Math. Commun. **31** (2026), 17–26
DOI: 10.64785/mc.31.1.3

Mathematical Communications

Existence of three solutions to a $p(\cdot)$-biharmonic problem via a local mountain pass theorem

Ghasem Afrouzi[1], John R. Graef[2,*], and A. R. Jalali[1]

[1] Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran
[2] University of Tennessee at Chattanooga, Chattanooga, 37 403 TN, USA

Abstract. The authors consider the problem of the existence of multiple weak solutions to $p(x)$-biharmonic equations with Navier boundary conditions. Using Ricceri's variational principle and a local mountain pass theorem, and without requiring the Palais-Smale condition, the authors establish sufficient conditions for the existence of at least three solutions to the problem.

AMS subject classifications: 35J60, 35J48
Keywords: p(x)-biharmonic; Neumann problem; embedding theorem; variational methods
Received November 14, 2024; accepted July 29, 2025

1. Introduction

In this work, we are interested in the existence of at least three weak solutions to the Navier boundary value problem

$$\begin{cases} \Delta^2_{p(x)} u = \lambda t(x, u(x)) + \mu k(x, u(x)), & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1}$$

The setting for the problem is as follows:

* $\lambda, \mu \geq 0$, and Ω is a bounded open domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$;
* $p \in C(\bar{\Omega})$ with $p(x) > 1$, $p^- > 1$, and $p^+ < \infty$, where $p^- := \inf_{x\in\Omega} p(x)$ and $p^+ := \sup_{x\in\Omega} p(x)$;
* the functions t and k belong to $C(\bar{\Omega} \times \mathbb{R})$.

Here, $\Delta^2_{p(x)} u := \Delta(|\Delta|^{p(x)-2} \Delta u)$ is the fourth order operator known as the $p(x)$-biharmonic operator.

In the last two decades, the literature on various mathematical problems with variable exponents has been increasing, especially the study of elastic mechanics, electro-rheological fluids, image processing, micro electro-mechanical systems, surface diffusion on solids, and flows in Hele-Shaw cells. One reason for such an increase in interest is the fact that different areas of applied mathematics and physical phenomena can be modeled by such equations. For example, applications in nonlinear elasticity and electro-rheological fluids can be found in [1, 5, 14, 21, 23, 34, 38]. Many authors have considered differential equations with variable exponents (see [9, 33, 7]) and have pointed out applications involving the $p(x)$-biharmonic operator. This operator allows growth conditions that involve more complicated nonlinearities than the constant order case. We refer the reader to [3, 2, 5, 10, 12, 22, 24, 26, 27, 29, 30, 32, 31, 7, 6, 39, 40, 41] and the references therein for details. Some classical tools such as the three critical points theorem of Ricceri [36, 37] have been used in solving such problems. Additional background on variational approaches to obtaining multiple solutions to boundary value problems can be found in the monograph by Graef and Kong [20].

Kong [29] used Ekeland's variational principle and some recent results on the generalized Lebesgue-Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{h,p(x)}(\Omega)$ to prove the existence of solutions to

$$\begin{cases} \Delta^2_{p(x)} u + a(x)|u|^{p(x)-2}u = \lambda w(x) f(u), & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega. \end{cases}$$

Email addresses: afrouzi@umz.ac.ir (G. Afrouzi), john-graef@utc.edu, johngraef9@gmail.com (J. R. Graef), jalali.atieh@yahoo.com (A. R. Jalali)
*Corresponding author.

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A class of quasilinear elliptic equations involving the $p(x)$-biharmonic operator with Navier boundary conditions was recently analyzed by Yin and Xu in [40]. Deng [13] applied a local mountain pass theorem, without the Palais-Smale condition, and Ricceri's variational principle to obtain the existence of multiple solutions to the $p(x)$-Laplacian doubly perturbed Neumann problem

$$\begin{cases} \Delta^2_{p(x)} u + a(x)|u|^{p(x)-2}u = f(x,u) + \lambda h_1(x,u) & \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial y} = g(x,u) + \mu h_2(x,u), & \text{on } \partial\Omega. \end{cases}$$

In the present paper, we discuss the existence of at least three weak solutions to problem (1). We first review some basic facts on variable exponent Lebesgue and Sobolev spaces and then describe Ricceri's variational principle (see Theorem 2 below). We will also make use of a local mountain pass theorem, without requiring the Palais Smale condition (Proposition 6), to prove our main results.

