx^2 & 1 - \beta h - 3hy^2 \end{bmatrix},$$

and characteristic polynomial

$$F(\lambda) = \lambda^2 + \left(3hy^2 + h\beta - 2\right)\lambda + 3h^2x^2 + h^2\alpha - 3hy^2 - h\beta + 1. \tag{4}$$

To study the stability of $E_0 = (0,0)$ and $E_\pm = (\pm\sqrt{-\alpha}, 0)$, we use [19, Lemma 2.2]. In the following proposition, the stability of system (3) at the fixed point $E_0 = (0,0)$ will be considered and used in the rest of the paper to find some conditions that eventuate bifurcations at the origin.

Proposition 1. *Assume that $\alpha, \beta \in \mathbb{R}$, and $h > 0$.*
Then the equilibrium point $E_0 = (0,0)$ of system (3) is:

(a) *a sink for* $\max\left\{0, \frac{2\beta h - 4}{h^2}\right\} < \alpha < \frac{\beta}{h}$ *and* $0 < \beta < \frac{4}{h}$,

(b) *a saddle for* $\frac{2\beta h - 4}{h^2} < \alpha < 0$ *and* $\beta < \frac{2}{h}$ *or* $0 < \alpha < \frac{2\beta h - 4}{h^2}$ *and* $\beta > \frac{2}{h}$,

(c) *a source for* $\beta \in \mathbb{R}$ *and* $\alpha < \min\left\{0, \frac{2\beta h - 4}{h^2}\right\}$ *or* $\alpha > \max\left\{0, \frac{2\beta h - 4}{h^2}, \frac{\beta}{h}\right\}$,

(d) *a non-hyperbolic with*

  1) $\lambda_{1,2}$ *are the conjugate roots and* $\left|\lambda_{1,2}\right| = 1$ *for* $\alpha = \frac{\beta}{h} \neq 0$ *and* $0 < \beta < \frac{4}{h}$,
  2) $\lambda_1 = -1$, $|\lambda_2| \neq 1$ *or* $|\lambda_1| \neq 1$, $\lambda_2 = -1$ *for* $\alpha = \frac{2\beta h - 4}{h^2}$, $\beta \neq \frac{2}{h}$, $\beta \neq \frac{4}{h}$ *and* $\beta \neq 0$,
  3) $\lambda_1 = 1$, $-1 \neq \lambda_2 < 1$ *for* $\alpha = 0$ *and* $\frac{2}{h} \neq \beta > 0$,
  4) $\lambda_1 = 1$, $\lambda_2 > 1$ *for* $\alpha = 0, \beta < 0$,
  5) $\lambda_1 = \lambda_2 = 1$ *for* $M_1 = (\beta, \alpha) = (0,0)$,
  6) $\lambda_1 = -1$, $\lambda_2 = 1$ *for* $M_2 = (\beta, \alpha) = \left(\frac{2}{h}, 0\right)$,
  7) $\lambda_1 = -1$, $\lambda_2 = -1$ *for* $M_3 = (\beta, \alpha) = \left(\frac{4}{h}, \frac{4}{h^2}\right)$.

See Figure 1.

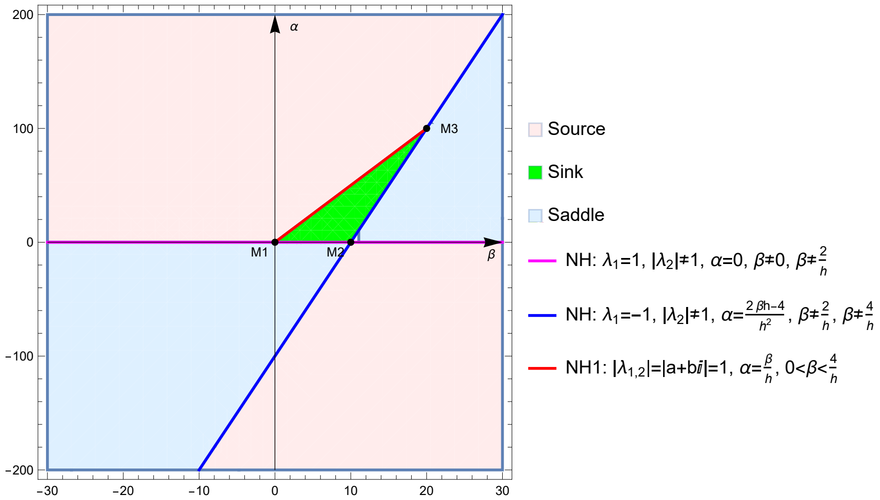

Figure 1: The stability of $E_0 = (0,0)$ in the (β, α)-plane, $M_1 = (0,0)$, $M_2 = \left(\frac{2}{h}, 0\right)$ and $M_3 = \left(\frac{4}{h}, \frac{4}{h^2}\right)$.

Proof. For $h > 0$ and $\alpha, \beta \in \mathbb{R}$, we have

$$J_T(E_0) = J_T(0,0) = \begin{bmatrix} 1 & h \\ -\alpha h & 1 - \beta h \end{bmatrix},$$

and the corresponding characteristic equation is

$$F(\lambda) = \lambda^2 + (\beta h - 2)\lambda + \alpha h^2 - \beta h + 1 = \lambda^2 + B\lambda + C, \tag{5}$$

from which

$$\lambda_1 = \frac{2 - \beta h - h\sqrt{\beta^2 - 4\alpha}}{2}, \quad \lambda_2 = \frac{2 - \beta h + h\sqrt{\beta^2 - 4\alpha}}{2}. \tag{6}$$

Since

$$F(1) = \alpha h^2 \begin{cases} < 0 \text{ for } \alpha < 0, \\ = 0 \text{ for } \alpha = 0, \\ > 0 \text{ for } \alpha > 0, \end{cases}$$

we have three different cases.

i) For $\alpha = 0$:

$$\lambda_1 = \frac{2 - \beta h - h|\beta|}{2} \begin{cases} = 1 \text{ for } \beta \leqslant 0, \\ < 1 \text{ for } \beta > 0, \end{cases}$$

and

$$\lambda_2 = \frac{2 - \beta h + h|\beta|}{2} \begin{cases} > 1 \text{ for } \beta < 0, \\ = 1 \text{ for } \beta \geqslant 0. \end{cases}$$

ii) For $\alpha < 0$:

$$\lambda_1 < 1 \Leftrightarrow -\beta h < h\sqrt{\beta^2 - 4\alpha},$$

which is satisfied for all $\beta \in \mathbb{R}$, because

$$-\beta h \leq |\beta| h < h\sqrt{\beta^2 - 4\alpha}.$$

Also,

$$\lambda_1 \geq -1 \Leftrightarrow \beta < \frac{4}{h} \text{ and } \alpha \geq \frac{2\beta h - 4}{h^2}.$$

Therefore,

$$\lambda_1 < -1 \Leftrightarrow \beta \geq \frac{4}{h} \text{ or } (\beta < \frac{4}{h} \text{ and } \alpha < \frac{2\beta h - 4}{h^2}),$$
$$\lambda_1 = -1 \Leftrightarrow \beta < \frac{4}{h} \text{ and } \alpha = \frac{2\beta h - 4}{h^2},$$
$$|\lambda_1| < 1 \Leftrightarrow \beta < \frac{4}{h} \text{ and } \alpha > \frac{2\beta h - 4}{h^2}.$$

It is clear that $\lambda_2 > 1$ for $\alpha < 0$ and $\beta \in \mathbb{R}$.

iii) If $\alpha > 0$, we have that $B^2 - 4C = \beta^2 - 4\alpha \geq 0$ for $0 < \alpha \leq \frac{\beta^2}{4}$ and

$$B = \beta h - 2 \neq \begin{cases} 0 \text{ for } \beta \neq \frac{2}{h}, \\ 2 \text{ for } \beta \neq \frac{4}{h}, \end{cases}$$

$$C = (\alpha h - \beta) h + 1 \begin{cases} = 1 \text{ for } \alpha = \frac{\beta}{h}, \\ > 1 \text{ for } \alpha > \frac{\beta}{h}, \\ < 1 \text{ for } \alpha < \frac{\beta}{h}, \end{cases}$$

$$C = 1 \text{ and } B^2 - 4C = \beta^2 - 4\alpha < 0 \text{ for } \alpha = \frac{\beta}{h} \text{ and } 0 < \beta < \frac{4}{h},$$

$$F(-1) = \alpha h^2 - 2\beta h + 4 \begin{cases} > 0 \text{ for } \alpha > \frac{2\beta h - 4}{h^2}, \\ = 0 \text{ for } \alpha = \frac{2\beta h - 4}{h^2}, \\ < 0 \text{ for } \alpha < \frac{2\beta h - 4}{h^2}, \end{cases}$$

and by using Lemma 2.2 [19], there follows the conclusion. □

Proposition 2. *Assume $\alpha < 0$, $\beta \in \mathbb{R}$, and $h > 0$.*
Then equilibrium points $E_{\pm} = \left(\pm\sqrt{-\alpha}, 0\right)$ of system (3) are:

