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## Mathematical Communications

# **On the Diophantine equation** $F_n^2 + F_m^2 = 2^a$

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**Abstract.** Let  $F_n$  be the *n*-th Fibonacci number. In this paper, we show that the only nonnegative solutions (n, m, a) of the Diophantine equation  $F_n^2 + F_m^2 = 2^a$  with  $n \ge m \ge 0$  are (1, 0, 0), (2, 0, 0), (3, 0, 2), (6, 0, 6), (1, 1, 1), (2, 1, 1), (2, 2, 1), (3, 3, 3), (6, 6, 7).

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## 1. Introduction

Let  $(F_n)_{\geq 0}$  be the Fibonacci sequence given by  $F_0 = 0$ ,  $F_1 = 1$ , and  $F_{n+2} = F_{n+1} + F_n$ , for all  $n \geq 0$ . Lots of mathematicians have studied the interesting properties of the Fibonacci sequence.

Bravo and Luca [3] showed that the Diophantine equation  $F_n + F_m = 2^a$  in nonnegative integers n, m, a has finitely many solutions. Bravo and Bravo [1] extended it so that the Diophantine equation  $F_n + F_m + F_\ell = 2^a$  also has finitely many solutions. The number of Fibonacci numbers on the LHS has been studied up to five [13, 14].

The equation  $F_n \pm F_m = y^a$  has also been studied [6, 8, 9, 10, 11, 5, 15]. Recently, Bravo, Díaz, and Luca [2] have found all nonnegative solutions of  $(F_n + F_m)/(F_r + F_s) = 2^a$ . As such, various forms of the equations are being studied.

In the present paper, we investigate the Diophantine equation

$$F_n^2 + F_m^2 = 2^a$$

in nonnegative integers n, m, a. As in the previous papers, we use the Matveev theorem to show the finiteness of the solutions in §2 and the Dujella-Pethő theorem to reduce the bound for n in §3. The result is as follows.

Theorem 1. The only solutions of the Diophantine equation

$$F_n^2 + F_m^2 = 2^a$$

in nonnegative integers n, m and a with  $n \ge m$  are given by

$$(n, m, a) \in \left\{ \begin{array}{c} (1, 0, 0), (2, 0, 0), (3, 0, 2), (6, 0, 6), (1, 1, 1), \\ (2, 1, 1), (2, 2, 1), (3, 3, 3), (6, 6, 7) \end{array} \right\}.$$

## 2. Finiteness

To find a bound for *n* we apply Matveev's theorem (Theorem 2) to  $\Lambda = \Lambda_1, \Lambda_2$  in two steps.

Step 1. Compute  $\Lambda_1 = 2^a \cdot \sqrt{5}^2 \cdot \alpha^{-2n}$  to obtain  $(n - m) \log \alpha < (\text{an expression with } \log 2n)$ .

Step 2. Compute  $\Lambda_2 = 2^a \cdot \sqrt{5}^2 \alpha^{-2m} (1 + \alpha^{2n-2m})^{-1}$  to obtain  $n \log \alpha < (\text{an expression with } (n-m)\log \alpha)$ .

Combining two inequalities, we obtain a bound of n. In this process, we need the logarithmic height.

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**Definition 1.** Let  $\eta$  be an algebraic number of degree *d* with minimal polynomial

$$a_d x^d + a_{d-1} x^{d-1} + \dots + a_0 = a_d \prod_{i=1}^d (x - \eta^{(i)}),$$

where  $a_i$ 's are relatively prime integers with  $a_d > 0$ , and  $\eta^{(i)}$ 's are conjugates of  $\eta$ . Then

$$h(\eta) = \frac{1}{d} \left( \log a_d + \sum_{i=1}^d \log \left( \max\{|\eta^{(i)}|, 1\} \right) \right)$$

is called the *logarithmic height* of  $\eta$ .

In particular, if  $\eta = p/q$  is a rational number with gcd(p,q) = 1, then  $h(\eta) = \log \max\{|p|, |q|\}$ . The following properties holds:

- $h(\eta \pm \gamma) \le h(\eta) + h(\gamma) + \log 2$ .
- $h(\eta \gamma^{\pm 1}) \le h(\eta) + h(\gamma)$ .
- $h(\eta^s) = |s|h(\eta)$ .