2. Preliminaries

We first review some definitions and facts about variable exponent Lebesgue and Sobolev spaces; detailed descriptions can be found in [15, 16, 19]. Set

$$C_+(\Omega) := \{r : r \in C(\bar{\Omega}) \quad \text{and} \quad r(x) > 1 \text{ for all } x \in \bar{\Omega}\}.$$

Define the Lebesgue space with variable exponent $p(\cdot) \in C(\bar{\Omega})$ by

$$L^{p(\cdot)}(\Omega) := \left\{u : \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega |u(x)|^{p(x)}\, dx < \infty\right\}.$$

This is a separable and reflexive Banach space if endowed with the Luxemburg norm

$$\| u \|_{L^{p(\cdot)}} = \inf\left\{\gamma > 0 : \int_\Omega \left|\frac{u(x)}{\gamma}\right|^{p(x)} dx \le 1\right\}.$$

Next, we recall the mapping $\rho : L^{p(x)} \to \mathbb{R}$ called the modular function for the space $L^{p(x)}(\Omega)$, as defined by

$$\rho(u) = \int_\Omega | u(x) |^{p(x)}\, dx.$$

The relationship between the Luxemburg norm and the modular function is given in the following proposition.

Proposition 1. ([18, 35]) *If $u \in L^{p(x)}$, then the following relationships hold:*

(1) $\| u \|_{L^{p(\cdot)}(\Omega)} < 1\ (= 1; > 1)$ *if and only if* $\rho(u) < 1\ (= 1; > 1)$;

(2) $\| u \|_{L^{p(\cdot)}(\Omega)} > 1$ *implies* $\| u \|^{p^-}_{L^{p(\cdot)}(\Omega)} \le \rho(u) \le \| u \|^{p^+}_{L^{p(\cdot)}(\Omega)}$;

(3) $\| u \|_{L^{p(\cdot)}(\Omega)} < 1$ *implies* $\| u \|^{p^+}_{L^{p(\cdot)}(\Omega)} \le \rho(u) \le \| u \|^{p^-}_{L^{p(\cdot)}(\Omega)}$;

(4) $\| u \|_{L^{p(\cdot)}} \to 0\ (\to \infty)$ *if and only if* $\rho(u) \to 0\ (\to \infty)$;

(5) *if* $(u_n)_n \subset L^{p(\cdot)}(\Omega)$, *then* $\lim_{n\to\infty} \|u_n - u\|_{L^{p(\cdot)}(\Omega)} = 0$ *if and only if* $\lim_{n\to\infty} \rho(u_n - u) = 0$ *if and only if* $(u_n)_n$ *converges to* u *in measure and* $\lim_{n\to\infty} \rho(u_n) = \rho(u)$.

From [19, 28], for any positive integer h, the Sobolev space with variable exponent $W^{h,p(x)}(\Omega)$ is defined as

$$W^{h,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : D^\eta u \in L^{p(x)}(\Omega),\ | \eta | \le h\},$$

equipped with the norm

$$\| u \|_{W^{h,p(x)}(\Omega)} = \sum_{|\eta| \le h} \| D^\eta u \|_{L^{p(\cdot)}(\Omega)},$$

where $D^\eta u = \frac{\partial^{|\eta|}}{\partial x_1^{\eta_1}\partial x_2^{\eta_2}...\partial x_N^{\eta_N}}u$ and $\eta = (\eta_1, ..., \eta_N)$ is a multi-index such that $|\eta| = \sum_{i=1}^N \eta_i$. We note that $W^{h,p(x)}(\Omega)$ is a separable, reflexive, and uniformly convex Banach space. Let $q(x)$ be the conjugate exponent of $p(x)$, i.e., $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$. Then the Hölder type inequality

$$\int_\Omega |uv|dx \leq \left(\frac{1}{p^-} + \frac{1}{q^-}\right) |u|_{p(x)} \, |v|_{q(x)}, \quad u \in L^{p(x)}(\Omega) \text{ and } v \in L^{q(x)}(\Omega),$$

holds.