- (a) *a source for $\alpha < 0$, $\beta \leqslant 0$ and $\alpha < -\frac{\beta}{2h}$ and $\alpha < \frac{2-h\beta}{h^2}$,*
- (b) *a saddle for $\alpha < 0$ and $\alpha > \frac{2-h\beta}{h^2}$, $\beta > \frac{2}{h}$,*
- (c) *a non-hyperbolic with*
  - *1) $\lambda_1 = -1$, $|\lambda_2| \neq 1$ for $\alpha < 0$ and $\alpha = \frac{2-h\beta}{h^2}$, $\beta > \frac{2}{h}$, $\beta \neq \frac{4}{h}$,*
  - *2) $\lambda_1 = -1$, $\lambda_2 = 1$ for $M_4 = (\beta, \alpha) = \left(\frac{4}{h}, -\frac{2}{h^2}\right)$,*
  - *3) $\lambda_{1,2}$ are the conjugate roots and $\left|\lambda_{1,2}\right| = 1$ for $\alpha = -\frac{\beta}{2h}$ and $0 < \beta < \frac{4}{h}$.*

See Figure 2.

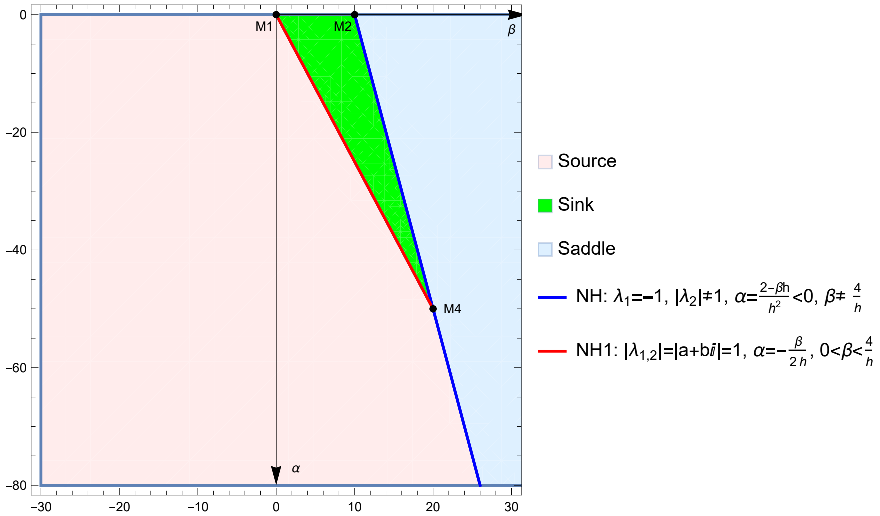

Figure 2: The stability of $E_{\pm}$ in the (β, α)-plane, $\alpha < 0$, $M_1 = (0, 0)$, $M_2 = \left(\frac{2}{h}, 0\right)$ and $M_4 = \left(\frac{4}{h}, -\frac{2}{h^2}\right)$

Proof. For $\alpha < 0$, $\beta \in \mathbb{R}$ and $h > 0$, we have $J_T(E_{\pm}) = \begin{bmatrix} 1 & h \\ 2\alpha h & 1 - \beta h \end{bmatrix}$, and

$$F(\lambda) = \lambda^2 + (\beta h - 2)\lambda - 2\alpha h^2 - \beta h + 1 = \lambda^2 + B\lambda + C.$$

For $\alpha < 0$ and $\beta \leq 0$, equilibriums $E_{\pm}$ are the source because

$$C = -2\alpha h^2 - \beta h + 1 > 1,$$

$$F(1) = -2h^2\alpha > 0, \quad F(-1) = 2\left(2 - h\beta - h^2\alpha\right) > 0.$$

For $\beta > 0$, we have

$$B = \beta h - 2 \begin{cases} \neq 0 \text{ for } \beta \neq \frac{2}{h}, \\ \neq 2 \text{ for } \beta \neq \frac{4}{h}, \end{cases}$$

$$C = 1 - h(\beta + 2h\alpha) \begin{cases} < 1 \text{ for } \alpha > -\frac{\beta}{2h}, \\ = 1 \text{ for } \alpha = -\frac{\beta}{2h}, \\ > 1 \text{ for } \alpha < -\frac{\beta}{2h}, \end{cases}$$

$$C = 1 \text{ and } B^2 - 4C = h\beta(h\beta - 4) < 0 \text{ for } 0 < \beta < \frac{4}{h}, \alpha = -\frac{\beta}{2h},$$

$$F(1) = -2h^2\alpha > 0,$$

$$F(-1) = 2\left(2 - h\beta - h^2\alpha\right) \begin{cases} < 0 \text{ for } \alpha > \frac{2-h\beta}{h^2}, \beta > \frac{2}{h}, \\ = 0 \text{ for } \alpha = \frac{2-h\beta}{h^2}, \beta > \frac{2}{h}, \\ > 0 \text{ for } \alpha < \frac{2-h\beta}{h^2}. \end{cases}$$

Using [19, Lemma 2.2], the above proves all the claims. □

Also, for the local stability of all equilibrium points, see Figure 3.

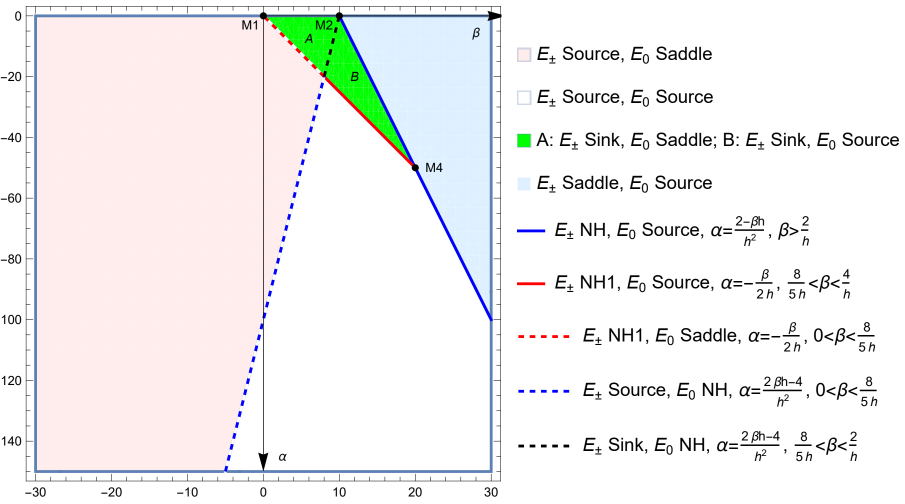

Figure 3: The stability of E_0 and $E_{\pm}$ in the (β,α)-plane, $M_1=(0,0)$, $M_2=\left(\frac{2}{h},0\right)$, $M_4=\left(\frac{4}{h},-\frac{2}{h^2}\right)$

Remark 1. Propositions 1 and 2 give us a clear idea of the local stability of the equilibrium points of system (3). From this, it can be concluded that system (3) dynamics are very complicated. Namely, in addition to Neimark-Sacker and period-doubling bifurcations, the equilibrium E_0 also has codimension-two bifurcations (1:1, 1:2, possibly 1:3 and 1:4, as well as the bifurcation that occurs in (d) 6) of Proposition 1). On the other hand, $E_{\pm}$ equilibria are unstable, except possibly in cases (c) 2), and 3) of Proposition 2). Examining all these cases requires much more time and would take up much more space in the manuscript so that it can be the subject of future research. It is very important in situations where a codimension-two bifurcation occurs and when the shape of the trajectory of the orbits that move between all equilibrium points needs to be precisely determined. For this reason, we decided only to examine Neimark-Sacker and period-doubling bifurcations of the equilibrium point E_0 and theoretical proof of the appearance of chaos in these situations.