**Theorem 2** (Matveev [12, Corollary 2.3], [4, Theorem 9.4]). Assume that  $\gamma_1, \ldots, \gamma_t$  are positive real algebraic numbers in a real algebraic number field K of degree D,  $b_1, \ldots, b_t \in \mathbb{Z}$ , and

$$\Lambda \coloneqq \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1$$

is not zero. Then

$$|\Lambda| > \exp(-1.4 \times 30^{t+3} \times t^{4.5} \times D^2(1 + \log D)(1 + \log B)A_1 \cdots A_t).$$

when the following holds:

- $B \ge \max\{|b_1|, ..., |b_t|\}$
- $A_i \ge \max\{Dh(\gamma_i), |\log \gamma_i|, 0.16\}$  for all i = 1, ..., t.

First, we compute Step 1. By the Binet formula

$$F_n = \frac{1}{\sqrt{5}} (\alpha^n - \beta^n) \text{ for } n \ge 0$$

where

$$\alpha = \frac{1+\sqrt{5}}{2}$$
 and  $\beta = \frac{1-\sqrt{5}}{2}$ 

Note that  $1 < \alpha < 2$  and

$$\alpha^{n-2} < F_n < \alpha^{n-1} < \alpha^n.$$

Thus, since

$$2^{a} = F_{n}^{2} + F_{m}^{2} < \alpha^{2n} + \alpha^{2m} < 2\alpha^{2n} < 2 \cdot 2^{2n} = 2^{2n+1},$$

we have  $a \leq 2n$ . Arranging the equation

$$\left(\frac{\alpha^n - \beta^n}{\sqrt{5}}\right)^2 + \left(\frac{\alpha^m - \beta^m}{\sqrt{5}}\right)^2 = 2^a,\tag{1}$$

we obtain

$$\begin{aligned} \left| \frac{\alpha^{2n}}{5} - 2^{\alpha} \right| &= \left| \frac{2\alpha^{n}\beta^{n}}{5} - \frac{\beta^{2n}}{5} - \frac{\alpha^{2m}}{5} + \frac{2\alpha^{m}\beta^{m}}{5} - \frac{\beta^{2m}}{5} \right| \\ &< \frac{2\alpha^{n+m}}{5} + \frac{\alpha^{n+m}}{5} + \frac{\alpha^{m+n}}{5} + \frac{2\alpha^{m+n}}{5} + \frac{\alpha^{n+m}}{5} \\ &= \frac{7}{5}\alpha^{n+m}, \end{aligned}$$

On the Diophantine equation  $F_n^2 + F_m^2 = 2^a$ 

since  $\alpha > 1$  and  $|\beta| < 1$ . Dividing by  $\frac{\alpha^{2n}}{5}$ , we obtain an inequality

$$\left|1 - 2^a \cdot \frac{5}{\alpha^{2n}}\right| < \frac{7}{\alpha^{n-m}}.$$
(2)

We apply the Matveev theorem to

$$\Lambda_1 = 2^a \cdot \frac{5}{\alpha^{2n}} - 1.$$

It is clear that  $\Lambda_1 \neq 0$ . Assume that t = 3 and

$$\gamma_1 = 2, \gamma_2 = \sqrt{5}, \gamma_3 = \alpha; \quad b_1 = a, b_2 = 2, b_3 = -2n.$$

Then

$$B \coloneqq 2n \ge \max\{|b_1|, |b_2|, |b_3|\}$$

and

$$A_1 := 1.4 \ge \max\{Dh(\gamma_1), |\log \gamma_1|, 0.16\}$$
$$= \max\{2 \log 2, |\log 2|, 0.16\}$$
$$= 2 \log 2 = 1.386...,$$

where  $D = [\mathbb{Q}(\sqrt{5}) : \mathbb{Q}] = 2$ . Next, we choose  $A_2$ . Note that the minimal polynomial of  $\gamma_2 = \sqrt{5}$  is  $x^2 - 5$  and

$$h(\gamma_2) = \frac{1}{2}(\log 1 + \log \sqrt{5} + \log |-\sqrt{5}|) = \frac{1}{2}\log 5.$$

