Math. Commun. **31** (2026), 43–59
DOI: 10.64785/mc.31.1.5

Mathematical Communications

Existence and asymptotic behavior of solutions for a class of Kirchhoff-type equations on the real half-line

Ishak Kettaf, Sofiane Khoutir*, and Hicham Kasri

Faculty of Mathematics, Laboratory AMNEDP, University of Science and Technology Houari Boumediene, PB 32 El-Alia, Bab Ezzouar 16 111, Algiers, Algeria

Abstract. This paper is concerned with the following Kirchhoff-type equation:

$$-\left(a+b\int_0^{+\infty}|u'(x)|^2dx\right)u''+p(x)u=q(x)f(u),\quad x\in(0,+\infty),$$

where $a>0$, $b\geq 0$, are constants, $f\in C(\mathbb{R})$, $p\in C(\mathbb{R}^+,\mathbb{R}^+_*)$ and $q\in L^1(0,+\infty)$. Firstly, by using Ekeland's variational principle, we show the existence of solutions to the above equation in the case where f is a sublinear function. Then, we establish the existence of solutions in the case where f is a superlinear function by using the mountain pass theorem. Moreover, we discuss the asymptotic behavior of the obtained solutions in both cases with respect to the parameter b. Some recent results are complemented and extended.

AMS subject classifications: 35J20, 35B38, 34B40
Keywords: Kirchhoff-type equation; sublinear; Ekeland's variational principle; superlinear; mountain pass theorem; asymptotic behavior
Received December 25, 2024; accepted May 4, 2025

1. Introduction

In this paper, we consider the following Kirchhoff-type equation:

$$\begin{cases}-\left(a+b\displaystyle\int_0^{+\infty}|u'(x)|^2dx\right)u''+p(x)u=q(x)f(u), & x\in(0,+\infty),\\ u(0)=0,\end{cases}\tag{$\mathcal{P}_b$}$$

where $a>0$ and $b\geq 0$ are constants, $f\in C(\mathbb{R})$, $p\in C(\mathbb{R}^+,\mathbb{R}^+_*)$, and $q\in L^1(0,+\infty)$.

The problem ($\mathcal{P}_b$) is related to the stationary analogue of the following Kirchhoff equation:

$$\frac{\partial^2u}{\partial t^2}-\left(\alpha+\beta\int_0^{+\infty}\left|\frac{\partial u}{\partial x}\right|^2dx\right)\frac{\partial^2u}{\partial x^2}=g(x,u),$$

which was introduced by Kirchhoff [16] as a generalization of the well-known D'Alembert's wave equation

$$\rho\frac{\partial^2u}{\partial t^2}-\left(\frac{\rho_0}{h}+\frac{E}{2L}\int_0^{L}\left|\frac{\partial u}{\partial x}\right|^2dx\right)\frac{\partial^2u}{\partial x^2}=g(x,u),\tag{1}$$

for free vibrations of elastic strings. Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations.

In (1), u denotes the displacement, $g(x,u)$ is the external force and the other parameters have the following meaning: L is the length of the string, h is the area of cross section, E is the Young modulus of the material, ρ is the mass density, and ρ_0 is the initial tension.

Email addresses: iKettaf@usthb.dz (I. Kettaf), skhoutir@usthb.dz (S. Khoutir), hkasri@usthb.dz(H. Kasri)
*Corresponding author.

https://www.mathos.unios.hr/mc

We notice that problem (1) appears in other fields such as biological systems, where u describes a process which depends on the average of the density itself (for instance, population density). For more information on the physical background of problem (1), we refer the readers to [1, 4, 17, 21] and the references therein.

In the recent decades, the following Kirchhoff-type equation:

$$-\left(a+b\int_{\Omega}|\nabla u|^2dx\right)\Delta u+V(x)u=f(x,u),\quad x\in\Omega,$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$ or $\Omega=\mathbb{R}^N$, has been extensively investigated, and many interesting results have been obtained by means of variational methods, see for instance [3, 9, 11, 14, 18, 20, 23, 24, 26] and the references therein.