3. Bifurcation analysis

In this section, we study the conditions for the existence of Neimark-Sacker and period-doubling bifurcations of system (3) at the fixed point $E_0=(0,0)$, see [7, 12, 15, 16, 17, 14, 24]. We choose the step size h as a bifurcation parameter to study these bifurcations.

3.1. Neimark-Sacker bifurcation

Based on statement (d), 1) of Proposition 1, system (3) has a pair of complex multipliers on the unit circle. We use the standard version of the Naimark-Sacker result, see [15, Theorems 15 and 16] and [9, 17].

Theorem 1. *Assume that* $h>0, h_0:=\frac{\beta}{\alpha}<\frac{4}{\beta}$, $a\left(h_0\right)=\frac{3\beta^3\left(\alpha^2-\beta\right)}{8\alpha^4}$, $\alpha>0$, $\beta>0$.

(i) *For* $\alpha\in\left(0,\sqrt[3]{4}\right)$ *and* $\beta\in\left(0,\alpha^2\right)$ *or* $\alpha\geq\sqrt[3]{4}$ *and* $\beta\in\left(0,2\sqrt{\alpha}\right)$, *i.e.,* $a\left(h_0\right)>0$, *there is a neighborhood* $\mathcal{U}$ *of the origin and a* $\delta>0$ *such that for* $|h-h_0|<\delta$ *and* $(x_0,y_0)\in\mathcal{U}$, *then the* ω*-limit set of* (x_0,y_0) *is the origin when* $h<h_0$ *and belongs to a closed invariant* C^1 *curve* $\Gamma(\lambda)$ *encircling the origin if* $h>h_0$. *Furthermore,* $\Gamma(h_0)=0$.

(ii) *For* $\alpha\in\left(0,\sqrt[3]{4}\right)$ *and* $\beta\in\left(\alpha^2,2\sqrt{\alpha}\right)$, *i.e.,* $a\left(h_0\right)<0$, *there is a neighborhood* $\mathcal{U}$ *of the origin and a* $\delta>0$ *such that for* $|h-h_0|<\delta$ *and* $(x_0,y_0)\in\mathcal{U}$, *then the* α*-limit set of* (x_0,y_0) *is the origin when* $h>h_0$ *and belongs to a closed invariant* C^1 *curve* $\Gamma(\lambda)$ *encircling the origin if* $h<h_0$. *Furthermore,* $\Gamma(h_0)=0$.

Proof. We rewrite system (3) in the following form:

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & h \\ -h\alpha & 1-h\beta \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} 0 \\ -h(x_n^3+y_n^3) \end{bmatrix}. \tag{7}$$

The Jacobian $J_T(0,0)$ has two non-real eigenvalues: $\lambda(h) = \frac{2-h\beta+ih\sqrt{4\alpha-\beta^2}}{2}$ and $\overline{\lambda(h)}$, for $4\alpha-\beta^2>0$, which implies that $|\lambda(h)|_{h=h_0}=1$. Since

$$\frac{d|\lambda(h)|}{dh} = \frac{1}{2}(2\alpha h-\beta)\left(1-h\beta+\alpha h^2\right)^{-\frac{1}{2}},$$

we obtain $d(h_0) := \left.\frac{d|\lambda(h)|}{dh}\right|_{h=h_0} = \frac{\beta}{2}>0$.

Also, we have that

$$\begin{aligned} \left.\lambda^k(h)\right|_{h=h_0} &= exp\left(ik\arctan\left(\frac{h\sqrt{4\alpha-\beta^2}}{2-h\beta}\right)\right)\Bigg|_{h=h_0} \\ &= exp\left(ik\arctan\left(\frac{\beta\sqrt{4\alpha-\beta^2}}{2\alpha-\beta^2}\right)\right), \end{aligned}$$

and

$$\left.\lambda^k(h)\right|_{h=h_0} \neq 1 \Leftrightarrow \arctan\left(\frac{\beta\sqrt{4\alpha-\beta^2}}{2\alpha-\beta^2}\right)\neq 0, \quad k=1,2,3,4,$$

(since $4\alpha-\beta^2>0$) which implies that the third condition of Poincare-Andronov-Hopf bifurcation for maps (see [9, p. 474, Theorem 15.31]) is valid.

Now, our goal is to obtain the normal form of system (7), when $h=h_0$.

First, denote $\lambda(h_0)=k+ie$ (with $k=\frac{2-h_0\beta}{2}$, $e=\frac{h_0\sqrt{4\alpha-\beta^2}}{2}$), and let

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = P\begin{bmatrix} \tilde{x}_n \\ \tilde{y}_n \end{bmatrix}, \tag{8}$$

where $\tilde{x}_n$, $\tilde{y}_n$ are new variables, and

$$P = \begin{bmatrix} h_0 & 0 \\ k-1 & -e \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} \frac{1}{h_0} & 0 \\ \frac{k-1}{h_0 e} & -\frac{1}{e} \end{bmatrix},$$

such that $P^{-1}J_T(0,0)P = \begin{bmatrix} k & -e \\ e & k \end{bmatrix}$, with $\begin{vmatrix} k & -e \\ e & k \end{vmatrix} = 1$.

Let us notice that from (8) we have that

$$x_n = h_0\tilde{x}_n, \quad y_n = (k-1)\tilde{x}_n - e\tilde{y}_n.$$

Then, using (8) and $h=h_0$, system (7) has the form

$$P\begin{bmatrix} \tilde{x}_{n+1} \\ \tilde{y}_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & h_0 \\ -h_0\alpha & 1-h_0\beta \end{bmatrix} P \begin{bmatrix} \tilde{x}_n \\ \tilde{y}_n \end{bmatrix} + \begin{bmatrix} \phi(\tilde{x}_n,\tilde{y}_n) \\ \psi(\tilde{x}_n,\tilde{y}_n) \end{bmatrix},$$

where

$$\begin{aligned} \phi(\tilde{x}_n,\tilde{y}_n) &= 0, \\ \psi(\tilde{x}_n,\tilde{y}_n) &= -h_0((h_0\tilde{x}_n)^3 + ((k-1)\tilde{x}_n - e\tilde{y}_n)^3), \end{aligned}$$

from which we obtain its normal Jordan form as

$$\begin{bmatrix} \tilde{x}_{n+1} \\ \tilde{y}_{n+1} \end{bmatrix} = P^{-1}\begin{bmatrix} 1 & h_0 \\ -h_0\alpha & 1-h_0\beta \end{bmatrix} P \begin{bmatrix} \tilde{x}_n \\ \tilde{y}_n \end{bmatrix} + P^{-1}\begin{bmatrix} \phi(\tilde{x}_n,\tilde{y}_n) \\ \psi(\tilde{x}_n,\tilde{y}_n) \end{bmatrix},$$

i.e.,

$$\begin{bmatrix}\tilde{x}_{n+1}\\ \tilde{y}_{n+1}\end{bmatrix} = \begin{bmatrix}k & -e\\ e & k\end{bmatrix}\begin{bmatrix}\tilde{x}_{n}\\ \tilde{y}_{n}\end{bmatrix} + \begin{bmatrix}f(\tilde{x}_n,\tilde{y}_n)\\ g(\tilde{x}_n,\tilde{y}_n)\end{bmatrix},$$

where

$$\begin{bmatrix}f(\tilde{x}_n,\tilde{y}_n)\\ g(\tilde{x}_n,\tilde{y}_n)\end{bmatrix} = P^{-1}\begin{bmatrix}\phi(\tilde{x}_n,\tilde{y}_n)\\ \psi(\tilde{x}_n,\tilde{y}_n)\end{bmatrix},$$

and

$$\begin{aligned} f(\tilde{x},\tilde{y}) &= 0\\ g(\tilde{x},\tilde{y}) &= h_0\frac{\left((k-1)^3+h_0^3\right)\tilde{x}^3 - 3e\,(k-1)^2\,\tilde{x}^2\tilde{y} + 3e^2\,(k-1)\,\tilde{x}\tilde{y}^2 - e^3\tilde{y}^3}{e}. \end{aligned}$$