Then

$$A_2 := 1.7 \ge \max\{Dh(\gamma_2), |\log \gamma_2|, 0.16\}$$
$$= \max\{2 \cdot \frac{1}{2}\log 5, |\log \sqrt{5}|, 0.16\}$$
$$= \log 5 = 1.609 \dots$$

Finally, let us choose  $A_3$  as follows:

$$A_3 \coloneqq 0.49 \ge \max\{Dh(\gamma_3), |\log \gamma_3|, 0.16\} \\ = \max\{2 \cdot \frac{1}{2} \log \alpha, |\log \alpha|, 0.16\} \\ = \log \alpha = 0.4812....$$

Let

$$C_1 = 1.4 \times 30^{t+3} \cdot t^{4.5} \cdot D^2 (1 + \log D) \cdot A_1 \cdot A_2 \cdot A_3$$
  
= 1.4 × 30<sup>6</sup> · 3<sup>4.5</sup> · 2<sup>2</sup> (1 + log 2) × 1.4 × 1.7 × 0.49  
= 1.13091 ... × 10<sup>12</sup>.

Then, by Theorem 2 and inequality (2), we obtain the inequality

$$\exp(-C_1 \cdot (1 + \log 2n)) < |\Lambda_1| < \frac{7}{\alpha^{n-m}}$$

Thus,

$$-C_1 \cdot (1 + \log 2n) < \log 7 - (n - m) \log \alpha$$

and then

$$(n-m)\log\alpha < \log 7 + C_1(1+\log 2n) < \log 7 + 1.131 \times 10^{12}(1+\log 2n) < 1.14 \times 10^{12}(1+\log 2n).$$
(3)

Now, we compute Step 2. By arranging (1), we obtain

$$\frac{\alpha^{2n}}{5} + \frac{\alpha^{2m}}{5} - 2^a = \frac{2\alpha^n \beta^n}{5} - \frac{\beta^{2n}}{5} + \frac{2\alpha^m \beta^m}{5} - \frac{\beta^{2m}}{5}$$

Since  $|\beta| < 1$ ,  $\alpha > 1$ , and  $n \ge m$ , the inequality

$$\left|\frac{\alpha^{2n}}{5} + \frac{\alpha^{2m}}{5} - 2^a\right| < \frac{2}{5}\alpha^n + \frac{1}{5}\alpha^n + \frac{2}{5}\alpha^n + \frac{1}{5}\alpha^n = \frac{6}{5}\alpha^n$$

holds. Dividing both sides by  $\frac{1}{5}(\alpha^{2m} + \alpha^{2n})$ , we obtain an inequality

$$\left|1 - 2^a \cdot \frac{5}{\alpha^{2m} + \alpha^{2n}}\right| < \frac{6}{\alpha^n}.$$
(4)

If we set t = 4 and

$$\gamma_1 = 2, \gamma_2 = \sqrt{5}, \gamma_3 = \alpha, \gamma_4 = 1 + \alpha^{2n-2m};$$
  
 $b_1 = a, b_2 = 2, b_3 = -2m, b_4 = -1,$ 

then the LHS of (4) becomes

$$\Lambda_2 \coloneqq 2^a \cdot \frac{5}{\alpha^{2m} + \alpha^{2n}} - 1 = \gamma_1^{b_1} \gamma_2^{b_2} \gamma_3^{b_3} \gamma_4^{b_4} - 1.$$

If  $\Lambda_2 = 0$ , by Galois conjugation  $\overline{\alpha} = \beta$  in  $\mathbb{Q}(\sqrt{5})$ ,

$$\alpha^{2n} \le \alpha^{2m} + \alpha^{2n} = 2^a \cdot 4 = |\beta^{2m} + \beta^{2n}| \le |\beta|^{2m} + |\beta|^{2n} \le 2$$

which yields a contradiction. So  $\Lambda_2 \neq 0$ . Note that

$$B \coloneqq 2n \ge \max\{|b_1|, |b_2|, |b_3|, |b_4|\},\$$

and we can set  $A_1 = 1.4$ ,  $A_2 = 1.7$ ,  $A_3 = 0.49$  as in the previous step for  $\Lambda_1$ . Since