However, there are a few papers dealing with the original case (1) posed on intervals $I\subseteq\mathbb{R}$, see [8, 13, 15]. In [13], the authors considered the following class of the Kirchhoff-type second-order impulsive differential problem:

$$\begin{cases} K\left(\int_0^{+\infty}\left(|u'(t)|^2+q(t)|u(t)|^2\right)dt\right)(-u''(t)+q(t)u(t))=\lambda f(t,u(t)), & t\in[0,+\infty),\quad t\neq t_j,\\ \Delta\left(u'\left(t_j\right)\right)=\lambda I_j\left(u\left(t_j\right)\right), & \\ u'\left(0^+\right)=g(u(0)),\quad u'(+\infty)=0, & \end{cases}$$

where $K:[0,+\infty)\to\mathbb{R}$ is a continuous function, $q\in L^\infty[0,+\infty)$, λ is a control parameter, $I_j\in C(\mathbb{R},\mathbb{R})$ for $1\leqslant j\leqslant p$, $0=t_0<t_1<t_2<\cdots<t_p<+\infty$, $\Delta\left(u'\left(t_j\right)\right)=u'\left(t_j^+\right)-u'\left(t_j^-\right)=\lim_{t\to t_j^+}u'(t)-\lim_{t\to t_j^-}u'(t)$, $f:[0,+\infty)\times\mathbb{R}\to\mathbb{R}$ is an L^2-Carathéodory function, and $g:\mathbb{R}\to\mathbb{R}$ is a Lipschitz continuous function. Using variational methods, they established the existence of at least one weak solution as well as infinitely many weak solutions for the above problem.

In [8], the authors investigated the following Kirchhoff-type second-order boundary value problem:

$$\begin{cases} \left(a+\lambda\int_0^{+\infty}\left(u'(t)^2+bu(t)^2\right)\mathrm{d}t\right)(-u''(t)+bu(t))=f(u(t)) \text{ for a.e. } t\in(0,+\infty),\\ u(0)=u(+\infty)=0, \end{cases}$$

where a and b are positive constants, $\lambda\geq 0$ is a parameter and $f\in C\left(\mathbb{R}^+,\mathbb{R}^+\right)$. By using the mountain pass lemma in combination with the Pohozaev identity, they established the existence of a positive solution for the above equation in the case where f is superlinear.

When $b=0$ and $a=1$ in ($\mathcal{P}_b$), equation ($\mathcal{P}_b$) becomes the following semilinear equation:

$$-u''+p(x)u=f(x,u). \tag{2}$$

Some interesting studies related to (2) by variational methods can be found in [5, 6, 7, 12, 19, 22].

Motivated by the above works, in the present paper, by using Ekeland's variational principle we first investigate the existence of solutions to problem ($\mathcal{P}_b$) with sublinear nonlinearity f. Then, we study the existence of solutions in the case where f is a superlinear function by using the mountain pass theorem. Finally, we discuss the asymptotic behavior of the obtained solutions in both cases with respect to the parameter b.

The outline of this paper is as follows. In Section 2, we introduce the variational framework associated with problem ($\mathcal{P}_b$). Section 3 is devoted to the study of the sublinear case and the proof of Theorems 1 and 2. In Section 4, we focus on the study of the superlinear case and establish the proof of Theorem 4.

2. Variational framework

Throughout this paper, we use the following notations:

- $\|.\|_r$ denotes the usual L^r−norm for $r\in[1,+\infty]$;
- X^* denotes the topological dual space of the Banach space X;

- $C_c(0,+\infty)$ denotes the space of continuous functions with compact support in $(0,+\infty)$;
- $\rightharpoonup$ denotes the weak convergence in X;
- B_ρ denotes the open ball of center 0 and radius ρ;
- $\overline{B}_\rho$ denotes the closed ball of center 0 and radius ρ;
- $\partial B_\rho := \{y \in X : \|y\| = \rho\}$;
- $o_n(1) \to 0$ as $n \to \infty$;