Now, for $(\tilde{x},\tilde{y}) = (0,0)$, we get

$$\begin{aligned} g_{\tilde{x}\tilde{x}} &= g_{\tilde{x}\tilde{y}} = g_{\tilde{y}\tilde{y}} = 0, \quad g_{\tilde{x}\tilde{y}\tilde{y}} = 6h_0e\,(k-1)\,, \quad g_{\tilde{y}\tilde{y}\tilde{y}} = -6h_0e^2,\\ g_{\tilde{x}\tilde{x}\tilde{x}} &= \frac{6h_0\left(h_0^3+(k-1)^3\right)}{e},\\ \xi_{20} &= \frac{1}{8}\left\{(f_{\tilde{x}\tilde{x}} - f_{\tilde{y}\tilde{y}} + 2g_{\tilde{x}\tilde{y}}) + i(g_{\tilde{x}\tilde{x}} - g_{\tilde{y}\tilde{y}} - 2f_{\tilde{x}\tilde{y}})\right\} = 0,\\ \xi_{11} &= \frac{1}{4}\left\{(f_{\tilde{x}\tilde{x}} + f_{\tilde{y}\tilde{y}}) + i(g_{\tilde{x}\tilde{x}} - g_{\tilde{y}\tilde{y}})\right\} = 0,\\ \xi_{02} &= (f_{\tilde{x}\tilde{x}} - f_{\tilde{y}\tilde{y}} - 2g_{\tilde{x}\tilde{y}}) + i(g_{\tilde{x}\tilde{x}} - g_{\tilde{y}\tilde{y}} + 2f_{\tilde{x}\tilde{y}}) = 0,\\ \xi_{21} &= \frac{1}{16}\left\{\left(f_{\tilde{x}\tilde{x}\tilde{x}} + f_{\tilde{x}\tilde{y}\tilde{y}} + g_{\tilde{x}\tilde{x}\tilde{y}} + g_{\tilde{y}\tilde{y}\tilde{y}}\right) + i\left(g_{\tilde{x}\tilde{x}\tilde{x}} + g_{\tilde{x}\tilde{y}\tilde{y}} - f_{\tilde{x}\tilde{x}\tilde{y}} - f_{\tilde{y}\tilde{y}\tilde{y}}\right)\right\},\\ \xi_{21} &= -\frac{3}{8}h_0\left(e^2 + (k-1)^2 + i\left((1-k)\,e - \frac{(k-1)^3+h_0^3}{e}\right)\right),\\ \operatorname{Re}\left\{\overline{\lambda}\xi_{21}\right\} &= \frac{3}{8}h_0\left(h_0^3 - (k-1)^2 - e^2\right). \end{aligned}$$

Finally, it is necessary to calculate the following coefficient:

$$a\,(h_0) = \operatorname{Re}\left\{\frac{(1-2\lambda)\overline{\lambda}^2}{1-\lambda}\xi_{11}\xi_{20}\right\} + \frac{1}{2}|\xi_{11}|^2 + |\xi_{02}|^2 - \operatorname{Re}\left\{\overline{\lambda}\xi_{21}\right\},$$

i.e.,

$$a\,(h_0) = a\left(\frac{\beta}{\alpha}\right) = \frac{3\beta^3\left(\alpha^2-\beta\right)}{8\alpha^4}, \quad 4\alpha-\beta^2>0, \alpha>0, \beta>0,$$

which is the coefficient of the cubic term in [7] (4.2) in polar coordinates.

It is easy to see that

$$a\,(h_0) = a\left(\frac{\beta}{\alpha}\right)\begin{cases} >0, & \text{for} \quad \begin{cases} \alpha\in\left(0,\sqrt[3]{4}\right) \text{ and } \beta\in\left(0,\alpha^2\right)\\ \text{or}\\ \alpha\geq\sqrt[3]{4} \text{ and } \beta\in\left(0,2\sqrt{\alpha}\right), \end{cases}\\ =0, & \text{for} \quad \alpha\in\left(0,\sqrt[3]{4}\right) \text{ and } \beta=\alpha^2,\\ <0, & \text{for} \quad \alpha\in\left(0,\sqrt[3]{4}\right) \text{ and } \beta\in\left(\alpha^2,2\sqrt{\alpha}\right). \end{cases}$$

This completes the proof of the theorem. □

3.2. Period-doubling bifurcation

The period-doubling bifurcation happens when $\lambda_i = -1$ and $|\lambda_j| \neq 1$ for $i, j \in \{1,2\}, i \neq j$. Then, from (6) we have $\lambda_1 = -1$ and $\lambda_2 = \frac{2-\alpha h_1^2}{2} = 3 - \beta h_1$, for $h_1 := \frac{\beta - \sqrt{\beta^2 - 4\alpha}}{\alpha}$ (and for $\alpha h_1^2 = 2\beta h_1 - 4$, $h_1 \neq \frac{2}{\beta}$, $h_1 \neq \frac{4}{\beta}$, $\beta \neq 0$, $\beta^2 > 4\alpha$, see (d) 2) of Proposition 1).

Now, we investigate the period-doubling bifurcation of system (3) at the origin when parameter h changes in the small neighborhood of h_1. Therefore, consider system (3) by choosing $\tilde{h}$ as a small bifurcation parameter ($h = h_1 + \tilde{h}$):

$$x_{n+1} = x_n + (h_1 + \tilde{h})y_n, \quad y_{n+1} = y_n - (h_1 + \tilde{h})(\alpha x_n + \beta y_n + x_n^3 + y_n^3), \tag{9}$$

Then, system (9) can be written in the following form:

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & h_1 \\ -h_1\alpha & 1 - h_1\beta \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} \Phi(x_n, y_n) \\ \Psi(x_n, y_n) \end{bmatrix}, \tag{10}$$

where

$$\begin{aligned} \Phi(x_n, y_n) &= \tilde{h} y_n \\ \Psi(x_n, y_n) &= -\tilde{h}(\alpha x_n + \beta y_n) - \left(h_1 + \tilde{h}\right)\left(x_n^3 + y_n^3\right), \end{aligned}$$

and $|\tilde{h}| << 1$ is a small perturbation parameter. To make the normal form of system (10), we need an invertible matrix B whose columns are eigenvectors of the corresponding $\lambda_1 = -1$ and $\lambda_2 = 3 - \beta h_1$:

$$B = \begin{bmatrix} h_1 & h_1 \\ -2 & \lambda_2 - 1 \end{bmatrix},$$

where

$$B^{-1} = \frac{1}{h_1(\lambda_2 + 1)} \begin{bmatrix} \lambda_2 - 1 & -h_1 \\ 2 & h_1 \end{bmatrix}, \quad B^{-1} \begin{bmatrix} 1 & h_1 \\ -h_1\alpha & 1 - h_1\beta \end{bmatrix} B = \begin{bmatrix} -1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

Now, let

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = B \begin{bmatrix} \tilde{x}_n \\ \tilde{y}_n \end{bmatrix} = \begin{bmatrix} h_1 & h_1 \\ -2 & \lambda_2 - 1 \end{bmatrix} \begin{bmatrix} \tilde{x}_n \\ \tilde{y}_n \end{bmatrix} = \begin{bmatrix} h_1(\tilde{x}_n + \tilde{y}_n) \\ \tilde{y}_n(\lambda_2 - 1) - 2\tilde{x}_n \end{bmatrix}.$$

Then, system (10) changes into

$$B \begin{bmatrix} \tilde{x}_{n+1} \\ \tilde{y}_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & h_1 \\ -h_1\alpha & 1 - h_1\beta \end{bmatrix} B \begin{bmatrix} \tilde{x}_n \\ \tilde{y}_n \end{bmatrix} + \begin{bmatrix} \Phi_1(\tilde{x}_n, \tilde{y}_n) \\ \Psi_1(\tilde{x}_n, \tilde{y}_n) \end{bmatrix},$$

where

$$\begin{aligned} \Phi_1(\tilde{x}_n, \tilde{y}_n) =& \tilde{h}(\tilde{y}_n(\lambda_2 - 1) - 2\tilde{x}_n), \\ \Psi_1(\tilde{x}_n, \tilde{y}_n) =& -\tilde{h}(\alpha h_1(\tilde{x}_n + \tilde{y}_n) + \beta(\tilde{y}_n(\lambda_2 - 1) - 2\tilde{x}_n)) \\ & - \left(h_1 + \tilde{h}\right)\left((h_1(\tilde{x}_n + \tilde{y}_n))^3 + (\tilde{y}_n(\lambda_2 - 1) - 2\tilde{x}_n)^3\right), \end{aligned}$$

which implies that

$$\begin{bmatrix} \tilde{x}_{n+1} \\ \tilde{y}_{n+1} \end{bmatrix} = B^{-1} \begin{bmatrix} 1 & h_1 \\ -h_1\alpha & 1 - h_1\beta \end{bmatrix} B \begin{bmatrix} \tilde{x}_n \\ \tilde{y}_n \end{bmatrix} + B^{-1} \begin{bmatrix} \Phi_1(\tilde{x}_n, \tilde{y}_n) \\ \Psi_1(\tilde{x}_n, \tilde{y}_n) \end{bmatrix},$$