Proposition 2. ([19, Theorem 2.3]) *Let $q \in C(\bar{\Omega}; \mathbb{R})$ satisfy $1 < q^- \leq q^+ < \infty$ and $q(x) < p_h^*(x)$ for all $x \in \bar{\Omega}$, where*

$$p_h^*(x) = \begin{cases} \frac{Np(x)}{N-hp(x)}, & \text{if } hp(x) < N, \\ +\infty, & \text{if } hp(x) \geq N, \end{cases}$$

for any $x \in \bar{\Omega}$ and $h \geq 1$. Then there is a continuous and compact embedding $W^{h,p(\cdot)}(\Omega)$ into $L^{q(\cdot)}(\Omega)$.

We denote by $W_0^{h,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{h,p(x)}(\Omega)$. We next recall some properties of the spaces $W^{2,p(\cdot)}(\Omega)$, $W_0^{1,p(\cdot)}(\Omega)$, and $W^{2,p(\cdot)}(\Omega) \cap W_0^{1,p(\cdot)}(\Omega)$.

Remark 1. ([8])

(a) For $u \in X = W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$, we define $\| \cdot \|_X$ by

$$\| u \|_X = \| u \|_{W_0^{1,p(x)}} + \| u \|_{W^{2,p(x)}} .$$

Note that X is a separable and reflexive Banach space. In [42], Zanga and Fu proved that the norms $\| \Delta u \|_{L^{p(\cdot)}}$ and $\| u \|_X$ are equivalent.

(b) The space $W^{2,p(x)}(\Omega)$ or $W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$ is equipped with the norm

$$\| u \|_a = \inf \left\{ n > 0 : \int_\Omega \left[\left|\frac{\Delta u}{n}\right|^{p(x)} + a(x)\left|\frac{u}{n}\right|^{p(x)} \right] dx \leq 1, \; a \in L^\infty \right\}.$$

This norm is equivalent to the norm $\| \Delta u \|_{L^{p(\cdot)}}$.

In [8, 42], the subspace $S = \{u \in W^{2,p(\cdot)}(\Omega) : u|_{\partial\Omega} \equiv constant\}$ is analyzed and can be considered as $\{S = u + b : u \in W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega), b \in \mathbb{R}\}$. Moreover, $(S, \| u \|_{W^{2,p(x)}})$ is a separable and reflexive Banach space and the norms $\| u \|_{W^{2,p(\cdot)}(\Omega)}$, $\| u \|_a$, and $\| \Delta u \|_{L^{p(x)}}$ are equivalent.

Throughout the remainder of this paper, for convenience we use $\| u \|$ instead of $\| u \|_{W^{2,p(\cdot)}(\Omega)}$ on X.

Next, we present some concepts and results needed to prove our main theorems in this paper.

Proposition 3. ([25, 35]) *If $u \in L^{p(\cdot)}(\Omega)$, then the following relations hold:*

(i) $\| u \|^{p^-} \leq \int_\Omega | \Delta u |^{p(x)} \, dx \leq \|u\|^{p^+}, \quad$ *if* $\|u\| \geq 1$.

(ii) $\| u \|^{p^+} \leq \int_\Omega | \Delta u |^{p(x)} \, dx \leq \|u\|^{p^-}, \quad$ *if* $\|u\| \leq 1$.

Here we are studying the existence of weak solutions to problem (1), so we next define what that means.