We consider the Sobolev space

$$H_0^1(0,+\infty) = \left\{u \in L^2(0,+\infty)\ : u' \in L^2(0,+\infty), u(0) = 0\right\},$$

and let H be the subspace of $H_0^1(0,+\infty)$ defined by

$$H = \left\{u \in H_0^1(0,+\infty) : \int_0^{+\infty} p(x)u^2 dx < +\infty\right\}.$$

Obviously, H is a Hilbert space with a scalar product and norm given by

$$(u,v) = \int_0^{+\infty} (au'v' + p(x)uv)dx \ \text{ and } \ \|u\|^2 = \int_0^{+\infty} (a|u'|^2 + p(x)u^2)dx,$$

for all $u, v \in H$.
Under the assumption (P), it is easy to show that H is embedded continuously into $H_0^1(0,+\infty)$ and therefore also into $L^r(0,+\infty)$ for $r \in [2,+\infty]$. Thus, there exists $\mu_r > 0$ such that

$$\|u\|_r \le \mu_r \|u\|, \qquad \forall u \in H. \tag{3}$$

We notice that if $u \in H$, then $\lim\limits_{x\to+\infty} u(x) = 0$.

For the problem $(\mathcal{P}_b)$, the associated energy functional is defined on H as follows:

$$I_b(u) = \frac{1}{2}\|u\|^2 + \frac{b}{4}\Big(\int_0^{+\infty} |u'|^2 dx\Big)^2 - \int_0^{+\infty} q(x)F(u)dx, \tag{4}$$

where $F(u) = \int_0^u f(s)ds$.
We have the following result.

Lemma 1. *The functional I_b is of class C^1 on H, and*

$$\langle I_b'(u), v\rangle = (u,v) + b\Big(\int_0^{+\infty} |u'|^2 dx\Big)\int_0^{+\infty} u'v' dx - \int_0^{+\infty} q(x)f(u)v dx,$$

for all $u, v \in H$.

Proof. We consider the functional J defined on H by

$$J(u) = \int_0^{+\infty} q(x)F(u)dx.$$

By combining (f_1) and (F_1) (which are introduced in Section 3 and Section 4) with (3), we can easily derive that for any $u \in H$, the inequality

$$|f(u)| \le C_{\|u\|}, \tag{5}$$

where $C_{\|u\|} > 0$ is the constant given by

$$C_{\|u\|} = \begin{cases} \alpha_1 \mu_\infty^{\theta-1} \|u\|^{\theta-1}, & \text{if } (f_1) \text{ holds}, \\ \alpha_2 + \beta \mu_\infty^{\theta-1} \|u\|^{\theta-1}, & \text{if } (F_1) \text{ holds}. \end{cases}$$

Then, it follows that

$$|F(u)| \leq \mu_\infty C_{\|u\|} \|u\|, \qquad \forall u \in H,$$

which implies that J is well-defined on H since $q \in L^1(0,+\infty)$.
To prove that I_b is of class C^1 on H, it is sufficient to prove this property only for J. For this purpose, firstly we prove that J is Gâteaux differentiable, and then we show that J'_G is continuous.

Claim 1. J is Gâteaux differentiable.
It is obvious that for all $u, v \in H$, and almost every $x \in (0,+\infty)$

$$\lim_{t \longrightarrow 0} q(x) \frac{F(u(x)+tv(x)) - F(u(x))}{t} = q(x) f(u(x)) v(x).$$