$$h(\gamma_4) = h(1 + \alpha^{2n-2m})$$
  

$$\leq h(1) + h(\alpha^{2n-2m}) + \log 2$$
  

$$= |2n - 2m|h(\alpha) + \log 2$$
  

$$= (n - m) \log \alpha + \log 2,$$

we have that

$$\begin{aligned} \max\{2h(\gamma_4), |\log \gamma_4|, 0.16\} &\leq \max\{2(n-m)\log \alpha + \log 4, \log(2\alpha^{2n-2m}), 0.16\} \\ &= 2(n-m)\log \alpha + \log 4 \\ &= 2(n-m)\log \alpha + 1.386\dots, \end{aligned}$$

and thus we can set

$$A_4 \coloneqq 2(n-m)\log\alpha + 1.4.$$

Then, by Theorem 2 and inequality (4), we obtain

$$\exp(-C_2 \cdot (1 + \log 2n) \cdot A_4) < |\Lambda_2| < \frac{6}{\alpha^n},$$

where

$$C_2 = 1.4 \times 30^7 \times 4^{4.5} \times 2^2 (1 + \log 2) \times 1.4 \times 1.7 \times 0.49$$
  
= 1.23815 ... × 10<sup>14</sup>.

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Taking logarithms on both sides, we obtain

$$n \log \alpha < \log 6 + C_2(1 + \log 2n) \times A_4$$
  

$$< \log 6 + (1.3 \cdot 10^{14})(1 + \log 2n) \times (2(n - m) \log \alpha + 1.4)$$
  

$$< \log 6 + (1.3 \cdot 10^{14})(1 + \log 2n)$$
  

$$\times (2 \cdot 1.14 \cdot 10^{12} \cdot (1 + \log 2n) + 1.4)$$
(5)

by using (3). This inequality does not hold for  $n > 3.18683 \times 10^{30}$ , i.e. the set of integer solutions of  $F_n^2 + F_m^2 = 2^a$  is finite.

## 3. Reduction on n

To lower the upper bound of n we use Dujella–Pethő's reduction method. The following lemma is an immediate variation of the result due to Dujella and Pethő from [7].

**Lemma 1.** Let *M* be a positive integer, let p/q be a convergent of the continued fraction of the irrational number  $\gamma$  such that q > 6M, and let  $A, B, \mu$  be some real numbers with A > 0 and B > 1. Let  $\epsilon := \|\mu q\| - M \|\gamma q\|$ , where  $\|\cdot\|$  denotes the distance from the nearest integer. If  $\epsilon > 0$ , then there exists no solution to the inequality

$$0 < |u\gamma - v + \mu| < AB^{-w}$$

in positive integers u, v and w with

$$u \le M$$
 and  $w \ge \frac{\log(Aq/\epsilon)}{\log B}$ 

We use this reduction in two steps.

Step 1. Compute the upper bound of  $w \coloneqq n - m$ .

Step 2. Compute the upper bound of w := n.

Now, we compute Step 1 of the reduction. Since

$$F_n^2 = \frac{\alpha^{2n}}{5} - \frac{2\alpha^n \beta^n}{5} + \frac{\beta^{2n}}{5},$$

we have that

$$\frac{\alpha^{2n}}{5} = F_n^2 + \frac{2\alpha^n \beta^n}{5} - \frac{\beta^2 n}{5}$$
$$< F_n^2 + \frac{2}{5} < F_n^2 + F_m^2 = 2^a.$$

Hence,  $\frac{\alpha^{2n}}{5} < 2^a$  or  $1 < 2^a \cdot 5 \cdot \alpha^{-2n}$ . Taking logarithms on both sides, we obtain

$$z_1 \coloneqq a \log 2 - 2n \log \alpha + \log 5 > 0.$$

Then, since

$$0 < z_1 \le e^{z_1} - 1 < \frac{7}{\alpha^{n-m}}$$

by inequality (2), we obtain an inequality

$$0 < a \log 2 - 2n \log \alpha + \log 5 < \frac{7}{\alpha^{n-m}}$$

and thus

$$0 < a\left(\frac{\log 2}{\log \alpha}\right) - 2n + \left(\frac{\log 5}{\log \alpha}\right) < \frac{7}{\log \alpha} \cdot \alpha^{-(n-m)} < 15\alpha^{-(n-m)}.$$