Definition 1. We say that $u \in S$ is a weak solution to (1) if for all $v \in S$,

$$\int_\Omega | \Delta u |^{p(x)-2} \Delta u \Delta v dx - \lambda \int_\Omega t(x, u(x))v(x)dx - \mu \int_\Omega k(x, u(x))v(x)dx = 0.$$

To be able to find weak solutions to (1), we consider the functional $I : S \longrightarrow \mathbb{R}$ defined by

$$I(u) = I_1 + \mu I_2,$$

where

$$I_1 = \int_\Omega \frac{1}{p(x)} | \Delta u |^{p(x)} \, dx - \lambda \int_\Omega T(x, u(x))dx, \quad I_2 = -\int_\Omega K(x, u(x))dx,$$

$$T(x, c) = \int_0^c t(x, s)ds \quad \text{and} \quad K(x, c) = \int_0^c k(x, s)ds.$$

Thus, weak solutions to problem (1) are exactly the critical points of the functional I.

3. Ricceri's variational principle

We will use a somewhat standard notation that $\rightharpoonup$ and $\to$ denote weak and strong convergence, respectively.

Definition 2. Let D be a bounded open subset of X. We say that D is a Ricceri block of I_1 of type κ if

$$\begin{cases} I_1(x) < \kappa, & x \in D, \\ I_1(x) = \kappa, & x \in \overline{\partial D} = \overline{D}^w \setminus D, \end{cases}$$

where D^w means in the weak topology on D. If $c < b$, D is said to be a Ricceri box of I_1 of type (c, b) if $c = \inf_D I_1 < \inf_{\overline{\partial D}} I_1 = b$.

Definition 3. Let D and D_0 be two bounded open subsets of the Banach space X. We say that (D_0, D) is a valley box of $\phi : X \to \mathbb{R}$ if $\overline{D_0} \subset D$ and

$$\sup_{D_0} \phi < \inf_{\partial D} \phi.$$

Proposition 4. ([4]) *Let $L : X \to \mathbb{R}$ be defined by*

$$L(u) = \int_\Omega \frac{1}{p(x)}(|\ \Delta x\ |^{p(x)} + a(x)\ |\ u\ |^{p(x)})dx.$$

Then:

(i) $L \in C^1(X, \mathbb{R})$ and L is a convex functional that is sequentially weakly lower semi-continuous.

(ii) The derivative operator $L' : X \to X^*$ of L is a bounded and strictly monotonic homeomorphism. Here, X^* is the dual space.

(iii) L' is a mapping of type (S_+), i.e., if $u_n \rightharpoonup u$ and $\limsup_{n\to\infty} L'(u_n)(u_n - u) \le 0$, then $u_n \to u$ $(strongly)$.

Theorem 1. ([11]) *Let X be a reflexive Banach space and let the functional $\varphi : X \to \mathbb{R}$ be coercive and sequentially weakly lower semicontinuous. Then φ is bounded from below and $\inf_{x\in X} \varphi(x) \in X$.*

Remark 2. ([17]) If $x_* \in X$ is a strictly local minimizer of I_1, then for $\theta > 0$ sufficiently small, we have $\inf_{\partial B(x_*,\theta)} I_1 > I_1(x_*)$. Moreover, the open ball $B(x_*, \theta) = \{x \in X : \|x - x_*\| \le \theta\}$ is a Ricceri box of I_1.

Remark 3. ([17]) For I_1 and I_2, the following assertions hold:

(i) If D is a Ricceri box of I_1 of type (c, b), then for each $\kappa \in (c, b]$, $I_1^{-1}(-\infty, \kappa) \bigcap D$ is a Ricceri block of I_1 of type κ.

(ii) I_1 and I_2 are sequentially weakly lower semi continuous, that is, for any $x \in X$ and any subsequence $x_n \subset X$ such that $x_n \rightharpoonup x$ weakly, we have $I_i(x) \le \liminf_{n\to\infty} I_i(x_n)$ for $i = 1, 2$.

(iii) The mapping I_2' is weakly-strongly continuous, i.e., if $x_n \rightharpoonup x$, then $I_2'(x_n) \to I_2'(x)$.

(iv) The sum of a type (S_+) mapping and a weakly-strongly continuous mappings is also of the type (S_+).