i.e.,

$$\begin{bmatrix} \tilde{x}_{n+1} \\ \tilde{y}_{n+1} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \tilde{x}_n \\ \tilde{y}_n \end{bmatrix} + \begin{bmatrix} f(\tilde{x}_n, \tilde{y}_n) \\ g(\tilde{x}_n, \tilde{y}_n) \end{bmatrix}, \tag{11}$$

where

$$\begin{aligned} f(\tilde{x},\tilde{y}) =& \frac{1}{\lambda_2+1}\Bigg[\left(\frac{-2\tilde{h}\left(\lambda_2+1\right)}{h_1}\right)\tilde{x} \\ &+ \left(h_1+\tilde{h}\right)\left(\left(h_1^3-8\right)\tilde{x}^3+\left(\left(\lambda_2-1\right)^3+h_1^3\right)\tilde{y}^3\right) \\ &+ 3\left(h_1+\tilde{h}\right)\left(\left(h_1^3-2\left(\lambda_2-1\right)^2\right)\tilde{x}\tilde{y}^2+\left(h_1^3+4\left(\lambda_2-1\right)\right)\tilde{x}^2\tilde{y}\right)\Bigg], \end{aligned}$$

$$\begin{aligned} g(\tilde{x},\tilde{y}) =& \frac{1}{\lambda_2+1}\Bigg[\left(\frac{\left(2-\beta h_1\right)\tilde{h}\left(\lambda_2+1\right)}{h_1}\right)\tilde{y} \\ &- \left(h_1+\tilde{h}\right)\left(\left(h_1^3-8\right)\tilde{x}^3+\left(\left(\lambda_2-1\right)^3+h_1^3\right)\tilde{y}^3\right) \\ &+ 3\left(h_1+\tilde{h}\right)\left(\left(2\left(\lambda_2-1\right)^2-h_1^3\right)\tilde{x}\tilde{y}^2-\left(h_1^3+4\lambda_2-4\right)\tilde{x}^2\tilde{y}\right)\Bigg]. \end{aligned}$$

Now, we compute the center manifold $W^c(0,0)$ at the fixed point $(0,0)$ in a small neighborhood of $\tilde{h}=0$ by using the center manifold theorem in [8, 18]. To compute the center manifold, one can consider it by a polynomial of $\tilde{x}$ and $\tilde{h}$ of degree four $\tilde{y}=\mathcal{C}\left(\tilde{x}\right)$, where

$$\begin{aligned} \mathcal{C}\left(\tilde{x}\right) &= A\tilde{x}^2+B\tilde{x}\tilde{h}+C\tilde{h}^2+D\tilde{x}^3+E\tilde{x}^2\tilde{h}+F\tilde{x}\tilde{h}^2 \\ &\quad +\sigma\tilde{h}^3+H\tilde{x}^4+\omega\tilde{x}^3\tilde{h}+J\tilde{x}^2\tilde{h}^2+N\tilde{x}\tilde{h}^3+M\tilde{h}^4, \end{aligned}$$

with the unknown coefficients A, B, C, D, E, F, σ, H, ω, J, N, M. Then, it must satisfy the following condition:

$$\mathcal{C}\left(-\tilde{x}+f\left(\tilde{x},\mathcal{C}\left(\tilde{x}\right)\right)\right)-\lambda_2\mathcal{C}\left(\tilde{x}\right)-g\left(\tilde{x},\mathcal{C}\left(\tilde{x}\right)\right)=0.$$

By approximate computation, for the center manifold, we obtain

$$A=B=C=E=F=G=H=J=N=M=0,$$

$$\sigma=\frac{h_1\left(2-h_1\right)\left(2h_1+h_1^2+4\right)}{\left(\lambda_2+1\right)^2},\quad \omega=\frac{2\left(h_1-2\right)\left(\lambda_2+3\right)\left(2h_1+h_1^2+4\right)}{\left(\lambda_2+1\right)^3}. \tag{12}$$

Therefore, the center manifold

$$W^c\left(0,0\right)=\left\{\left(\tilde{x},\tilde{y}\right)\mid\ \tilde{y}=\mathcal{C}\left(\tilde{x}\right),\ \mathcal{C}\left(0\right)=\mathcal{C}'(0)=0\right\},$$

near $\tilde{h}=0$, can be approximately written in the following form:

$$W^c\left(0,0\right)=\left\{\left(\tilde{x},\tilde{y}\right)\mid\ \tilde{y}=\sigma\tilde{x}^3+\omega\tilde{x}^3\tilde{h}+\mathcal{O}\left(\left(|\tilde{x}|+|\tilde{h}|\right)^5\right)\right\},$$

where σ and ω are given by (12). Now, we restrict system (11) (the first equation) to the center manifold:

$$\widetilde{F}\left(\tilde{x}\right)=-\tilde{x}+f\left(\tilde{x},\mathcal{C}(\tilde{x})\right)=-\tilde{x}+f\left(\tilde{x},\sigma\tilde{x}^3+\omega\tilde{x}^3\tilde{h}\right),$$

i.e.,

$$\widetilde{F}\left(\tilde{x}\right)=-\tilde{x}+\delta\tilde{x}\tilde{h}+\eta\tilde{x}^3+\mathcal{O}\left(\left(|\tilde{x}|+|\tilde{h}|\right)^4\right),$$

where $\delta=-\frac{2}{h_1}$ and $\eta=\frac{\left(h_1+\tilde{h}\right)\left(h_1-2\right)\left(h_1^2+2h_1+4\right)}{\lambda_2+1}$.

The following three conditions are satisfied (see [25, Theorem 3.1, p. 246]):

- $\widetilde{F}(0)=0$,
- $\widetilde{F}'(0)=-1$ and

- $\kappa_1 = \left(\frac{\partial^2 \widetilde{F}}{\partial \tilde{x} \partial \tilde{h}} + \frac{1}{2} \frac{\partial \widetilde{F}}{\partial \tilde{h}} \frac{\partial^2 \widetilde{F}}{\partial \tilde{x}^2} \right) \Bigg|_{(0,0)} = \delta = -\frac{2}{h_1} \neq 0,$

  $\kappa_2 = \left(\frac{1}{6} \frac{\partial^3 \widetilde{F}}{\partial \tilde{x}^3} + \left(\frac{1}{2} \frac{\partial^2 \widetilde{F}}{\partial \tilde{x}^2} \right)^2 \right) \Bigg|_{(0,0)} = \frac{h_1 (h_1 - 2) \left(h_1^2 + 2h_1 + 4 \right)}{\lambda_2 + 1} \neq 0,$

  when $h_1 \neq 2$.

With the above arguments, we have the following theorem, which shows that system (3) experiences a period-doubling bifurcation at the origin, as the parameter $\tilde{h}$ varies through the bifurcation value $\tilde{h} = 0$.

Theorem 2. *System* (3) *undergoes a period-doubling bifurcation at the fixed point* $E_0 = (0, 0)$ *if* $h = h_1 := \frac{\beta - \sqrt{\beta^2 - 4\alpha}}{\alpha} \neq 2$ *and* $\beta^2 - 4\alpha > 0$.

*Furthermore, the orbits of period-*2 *that bifurcate from the origin are stable if* $h_1 < 2$, *and unstable if* $h_1 > 2$ *for* $\beta^2 - 4\alpha > 0$.

4. The existence of Li-Yorke chaos

In this section, by using Marotto's theorem [5, 6, 21, 22], we prove that system (3) has chaotic behavior for particular values of coefficients. In [22], Marotto extended the definition of chaos in the sense of Li-Yorke from one dimension to multiple dimensions by introducing the concept of a snap-back repeller. In [21], Marotto redefined the definition of a snap-back repeller. He proved that the existence of a snap-back repeller is an indicator of chaos. We present the concept of a snap-back repeller and Marotto's theorem.

We will prove that system (3) possesses chaotic behavior in the sense of Marotto's definition whenever the origin is a source and has a pair of complex multipliers.