Now, let

$$u = a, \quad \gamma = \frac{\log 2}{\log \alpha} \notin \mathbb{Q}, \quad v = 2n, \quad \mu = \frac{\log 5}{\log \alpha},$$
  
$$A = 15 > 0, \quad B = \alpha > 1, \quad w = n - m.$$

Since  $n < 3.18683 \times 10^{30}$  in the previous section and

$$u = a \le 2n < 2 \cdot 3.18683 \times 10^{30} < 6.4 \times 10^{30},$$

we can set  $M := 6.4 \times 10^{30}$ . Then  $q = q_{70} > 6M$  from the 70th convergent

$$\frac{p_{70}}{q_{70}} = \frac{228666343422267608843910896109913}{158749759840390984049158390593424}$$

of  $\gamma$  and

$$\epsilon = \|\mu q\| - M \|\gamma q\|$$
  
= 0.050737... - 0.036496... = 0.01424...

Then, by Lemma 1

$$m-m < rac{\log(Aq/\epsilon)}{\log B} < 168.542\dots$$

and, by inequality (5) of Step 2 in the previous section,

$$n \log \alpha < \log 6 + (1.3 \times 10^{14})(1 + \log 2n)(2(n - m) \log \alpha + 1.4)$$
  
$$\leq \log 6 + (1.3 \times 10^{14})(1 + \log 2n)(2 \cdot 168 \cdot \log \alpha + 1.4).$$

This inequality yields  $1 \le n < 1.92962 \times 10^{18}$ . Now, we compute Step 2 of the reduction. By inequality (4),

$$|1 - e^{z_2}| < \frac{6}{\alpha^n}$$

holds if we set  $\varphi(t) = 5(1 + \alpha^{2t})^{-1}$  and

$$z_2 \coloneqq a \log 2 - 2m \log \alpha + \log \varphi(n - m).$$

If  $z_2 > 0$ , then

$$0 < z_2 < e^{z_2} - 1 < \frac{6}{\alpha^n}.$$

On the other hand, if  $z_2 < 0$ , then

$$|1 - e^{z_2}| < \frac{6}{\alpha^n} < 6 \cdot \frac{1}{12} = \frac{1}{2}$$

for n > 6, and thus

$$0 < |z_2| \le e^{|z_2|} - 1 = e^{|z_2|} |e^{z_2} - 1| < 2 \cdot \frac{6}{\alpha^n} = \frac{12}{\alpha^n}$$

holds. Hence we have

$$0 < |z_2| < \frac{12}{\alpha^n}.$$

Dividing both sides by  $\log \alpha$ , we obtain

$$0 < \left| a \left( \frac{\log 2}{\log \alpha} \right) - 2m + \left( \frac{\log \varphi(n-m)}{\log \alpha} \right) \right| < \frac{12}{\log \alpha} \cdot \alpha^{-n} < 25\alpha^{-n}$$

and set

$$u = a, \quad \gamma = \frac{\log 2}{\log \alpha} \notin \mathbb{Q}, \quad v = 2m, \quad \mu = \frac{\log \varphi(n - m)}{\log \alpha},$$
  
$$A = 25 > 0, \quad B = \alpha > 1, \quad w = n.$$

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Since

$$u = a \le 2n < 2 \cdot 1.92962 \times 10^{18} < 4 \cdot 10^{18},$$

we can set  $M := 4 \cdot 10^{18}$ . Then  $q = q_{92} > 6M$  from the 92nd convergent

$$\frac{p_{92}}{q_{92}} = \frac{11050488518187223769675429814265137882796096777}{7671712295419462774253251611379830294430010977}$$

of  $\gamma$  and  $\epsilon = \|\mu q\| - M \|\gamma q\| > 0$  for each  $n - m = 1, 2, \dots, 168$ . Then, by Lemma 1,

$$w = n < \frac{\log(Aq/\epsilon)}{\log B}$$

holds for each value of n - m. The largest value of the RHS is 230.46555... when n - m = 35. So, we may check  $F_n^2 + F_m^2 = 2^a$  for  $0 \le m \le n \le 230$ . We can find all nonnegative solutions of the equation with the help of computers for this feasible bound.

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