Theorem 2. ([17, Theorem 3.2]) *Assume that $\mu_* := \sup_{x\in D} \frac{\kappa - I_1(x)}{I_2(x) - \inf_{\overline{D}^w} I_2(x)}$, where D is a Ricceri block of I_1 of type κ. Then:*

(i) For each $\mu \in (0, \mu_*)$, the restriction of $I_1 + \mu I_2$ to $\overline{D}^w$ attains its infimum at some $x_* \in D$, so that x_* is a local minimizer of $I_1 + \mu I_2$.

(ii) D is a Ricceri box of $I_1 + \mu I_2$.

Proposition 5. ([13]) *Assume that:*

(i) For $r > 0$ and $x_1 \in B(x_0, r)$, we have $I_1(x_0) = \inf_{B(x_0,r)} I_1 = c_0$ and $\inf_{\partial B(x_0,r)} I_1 = b > c_0$;

(ii) *x_1 is a strictly local minimizer of I_1 and $I_1(x_1) = c_1 > c_0$.*

Then for $\theta > 0$ sufficient small, $\kappa_1 > c_1$, $\kappa_0 \in (c_0, \min\{b, c_1\})$, and for all $\mu \in (0, \mu_*)$, $I_1 + \mu I_2$ has at least two local minima x_0^* and $x_1^* \in B(x_0, r)$, where $x_0^* \in I_1^{-1}((-\infty, \kappa_0)) \bigcap B(x_0, r)$, $x_0^* \notin \overline{B(x_1, \theta)}$, and $x_1^* \in I_1^{-1}((-\infty, \kappa_1)) \bigcap B(x_1, \theta)$.

Theorem 3. ([13]) *Let Y be a reflexive Banach space and assume that:*

(i) *For $\phi \in C^1(Y, \mathbb{R})$, the mapping $\phi' : Y \to Y'$ is of type (S_+).*

(ii) *(D_0, D) is a valley box of ϕ with D_0 and D being connected and $0 \in D_0$.*

(iii) *There exist $e \in D_0$ and $r > 0$ such that $\|e\| > r$ and $\inf_{\partial B(0,r)} \phi > \{\max \phi(0), \phi(e)\}$.*

Then, the functional ϕ has at least one critical point $x_0 \in \overline{D}$ with $\phi(x_0) = d$, where $d = \inf_{\beta\in\Gamma} \sup_{s\in[0,1]} \phi(\beta(s))$ and $\Gamma = \{\beta \in C([0, 1], D) : \beta(0) = 0,\ \beta(1) = e\}$.

Proposition 6. ([4, Corollary 2.1]) *If $I_2' : Y \to Y'$ is weakly-strongly continuous, and assumptions (i)–(iii) of Theorem 3 hold, then there exist $\mu \in (0, \mu_*)$ such that $I_1 + \mu I_2$ has a mountain pass type critical point $x_2 \in \overline{D}$.*

4. Main results

The following assumptions will be needed in our main theorems:

(a) $\lim_{|z|\to\infty} \frac{t(x,z)}{|t|^{p(x)-1}} = 0$ uniformly for $x \in \Omega$.

(b) $p^+ < p^*(x)$ and there exist $b_0 > 0$ and $\delta > 0$ such that

$$| T(x, z) | \leq b_0 \mid z \mid^{q_0(x)} \quad \text{for all} \quad x \in \Omega \quad \text{and} \quad |z| < \delta.$$

(c) Let D be a ball with $\overline{D} \subset \Omega$. For any given *Carathéodory* function $t : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$, there exists $\gamma_0 > 0$ such that

$$\int_D T(x, \gamma_0) dx > 0.$$

(d) There exist $\beta(x) \in C(\overline{\Omega})$ with $1 \leq \beta(x) \leq \beta^+ \leq p^-$, $b > 0$, and $\gamma > 0$ such that

$$K(x, z) \geq b \mid z \mid^{\beta^+} \quad \text{for all} \quad x \in \overline{\Omega} \quad \text{and} \quad |z| < \gamma.$$

(e) $\lim_{|z|\to\infty} \frac{T(x,z)}{|z|^{p^+}} = +\infty$ uniformly for $x \in \Omega$.