Indeed, by the mean value theorem there exists a real number $0 < \theta_t < |t|$ with $|t| \leq 1$ such that

$$q(x)\Big(F\big(u(x)+tv(x)\big) - F\big(u(x)\big)\Big) = tq(x) f\big(u(x) + \theta_t v(x)\big) v(x), \tag{6}$$

and by continuity of f we obtain the result.
Once again, from (f_1), (F_1), (3) and the inequality $|a+b|^r \leq \gamma_r(|a|^r + |b|^r)$ with $a, b \in \mathbb{R}$ we check that

$$\Big| f\big(u(x) + \theta_t v(x)\big) v(x) \Big| \leq C_{\|u\|,\|v\|}, \tag{7}$$

where $C_{\|u\|,\|v\|} > 0$ denotes the constant given by

$$C_{\|u\|,\|v\|} = \mu_\infty \|v\| \begin{cases} \alpha_1 \gamma_{\theta-1} \mu_\infty^{\theta-1} \left(\|u\|^{\theta-1} + \|v\|^{\theta-1} \right), & \text{if } (f_1) \text{ holds}, \\ \alpha_2 + \beta \gamma_{\theta-1} \mu_\infty^{\theta-1} \left(\|u\|^{\theta-1} + \|v\|^{\theta-1} \right), & \text{if } (F_1) \text{ holds}. \end{cases}$$

Hence, from (6) and (7), we conclude that

$$\begin{aligned} q(x) \left| \frac{F\big(u(x)+tv(x)\big) - F\big(u(x)\big)}{t} \right| &= q(x) \Big| f\big(u(x) + \theta_t v(x)\big) v(x) \Big| \\ &\leq C_{\|u\|,\|v\|} q(x). \end{aligned}$$

As the function $q \in L^1(0,+\infty)$, by the Lebesgue dominated convergence theorem we have

$$\lim_{t \longrightarrow 0} \int_0^{+\infty} q(x) \frac{F(u+tv) - F(u)}{t} dx = \int_0^{+\infty} q(x) f(u) v dx.$$

Since the right-hand side, as a function of v, is a continuous and linear functional on H, it is the Gâteaux differential J'_G of J.

Claim 2. J'_G is continuous.
We complete the proof by checking that the function J'_G is continuous on H^*. For this purpose, let take $\{u_n\}$ in H such that $u_n \longrightarrow u$ as $n \longrightarrow +\infty$. Up to a subsequence, we may assume that

1. $u_n \longrightarrow u$ in $L^r(0,+\infty)$, $\forall r \in [2,+\infty]$;
2. $u_n(x) \longrightarrow u(x)$ a.e in $(0,+\infty)$.

We have for all $v \in H$

$$\Big| \langle J'_G(u_n) - J'_G(u), v \rangle \Big| \leq \int_0^{+\infty} q(x) |f(u_n) - f(u)| |v| dx.$$

By continuity of f, it is clear that

$$f\big(u_n(x)\big) \longrightarrow f\big(u(x)\big), \qquad \text{a.e } x \in (0,+\infty).$$

Since the sequence $\{u_n\}$ is convergent, it is bounded in H, and therefore there exists a constant $M > 0$ such that

$$\|u_n\| \leq M, \qquad \forall n \in \mathbb{N}. \tag{8}$$

Thus, taking into account (5) and (8), one has

$$q(x)\Big|f\big(u_n(x)\big)\Big| \leq Cq(x) \in L^1(0,+\infty),$$

where $C > 0$ is defined as

$$C = \begin{cases} \alpha_1 \mu_\infty^{\theta-1} M^{\theta-1}, & \text{if } (f_1) \text{ holds},\\ \alpha_2 + \beta \mu_\infty^{\theta-1} M^{\theta-1}, & \text{if } (F_1) \text{ holds}, \end{cases}$$

and once again by the Lebesgue dominated convergence theorem, we get

$$\int_0^{+\infty} q(x) f(u_n) dx \longrightarrow \int_0^{+\infty} q(x) f(u) dx \qquad \text{as } n \longrightarrow +\infty.$$

Hence

$$\begin{aligned} \Big|\langle J'_G(u_n) - J'_G(u), v\rangle\Big| &\leq \int_0^{+\infty} q(x)|f(u_n) - f(u)||v| dx,\\ &\leq \mu_\infty \|v\| \int_0^{+\infty} q(x)|f(u_n) - f(u)| dx, \qquad \forall v \in H, \end{aligned}$$

and this implies that

$$\sup_{\|v\|\leq 1} \Big|\langle J'_G(u_n) - J'_G(u), v\rangle\Big| \leq \mu_\infty \int_0^{+\infty} q(x)|f(u_n) - f(u)| dx \longrightarrow 0, \text{ as } n \longrightarrow +\infty.$$