From (5) we see that it is $|\lambda_{1,2}| > 1$ if $C > 1$ and $B^2 - 4C < 0$, i.e., if $h\alpha - \beta > 0$ and $4\alpha - \beta^2 > 0$. Our next step is to determine a neighborhood $U_r(0, 0)$ in which the norms of eigenvalues exceed 1 for all $(x, y) \in U_r(0, 0)$. Then from (4) conditions $C > 1$ and $B^2 - 4C < 0$ are equivalent to the conditions

$$\begin{aligned} \left(3hy^2 + h\beta - 2\right)^2 - 4\left(3h^2x^2 + \alpha h^2 - 3hy^2 - \beta h + 1\right) &< 0, \\ 3h^2x^2 + \alpha h^2 - 3hy^2 - \beta h + 1 &> 1, \end{aligned}$$

or

$$\delta(x) = 12x^2 - 9y^4 - 6y^2\beta + 4\alpha - \beta^2 > 0,$$

$$\gamma(x) = 3hx^2 - 3y^2 + h\alpha - \beta > 0.$$

These are two hyperbolas that have local extremes at zero.

$$\begin{aligned} \delta_{\min}(x) &= \delta(0) = -9y^4 - 6y^2\beta + 4\alpha - \beta^2 > 0, \\ \gamma_{\min}(x) &= \gamma(0) = -3y^2 + h\alpha - \beta > 0. \end{aligned}$$

For $4\alpha - \beta^2 > 0$ and $h\alpha - \beta > 0$, we have

$$\delta_{\min}(x) > 0 \Leftrightarrow y \in \left(-\sqrt{\tfrac{2\sqrt{\alpha} - \beta}{3}}, \sqrt{\tfrac{2\sqrt{\alpha} - \beta}{3}} \right),$$

$$\gamma_{\min}(x) > 0 \Leftrightarrow y \in \left(-\sqrt{\tfrac{h\alpha - \beta}{3}}, \sqrt{\tfrac{h\alpha - \beta}{3}} \right),$$

and $r = \min\left\{ \sqrt{\frac{2\sqrt{\alpha} - \beta}{3}}, \sqrt{\frac{h\alpha - \beta}{3}} \right\}$, $U_r(0, 0) = \left\{ (x, y) |\ x^2 + y^2 < r^2 \right\}$.

We need to find the preimage (x_0, y_0) of $(0, 0)$ in $U_r(0, 0)$ with

$$(x_0, y_0) \neq (0, 0), \quad T^M(x_0, y_0) = (0, 0), \quad |J_T(x_k, y_k)| \neq 0$$

for $0 \leq k \leq M-1$ hold.

For $M = 2$, we get two systems

$$x_1 = x_0 + hy_0, \qquad y_1 = y_0 - h(\alpha x_0 + \beta y_0 + x_0^3 + y_0^3), \tag{13}$$

$$0 = x_1 + hy_1, \qquad 0 = y_1 - h(\alpha x_1 + \beta y_1 + x_1^3 + y_1^3). \tag{14}$$

From the first equation of system (14) we get $y_1 = -\frac{x_1}{h}$, so the second equation is equivalent to $x_1^2\left(1-h^3\right) = h^2\left(-\beta + h\alpha\right) + h$. For $h \ll 1$ and $h\alpha - \beta > 0$, we have $x_1 = \pm\sqrt{\frac{(h(h\alpha-\beta)+1)h}{1-h^3}} \neq 0$. For

$$(x_1, y_1) = \left(\sqrt{\tfrac{h^2(h\alpha-\beta)+h}{1-h^3}}, -\frac{x_1}{h}\right) \neq (0,0),$$

we have $|J_T(x_1, y_1)| \neq 0$ because it is

$$|J_T(x_1,y_1)| = -h\beta - 3hy^2 + h^2\alpha + 3h^2x^2 + 1\Big|_{(x_1,y_1)} = -2\left(h\left(h\alpha - \beta\right) + 1\right) < 0.$$

Using $y_1 = -\frac{x_1}{h}$ from (13) we get that x_0 is the solution of the equation

$$Ax_0^3 + Bx_0^2 + Cx_0 + D = 0, \tag{15}$$

where $A = \left(1 - h^3\right) > 0$, $B = -3x_1 < 0$,

$$C = -\alpha h^3 + \beta h^2 - h + 3x_1^2 = \frac{h\left(h^3+2\right)\left(1 + h\left(h\alpha - \beta\right)\right)}{\left(1-h\right)\left(h + h^2 + 1\right)} > 0,$$

$$D = -\beta h^2 x_1 + 2hx_1 - x_1^3 = \frac{\left(h^4\beta - 2h^3 - h^2\alpha + 1\right)x_1 h}{\left(1-h\right)\left(h + h^2 + 1\right)},$$

for $x_1 = \sqrt{\frac{h^2(h\alpha-\beta)+h}{1-h^3}}$ and a small enough h.

Example 1. For $\alpha = 20$, $\beta = 2$ and $h = 0.2$, we find a neighborhood

$$U_r(0,0) = \left\{(x,y) : x^2 + y^2 < r^2\right\},$$

for $r = \min\left\{\sqrt{\frac{2\sqrt{\alpha}-\beta}{3}}, \sqrt{\frac{h\alpha-\beta}{3}}\right\} = \frac{\sqrt{6}}{3} \approx 0.816\,50$, where all eigenvalues of $J_T(0,0)$ exceed 1 in norm and

$$(x_1, y_1) = (0.5312796481290517`, -2.6563982406452586`) \notin U_r(0,0),$$

because $|\lambda_1| = 3.4463 > 1$, $|\lambda_2| = 0.81246 < 1$.
Equation (15) has three real solutions:

$$-0.03237199632171373`, 0.6022113936915484`, 1.0368530868914105`.$$

Only for $x_0 = 0.6022113936915484`$ it is true that

$$(x_0, y_0) = (0.6022113936915484`, -0.3546587278124836`) \in U_r(0,0),$$

$$T^2(x_0, y_0) = \left(0., -8.881784197001252` \cdot 10^{-16}\right) \approx (0,0),$$

$$|J_T(x_1, y_1)| \approx 1.368\,1 \neq 0$$

and

$$\left|\lambda_{1,2}\right| = |0.76227 \pm 0.88713i| = 1.1696 > 1.$$

Then $(x_0, y_0) = (0.6022113936915484`, -0.3546587278124836`)$ is a snap-back repeller and in this case, system (3) is chaotic in the sense of Li-Yorke.

5. Numerical simulations

To justify the theoretical analysis for system (3), we present the phase portraits for $\alpha = 10$, $\beta = 0.1$, and initial conditions $(10^{-3}, 10^{-3})$ by choosing a different value of parameter h ($h = 0.001$, $h = 0.1$, and $h = 0.01$). Based on Proposition 1, when $\alpha < \frac{\beta}{h}$, $\alpha > \frac{\beta}{h}$ and $\alpha = \frac{\beta}{h}$, the fixed point $(0, 0)$ is a sink, source, and non-hyperbolic (NH), respectively. See Theorem 1 (i), for $\alpha = 10 > \sqrt[3]{4}$, $\beta = 0.1 \in \left(0, 2\sqrt{\alpha}\right)$, $h_0 = \frac{\beta}{\alpha} = 0.01$ and $a\left(h_0\right) = \frac{3\beta^3\left(\alpha^2-\beta\right)}{8\alpha^4} = \frac{2997}{800\,000\,000} > 0$.

For illustration see:

- Figure 4: 10000 and 100000 iterations for $(x_0, y_0) = (0.001, 0.001)$ and $h < h_0$;
- Figure 5: 1400 iterations for $(x_0, y_0) = (0.001, 0.001)$ and 1500 iterations for $(x_0, y_0) = (0.8, 1.1)$ for $h > h_0$;
- Figure 6: 350 iterations for $(x_0, y_0) = (0.001, 0.001)$ and ellipse $a^2x^2 + b^2y^2 = a^2b^2$ with $a = 0.0033232476573834087$ and $b = 0.0010486001027132572$ for $h = h_0$.