(f) $\limsup_{z\to 0} \frac{\inf_{x\in\Omega} K(x,z)}{|z|^{p^-}} = +\infty$.

(k) There exists $\xi \in \mathbb{R}$ such that $T(x, \xi) > \int_\Omega \frac{|\xi|^{p(x)} + |\Delta\xi|^{p(x)}}{p^-} dx$.

Our first existence result in this section is contained in the following theorem.

Theorem 4. *Assume that conditions (a)–(e) hold. Then there exists a constant $\lambda_0 > 0$ such that, for all $\lambda_0 < \lambda$ and $\mu \in (0, \mu_*)$, problem (1) has at least three nontrivial weak solutions.*

Proof. The proof will be divided into five steps.

Step 1: *$w_0 = 0$ is a strictly local minimizer of I_1.* From (a), we can conclude that for every $\epsilon > 0$ there is a $\delta_\epsilon > 0$ such that

$$| t(x, z) | \leq \epsilon \mid z \mid^{p(x)-1} \quad \text{for all} \quad |z| > \delta_\epsilon \quad \text{and} \quad x \in \Omega.$$

Then, from the continuity of t, there exists $c_0 > 0$ such that

$$|t(x,z)| \leq c_0 + \epsilon |z|^{p(x)-1} \tag{2}$$

for all $z \in \mathbb{R}$ and all $x \in \Omega$. From (2) and (b), it follows that there exists $q_1 \in C(\bar{\Omega})$ with $p^+ < q_1^- \leq q_1(x) \leq p^*(x)$ such that

$$|\, T(x,z) \,| \leq c_1 \,|\, z \,|^{q_1(x)} \quad \text{for all} \quad x \in \Omega \quad \text{and} \quad z \in \mathbb{R}.$$

Since $\| \, u \, \| < 1$, we see that

$$I_1(u) \geq \frac{1}{p^+} \int_\Omega |\, \Delta u \,|^{p(x)} \, dx - \lambda c_1 \int_\Omega |\, u \,|^{q_1(x)} \, dx \geq \frac{1}{p^+} \| \, u \, \|^{p^+} - \lambda c_1 \| \, u \, \|^{q_1^-} .$$

Since $q_1^- > p^+$, there exists $\theta > 0$ such that

$$I_1(u) > 0 \quad \text{for all} \quad u \in \overline{B(0,\theta)} \setminus 0.$$

Step 2: *I_1 has a global minimizer $w_1 \neq 0$.* By (2) and the definition of I_1,

$$I_1(u) \geq \frac{1}{p^+} \int_\Omega [|\, \Delta u \,|^{p(x)} - \lambda \epsilon \,|\, u \,|^{p(x)}] dx - \lambda c_0 \| \, u \, \|_{L^1} .$$

Choosing $\epsilon < \frac{1}{\lambda}$, by Proposition 2, for $\| \, u \, \| \geq 1$ and $1 < p_2^*(x)$, we have

$$\begin{aligned} I_1(u) &\geq \frac{1}{p^+} \int_\Omega (|\, \Delta u \,|^{p(x)} \, dx - \lambda \epsilon \left(\int_\Omega (|\, \Delta u \,|^{p(x)} + |\, u \,|^{p(x)}) dx \right) - \lambda c_0 \| \, u \, \|_{L^1} \\ &\geq \frac{1}{p^+} (1 - \lambda \epsilon) \| \, u \, \|_a^{p^-} - \lambda c_0 \| \, u \, \|_a . \end{aligned}$$

This shows that I_1 is coercive and has a global minimizer w_1.