Consequently, we conclude that

$$\|J'_G(v_n) - J'_G(v)\|_{H^*} \longrightarrow 0, \qquad \text{as} \quad n \longrightarrow +\infty,$$

which implies the continuity of J'_G. The proof is completed. □

Definition 1. We say that u is a weak solution of problem $(\mathcal{P}_b)$ if for any $v \in H$ we have

$$(u,v) + b\Big(\int_0^{+\infty} |u'|^2 dx\Big) \int_0^{+\infty} u'v' dx - \int_0^{+\infty} q(x) f(u) v dx = 0.$$

Remark 1. From Lemma 1 and Definition 1, we deduce that the critical points of I_b correspond to the weak solutions of $(\mathcal{P}_b)$.

3. The sublinear case

In this section, we investigate problem $(\mathcal{P}_b)$ in the case where the nonlinear term f has sublinear growth. The main results of this section are stated as follows.

Theorem 1. *Assume that*

- *(P)* $p \in C(\mathbb{R}^+)$ *and* $\inf\limits_{x\in\mathbb{R}^+} p(x) \geq p_0 > 0$*;*
- *(Q)* $q : \mathbb{R}^+ \to \mathbb{R}^+_*$ *such that* $q \in L^1(0,+\infty)$*.*

(f_1) There exist $\alpha_1 > 0$ and $\theta \in (1,2)$ such that

$$|f(s)| \leq \alpha_1 |s|^{\theta-1}, \qquad \forall s \in \mathbb{R};$$

(f_2) $\lim_{s\to 0} \frac{f(s)}{|s|} = +\infty$.

Then problem $(\mathcal{P}_b)$ has at least one nontrivial solution.

Theorem 2. *Let (P), (Q), (f_1) and (f_2) hold. Then, for any sequence $\{b_n\} \subset (0,+\infty)$ with $b_n \longrightarrow 0$ as $n \longrightarrow +\infty$, there exist a subsequence, still denoted by $\{b_n\}$, and $u_* \in H$ such that $u_n \longrightarrow u_*$ in H, where u_* is a solution of the equation*

$$\begin{cases} -au'' + p(x)u = q(x)f(u), & x \in (0,+\infty), \\ u(0) = 0. \end{cases} \tag{$\mathcal{P}_0$}$$

Remark 2. Since equation $(\mathcal{P}_b)$ is set on the unbounded interval $[0,\infty)$, the main difficulty is the lack of compactness of the embedding $H_0^1(0,+\infty) \hookrightarrow L^r(0,+\infty)$ for $r \geq 2$. Indeed, when using variational methods, we need to prove that the energy functional associated to $(\mathcal{P}_b)$ satisfies the Palais-Smale compactness condition, that is, any sequence $\{u_n\} \subset H^1(0,+\infty)$ satisfying (27) has a convergent subsequence. To this end, a careful analysis of the energy functional and its derivative is given (see pp. 14-15) to prove the convergence of the (PS) sequence.

3.1. Technical lemmas

In this subsection, we will prove some lemmas which will be used for proving theorem 1.

Lemma 2. *The functional I_b is bounded from below on $\overline{B}_\rho$, where $\rho > 0$.*

Proof. From (f_1), one has

$$\frac{|f(s)|}{|s|} \leq \alpha_1 |s|^{\theta-2}, \qquad \forall s \in \mathbb{R}^*.$$

Since $1 < \theta < 2$, we deduce that

$$\lim_{s\to+\infty} \frac{f(s)}{s} = 0,$$

and then, for all $\varepsilon > 0$, we get

$$|f(u)| \leq \varepsilon |u| + \alpha_1 |u|^{\theta-1} \text{ and } |F(u)| \leq \frac{\varepsilon}{2}|u|^2 + \frac{\alpha_1}{\theta}|u|^\theta, \qquad \forall u \in \mathbb{R}. \tag{9}$$