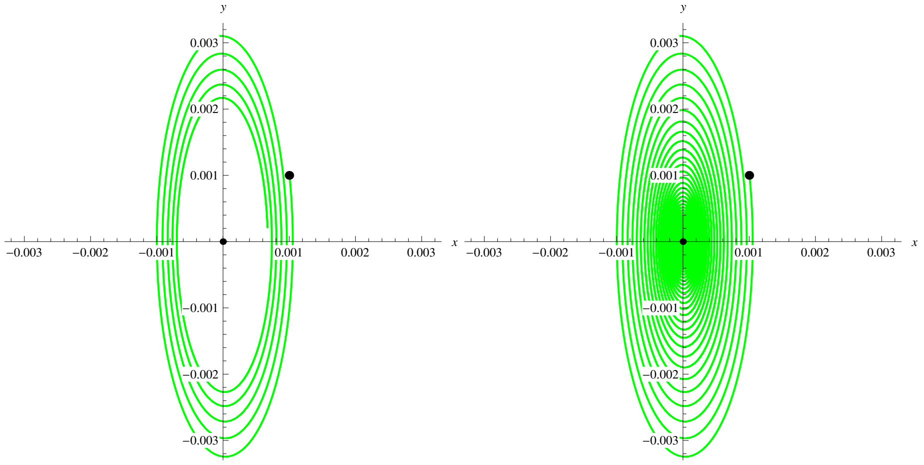

Figure 4: The phase portrait for $\alpha = 10 < \frac{\beta}{h} = \frac{0.1}{0.001} = 100$, (a sink for $h = 0.001$) for $h < h_0$

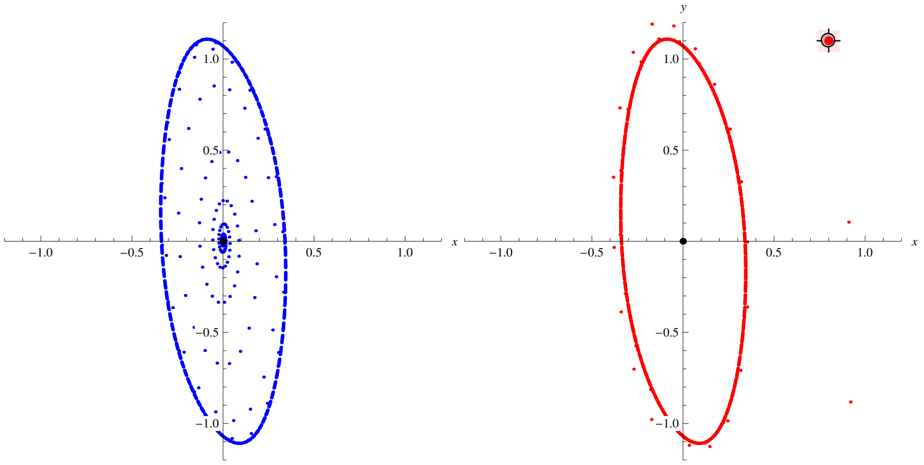

Figure 5: The phase portrait for $\alpha = 10 > \frac{\beta}{h} = \frac{0.1}{0.1} = 1$, (a source for $h = 0.1$) for $h > h_0$

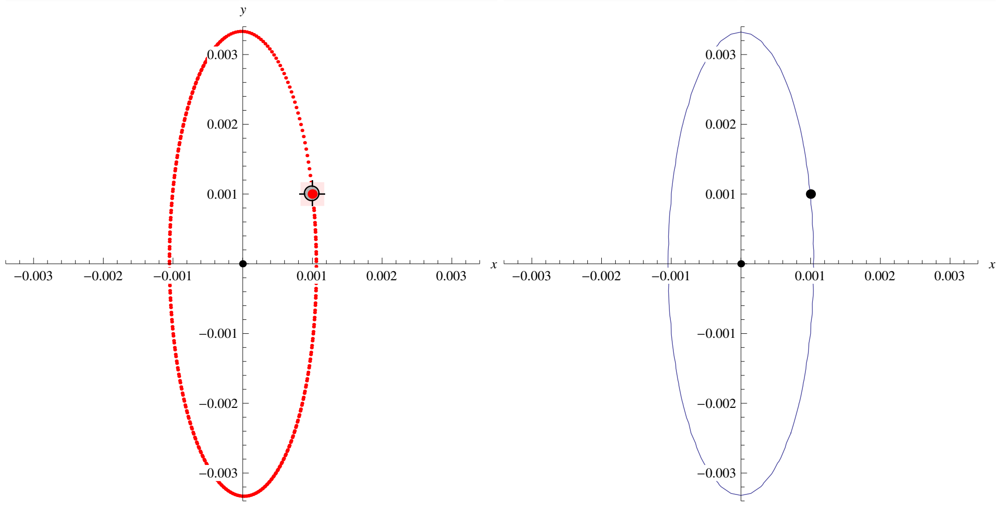

Figure 6: The phase portrait for $\alpha = \frac{\beta}{h} = \frac{0.1}{0.01} = 10$, (NH) and ellipse $a^2x^2 + b^2y^2 = a^2b^2$ with $a = 0.0033232476573834087$ and $b = 0.0010486001027132572$ for $h = h_0$

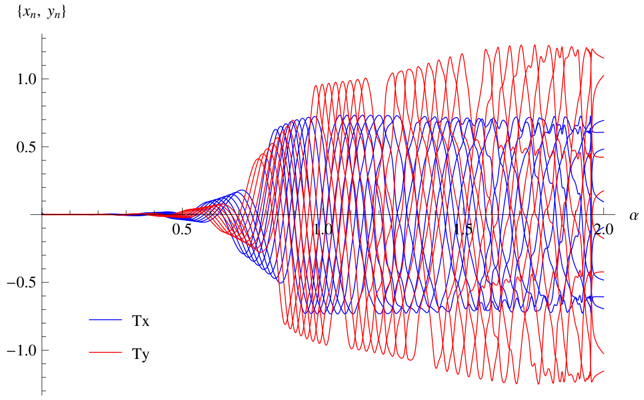

Figure 7: Bifurcation diagram generated by Code Bif2D [27]

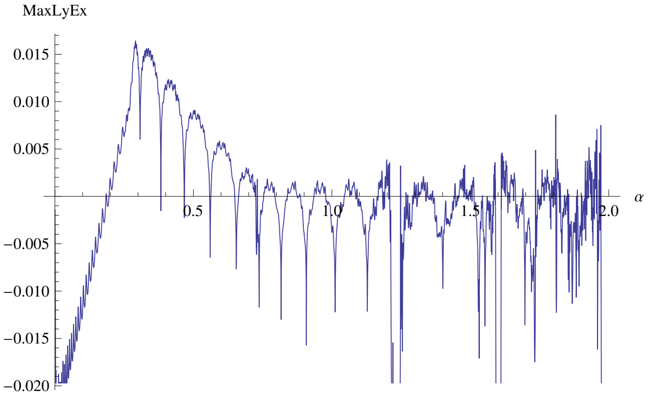

Figure 8: The Lyapunov exponent

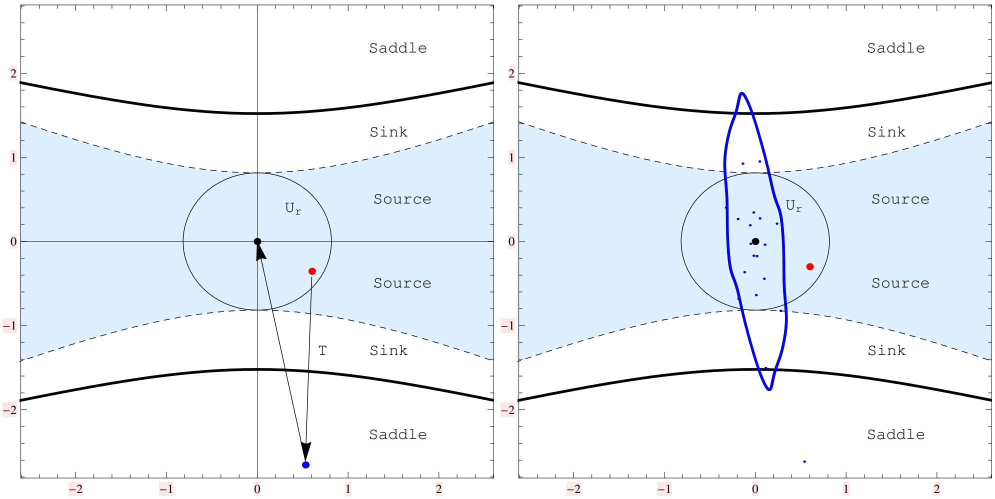

Figure 9: The snap-back repeller and chaotic attractor

Equation (15) has at least one real solution (it can have three real solutions). If there is such x_0 that $(x_0, y_0) \in U_r(0,0)$, $|J_T(x_1, y_1)| \neq 0$ and $(x_1, y_1) \notin U_r(0,0)$, then (x_0, y_0) is a snap-back repeller. In Figure 7, we plot the bifurcation diagram of system 3 in (α, x), (α, y) plane for $h = 0.5$, $\beta = 0.1$, and initial conditions $(10^{-3}, 10^{-3})$ are generated by Code Bif2D [27].