To show that I_1 is weakly lower semicontinuous, let $u_n \rightharpoonup u$ in S. Since S is a closed subspace of $W^{2,p(\cdot)}(\Omega)$ (see [8]), the compact embedding obtained in Proposition 2 implies

$$u_n \to u \text{ in } L^{p(\cdot)}(\Omega) \quad \text{and} \quad u_n \to u \text{ in } L^1(\Omega). \tag{3}$$

By the Mean Value Theorem, a straightforward computation shows that

$$\begin{aligned} \left| \int_\Omega T(x,u_n(x)) dx - \int_\Omega T(x,u(x)) dx \right| &\leq \int_\Omega |\, T(x,u_n(x)) - T(x,u(x)) \,| \, dx \\ &\leq \int_\Omega \sup_\Omega t(x,v(x)) \,|\, u_n - u \,| \, dx. \end{aligned}$$

Therefore, by (2), (3), and Proposition 4, the functional I_1 is weakly lower semicontinuous. Thus, the hypotheses of Theorem 1 hold, and I_1 has a infimum $w_1 \in S$.
By condition (c), for all $\theta > 0$ sufficiently small, we can take

$$\overline{B_\theta} := \overline{\{x \in \Omega : \text{dist}(x,B) \leq \theta\}} \subset \Omega.$$

Define the function

$$u_\theta(x) := \begin{cases} t_0, & x \in B, \\ 0, & x \in \Omega \backslash B_\theta. \end{cases}$$

Then,

$$I_1(w_1) \leq \int_\Omega \frac{1}{p(x)} \,|\, \Delta u_\theta \,|^{p(x)} \, dx - \lambda \int_B T(x,t_0) dx - \lambda \int_{B_\theta \backslash B} T(x,u_\theta) dx.$$

Hence, take θ_0 sufficiently small so that there is a positive constant α_0 such that

$$I_1(w_1) \leq \int_\Omega \frac{1}{p(x)} \,|\, \Delta u_{\theta_0} \,|^{p(x)} \, dx - \lambda \alpha_0 \int_B T(x,t_0) dx,$$

and set

$$\lambda_0 := \frac{\int_\Omega \frac{1}{p(x)} \mid \Delta u_{\theta_0} \mid^{p(x)} dx}{\alpha_0 \int_B T(x, t_0)} > 0.$$

Then, $I_1(w_1) < 0$ for all $\lambda_0 < \lambda$. Therefore, $w_1 \neq 0$.

Step 3: *$I = I_1 + \mu I_2$ has two local minima.* Since I_1 is coercive, we can find $r > 0$ large enough such that $w_0, w_1 \in B(0, r)$ and $\inf_{\partial B(0,r)} I_1 > I_1(w_0) > I_1(w_1)$. By Proposition 5, given any $\theta > 0$, $\kappa_1 \in (I_1(w_1), 0)$, and $\kappa_2 > 0$, we see that for all $\mu \in (0, \mu_*)$, I has at least two local minima $u_0 \in B(0, \theta) \bigcap I_1^{-1}((-\infty, \kappa_2))$ and $u_1 \in I_1^{-1}((-\infty, \kappa_1))$ with $u_1 \notin \overline{B(0, \theta)}$.

Step 4: *$I = I_1 + \mu I_2$ has a mountain pass type critical point.* Take a ball $B(0, r_1) \subset S$ such that $B(0, r_1) \supset I_1^{-1}(-\infty, \kappa_1) \bigcup B(0, \theta)$. Since I_1 is coercive, we can find $r_2 > r_1$ with

$$\inf_{\partial B(0, r_2)} I_1 > \sup_{B(0, r_1)} I_1.$$

Then, $(B(0, r_1), B(0, r_2))$ is a valley box of I_1. Since $I_1(w_1) < 0$, by **Step 1**, we have that for some $\epsilon_0 > 0$ with $\epsilon_0 <\| w_1 \|$, $\inf_{\partial B(0, \epsilon_0)} I_1 > \max\{I_1(0), I_1(w_1)\} = 0$. From Proposition 6, we can conclude that for all $\mu \in (0, \mu_*)$, I admits a mountain pass critical point u_2.