Therefore, by (3), (4), (9), and the fact that $b \geq 0$, for all $u \in H$ we get

$$\begin{aligned} I_b(u) &= \frac{1}{2}\|u\|^2 + \frac{b}{4}\Big(\int_0^{+\infty} |u'|^2 dx\Big)^2 - \int_0^{+\infty} q(x)F(u)dx \\ &\geq \frac{1}{2}\|u\|^2 - \int_0^{+\infty} q(x)\Big[\frac{\varepsilon}{2}|u|^2 + \frac{\alpha_1}{\theta}|u|^\theta\Big]dx \\ &= \frac{1}{2}\|u\|^2 - \frac{\varepsilon}{2}\int_0^{+\infty} q(x)|u|^2 dx - \frac{\alpha_1}{\theta}\int_0^{+\infty} q(x)|u|^\theta dx \\ &\geq \frac{1}{2}\|u\|^2 - \frac{\mu_\infty^2 \varepsilon}{2}\|q\|_1 \|u\|^2 - \frac{\mu_\infty^\theta \alpha_1}{\theta}\|q\|_1 \|u\|^\theta \\ &= \frac{1}{2}\Big(1 - \varepsilon\mu_\infty^2 \|q\|_1\Big)\|u\|^2 - \frac{\alpha_1 \mu_\infty^\theta}{\theta}\|q\|_1 \|u\|^\theta. \end{aligned}$$

By choosing $0 < \varepsilon \leq \dfrac{1}{2\mu_\infty^2 \|q\|_1}$, we obtain

$$\begin{aligned} I_b(u) &\geq \frac{1}{4}\|u\|^2 - \frac{\alpha_1 \mu_\infty^\theta}{\theta}\|q\|_1 \|u\|^\theta \\ &= \frac{1}{4}\left(1 - C\|u\|^{\theta-2}\right)\|u\|^2. \end{aligned}$$

Since $1 < \theta < 2$, choosing $\rho > 0$ sufficiently large, we conclude that there exists a constant $\alpha > 0$ such that

$$I_b(u) \geq \alpha > 0 \quad \text{whenever} \quad \|u\| = \rho, \quad \text{and then} \quad \inf_{u \in \partial B_\rho} I_b(u) > 0.$$

If $\|u\| \leq \rho$, then

$$I_b(u) \geq \frac{1}{4}\left(\|u\|^2 - C\|u\|^\theta\right) \geq -\frac{C}{4}\rho^\theta := -C_1.$$

Thus, the functional I_b is bounded from below on $\overline{B}_\rho$.
Therefore, $\inf\limits_{u \in \overline{B}_\rho} I_b(u)$ exists, and we put

$$c_b := \inf_{u \in \overline{B}_\rho} I_b(u).$$

□

Lemma 3. *We have $c_b < 0$.*

Proof. By combining (f_1) and (f_2), it follows that for all $M > 0$

$$F(u) \geq M|u|^2 - \frac{\alpha_1}{\theta}|u|^\theta, \qquad \forall u \in \mathbb{R}. \tag{10}$$

Let $R > 1$ and let ψ_R be the function defined on $[0, +\infty)$ by

$$\psi_R(x) = \begin{cases} \dfrac{x}{R^2}, & \text{if } x \in [0, R], \\ \dfrac{1}{R}, & \text{if } x \in [R, 2R], \\ -\dfrac{x}{R^2} + \dfrac{3}{R}, & \text{if } x \in [2R, 3R], \\ 0, & \text{if } x \in [3R, +\infty). \end{cases} \tag{11}$$

Since $\psi_R \in C_c(0, +\infty)$, it follows that $\psi_R \in H$. Let

$$\varphi_R = \frac{\rho}{2\|\psi_R\|}\psi_R.$$

It is clear that $\varphi_R \in H$ and by a straightforward computation we get

$$\|\varphi_R\| = \frac{\rho}{2}, \qquad \|\psi'\|_2^2 = \frac{2}{R^3} \qquad \text{and} \qquad \|\varphi_R'\|_2^2 = \frac{\rho^2}{2\|\psi_R\|^2 R^3}, \tag{12}$$

which implies that $\varphi_R \in \overline{B}_\rho$ for all $R > 1$.