Finally, the diagram of the Lyapunov exponent for $h = 0.5$, $\alpha < 2$, $\beta = 0.1$, and initial conditions $(10^{-3}, 10^{-3})$ are shown in Figure 8. The Lyapunov exponents (based on the computational algorithm in [10, 26]) corresponding to Figure 7 are shown in Figure 8, verifying the existence of chaos. In Section 4, we proved that system (3) has chaotic behavior in the sense of Marotto's definition. So, we guess it has chaotic behavior in the other sense. Namely, the point

$$(x_0, y_0) = (0.6022113936915484`, -0.3546587278124836`)$$

is a snap-back repeller for $\alpha = 20$, $\beta = 2$ and $h = 0.2$. In this case, system (3) is chaotic in the sense of Li-Yorke. For $(x_0, y_0) = (0.6, -0.3)$, we have a chaotic attractor (see Figure 9).

6. Discussion

The general form of the Rayleigh-Duffing oscillator,

$$\ddot{x} + a\dot{x} + b\dot{x}^3 + cx + dx^3 = 0, \tag{16}$$

was investigated by Chen et al. in [2]. This model incorporates both nonlinear damping and external periodic forcing, and its complex dynamical behavior, including bifurcations (pitchfork, Hopf, and heteroclinic), chaotic attractors, and multistability, is analyzed using both analytical and numerical techniques. The continuous system (2) is obtained from system (16) for $\dot{x} = y$, $c = \alpha$, $a = \beta$, $d = b = 1$, and it exhibits Hopf bifurcation, a homoclinic bifurcation, and a double-limit cycle bifurcation. In particular, the homoclinic bifurcation, also known as a gluing bifurcation, is notable for marking a critical point where two symmetric periodic orbits merge into a single, more complex orbit. This process leads to a sudden qualitative change in the system's dynamics, often associated with the emergence or disappearance of chaotic attractors, and plays a crucial role in shaping the system's global bifurcation structure (see [3]). By comparing the continuous system (2) with its discrete counterpart (3), we show that the fixed point at the origin $(0,0)$ undergoes both a Neimark–Sacker bifurcation and a period-doubling bifurcation. Moreover, when the origin of system (3) is a source with a pair of complex multipliers, the system exhibits chaotic behavior in the sense of Marotto's definition, an extension of Li–Yorke chaos from one-dimensional to multidimensional systems via the concept of a snap-back repeller.

Acknowledgements

The authors sincerely thank the anonymous referees for their valuable comments and suggestions, which significantly improved this paper.

References

[1] M. Bikdash, B. Balachandran, and A. Navfeh, Melnikov analysis for a ship with a general roll-damping model, *Nonlinear Dyn.* **6** (1994), no. 1, 101–124.

[2] H. Chen, D. Huang, and Y. Jian, The saddle case of Rayleigh-Duffing oscillators, *Nonlinear Dyn.* **93** (2018), no. 4, 2283–2300.

[3] H. Chen and L. Zou, Global study of Rayleigh-Duffing oscillators, *J. Phys. A, Math. Theor.* **49** (2016), no. 16, 25, Id/No 165202.

[4] E. Freire, E. Gamero, and A. J. Rondríguez-Luis, Study of a degenerate Bogdanov-Takens bifurcation in a family of mechanical oscillators, *Mech. Res. Commun.* **25** (1998), no. 3, 287–297.

[5] M. Garić-Demirović, S. Hrustić, S. Moranjkić, M. Nurkanović, and Z. Nurkanović, The existence of Li-Yorke chaos in certain predator-prey system of difference equations, *Sarajevo J. Math.* **18** (2022), no. 1, 45–62.

[6] M. Garić-Demirović, M. R. S. Kulenović, M. Nurkanović, and Z. Nurkanović, The existence of Li–Yorke chaos in a discrete-time glycolytic oscillator model, *Axioms* **13** (2024), no. 4, 1–17.

[7] M. Garić-Demirović, M. Nurkanović, and Z. Nurkanović, Stability, periodicity and Neimark-Sacker bifurcation of certain homogeneous fractional difference equations, *Int. J. Difference Equ.* **12** (2017), no. 1, 27–53.

[8] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, *Appl. Math. Sci.* **42**, Springer, Cham, 1983.

[9] J. Hale, H. Buttanri, and H. Kocak, *Dynamics and Bifurcations*, *Texts in Applied Mathematics*, Springer, New York, 2012. Available at `https://books.google.ba/books?id=xWLdBwAAQBAJ`.

[10] J. He, S. Yu, and J. Cai, Numerical analysis and improved algorithms for Lyapunov-exponent calculation of discrete-time chaotic systems, *Int. J. Bifurcation Chaos Appl. Sci. Eng.* **26** (2016), no. 13, 11, Id/No 1650219.

[11] W. Herfort and H. Troger, Robust modelling of flow induced oscillations of bluff bodies, *Math. Modelling* **8** (1987), 251–255.

[12] S. Hrustić, S. Moranjkić, and Z. Nurkanović, Local stability, period-doubling and 1:2 resonance bifurcation for a discretized chemical reaction system, *MATCH Commun. Math. Comput. Chem.* **93** (2025), no. 2, 349–378.

[13] Y. Kanai and H. Yabuno, Creation-annihilation process of limit cycles in the Rayleigh-Duffing oscillator, *Nonlinear Dyn.* **70** (2012), no. 2, 1007–1016.

[14] M. R. S. Kulenović and O. Merino, *Discrete Dynamical Systems and Difference Equations with Mathematica*, Chapman & Hall/CRC, Boca Raton, FL, 2002.

[15] M. R. S. Kulenović, S. Moranjkić, M. Nurkanović, and Z. Nurkanović, Global asymptotic stability and Naimark-Sacker bifurcation of certain mix monotone difference equation, *Discrete Dyn. Nat. Soc.* **2018** (2018), 1–22, Id/No 7052935.

[16] M. R. S. Kulenović, S. Moranjkić, and Z. Nurkanović, Global dynamics and bifurcation of a perturbed sigmoid Beverton-Holt difference equation, *Math. Methods Appl. Sci.* **39** (2016), no. 10, 2696–2715.

[17] M. R. S. Kulenović, S. Moranjkić, and Z. Nurkanović, Naimark-Sacker bifurcation of second order rational difference equation with quadratic terms, *J. Nonlinear Sci. Appl.* **10** (2017), no. 7, 3477–3489.

[18] Y. A. Kuznetsov, *Elements of Applied Bifurcation Theory*, 3rd ed., *Appl. Math. Sci.* **112**, Springer, New York, 2004.

[19] X. Liu and D. Xiao, Complex dynamic behaviors of a discrete-time predator–prey system, *Chaos, Solitons & Fractals* **32** (2007), no. 1, 80–94.

[20] A. C. J. Luo, *Discretization and Implicit Mapping Dynamics*, *Nonlinear Phys. Sci.*, Higher Education Press, Beijing; Springer, Berlin, 2015.

[21] F. Marotto, On redefining a snap-back repeller, *Chaos, Solitons & Fractals* **25** (2005), no. 1, 25–28.

[22] F. R. Marotto, Snap-back repellers imply chaos in rn, *J. Math. Anal. Appl.* **63** (1978), no. 1, 199–223.

[23] A. H. Nayfeh and D. T. Mook, *Nonlinear Oscillations*, John Wiley & Sons, New York, 1979.

[24] Z. Nurkanović, M. Nurkanović, and M. Garić-Demirović, Stability and Neimark-Sacker bifurcation of certain mixed monotone rational second-order difference equation, *Qual. Theory Dyn. Syst.* **20** (2021), no. 3, 41, Id/No 75.

[25] C. Robinson, *Dynamical Systems. Stability, Symbolic Dynamics, and Chaos*, CRC Press, Boca Raton, FL, 1995.

[26] M. Sandri, Numerical calculation of Lyapunov exponents, *Math.Physica A Stat. Mech. Appl.Phys J.* **6** (1996), no. 3, 78–84.

[27] U. Ufuktepe and S. Kapçak, Applications of discrete dynamical systems with mathematica, *Kurenai* **1909** (2014), 207–216.

[28] R. Yamapi, Synchronization dynamics in a ring of four mutually inertia coupled self-sustained electrical systems, *Physica A Stat. Mech. Appl.* **366** (2006), 187–196.