Step 5: $u_0 \neq 0$. From condition (e), for all $L > 0$ there exist $c_L > 0$ such that

$$T(x, z) \geq L z^{p^+} - c_L \quad \text{for all} \quad x \in \Omega \quad \text{and} \quad z \geq 0.$$

Moreover, by (d), (e), and choosing $\alpha \in (0, 1)$, we can immediately see that

$$\begin{aligned} I_1(z\alpha) &\leq \alpha^{p^+} \int_\Omega \mid \Delta z \mid^{p(x)} dx - \lambda L \alpha^{p^+} \int_\Omega \mid z \mid^{p(x)} dx + \lambda c_L \mid \Omega \mid \\ &= \alpha^{p^+} (\int_\Omega \mid \Delta z \mid^{p(x)} dx - \lambda L \int_\Omega \mid z \mid^{p(x)} dx) + \lambda c_L \mid \Omega \mid . \end{aligned}$$

Since L is arbitrary and large enough, we can obtain that

$$I_1(z\alpha) < 0,$$

and hence,

$$z\alpha \in B(0, \theta) \bigcap I_1^{-1}((-\infty, \kappa_2)).$$

On the other hand,

$$I_2(z\alpha) \leq -b \int_\Omega \mid z\alpha \mid^{\beta^+} dx = -b\alpha^{\beta^+} \int_\Omega \mid z \mid^{\beta^+} dx < 0.$$

Therefore,

$$I_1(u_0) + \mu I_2(u_0) < I_1(z\alpha) + \mu I_2(z\alpha) < 0,$$

i.e., $u_0 \neq 0$, and thus we obtain at least three nontrivial solutions to problem (1), which proves the theorem. □

In our next theorem, we replace assumptions (d) and (e) with (f) and (k).

Theorem 5. *Assume that conditions (a)–(c), (f), and (k) hold. Then, there exists a constant $\lambda_0 > 0$ such that for $1 < \lambda_0 < \lambda$ and $\mu \in (0, \mu_*)$, problem (1) admits at least three nontrivial weak solutions.*

Proof. The proof is similar to the proof of the previous theorem. First we show that $I_1(w_1) < 0$, and by condition (f), taking $z_n \to 0$ in **Step 5**, we can show that $u_0 \neq 0$.

We may assume that $\| \xi \|< 1$; then

$$I_1(\xi) < \frac{1}{p^-} \int_\Omega \mid \Delta\xi \mid^{p(x)} dx - \frac{\lambda}{p^-} \int_\Omega \mid \xi \mid^{p(x)} dx \leq (\frac{1}{p^-} - \frac{\lambda}{p^-}) \| \xi \|^{p^-} < 0.$$

Thus,

$$I_1(w_1) < I_1(\xi) < 0.$$

On the other hand, we have that for all $\mu \in (0, \mu_*)$,

$$\alpha_n = z_n \quad \text{and} \quad \frac{\inf_{x\in\Omega} K(x, z_n)}{| z_n |^{p^-}} \longrightarrow +\infty,$$

so

$$\begin{aligned} I(\alpha_n) &\leq \lambda \int_\Omega T(x, z_n) - \mu \int_\Omega K(x, z_n) \\ &\leq \lambda c_1 \int_\Omega | z_n |^{q_1(x)} \, dx - \mu \, | z_n |^{p^-} \int_\Omega \frac{K(x, z_n)}{| z_n |^{p^-}} dx \\ &\leq |z_n|^{p_1^-} n_1 - \mu \, | z_n |^{p^-} \int_\Omega \frac{K(x, z_n)}{| z_n |^{p^-}} dx < 0. \end{aligned}$$

Thus, $\alpha_n \in B(0, \theta) \bigcap I_1^{-1}(-\infty, \kappa_2)$, and so $I(u_0) \leq I(\alpha_n) < 0$, i.e., $u_0 \neq 0$.

This completes the proof of the theorem. □

By way of examples, it is easy to see that the equations considered by Kong [29] and Deng [13] described above are special cases of the equation considered in this paper.

5. Conclusions

Here we studied the problem of the existence of multiple weak solutions to $p(x)$-biharmonic equations with Navier boundary conditions. By applying Ricceri's variational principle and a local mountain pass theorem, we gave sufficient conditions for the existence of at least three solutions to the problem. We did so without requiring the Palais-Smale condition which is often required by other authors.

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