Moreover, by (10), (11) and (12), we have

$$\begin{aligned} I_b(\varphi_R) &= \frac{1}{2}\|\varphi_R\|^2 + \frac{b}{4}\|\varphi_R'\|_2^4 - \int_0^{+\infty} q(x)F(\varphi_R)dx \\ &= \frac{\rho^2}{8} + \frac{b}{4}\|\varphi_R'\|_2^4 - \int_0^{+\infty} q(x)F(\varphi_R)dx \\ &\leq \frac{\rho^2}{8} + \frac{b\rho^4}{16\|\psi_R\|^4 R^6} - M\int_0^{+\infty} q(x)\varphi_R^2(x)dx + \frac{\alpha_1}{\theta}\int_0^{+\infty} q(x)\varphi_R^\theta(x)dx \\ &\leq \frac{\rho^2}{8} + \frac{b\rho^4}{64a^2} - M\int_0^{+\infty} q(x)\varphi_R^2(x)dx + \frac{\alpha_1 \rho^\theta}{2^\theta \theta}\mu_\infty^\theta \|q\|_1, \end{aligned}$$

and then

$$I_b(\varphi_R) \leq g(\rho) - M \int_0^{+\infty} q(x)\varphi_R^2(x)dx,$$

where $g(\rho) = \dfrac{\rho^2}{8} + \dfrac{b\rho^4}{64a^2} + \dfrac{\alpha_1 \mu_\infty^\theta}{2^\theta \theta}\|q\|_1 \rho^\theta > 0.$
By choosing

$$M > \frac{g(\rho)}{\int_0^{+\infty} q(x)\varphi_R^2(x)dx},$$

one has

$$I_b(\varphi_R) < 0.$$

Consequently, we get

$$c_b = \inf_{u \in \overline{B}_\rho} I_b(u) \leq I_b(\varphi_R) < 0.$$

This completes the proof. □

From Lemma 2 and Lemma 3, we deduce that

$$\inf_{u \in \overline{B}_\rho} I_b(u) < 0 < \inf_{u \in \partial B_\rho} I_b(u).$$

3.2. Proof of the main result

In this subsection, we shall give the proof of Theorem 1 with the use of the following variational principle.

Theorem 3 (Ekeland's variational principle, [10])**.** *Let M be a complete metric space with metric d and let $J : M \longrightarrow \mathbb{R}$ a lower semicontinuous functional bounded from below. Then, for each $\varepsilon > 0$, there exists $u_\varepsilon \in M$ such that*

$$J(u_\varepsilon) \leq \inf_M J + \varepsilon,$$

and whenever $w \in M$

$$J(u_\varepsilon) \leq J(w) + \varepsilon d(u_\varepsilon, w).$$

Proof of Theorem 1. From Lemma 1 and Lemma 2, it is clear that the functional I_b is lower semicontinuous and bounded from below in the complete metric space $\overline{B}_\rho$. Then, by applying Ekeland's variational principle, there exists a sequence $\{u_n\} \subset \overline{B}_\rho$ such that

$$I_b(u_n) \leq c_b + \frac{1}{n} \quad \text{and} \quad I_b(u_n) \leq I_b(w) + \frac{1}{n}\|w - u_n\|, \qquad \forall w \in \overline{B}_\rho.$$

Note that for $n \in \mathbb{N}$ with

$$\frac{1}{n} \in \left(0, \inf_{\partial B_\rho} I_b - \inf_{\overline{B}_\rho} I_b\right),$$

one has

$$I_b(u_n) \leq c_b + \frac{1}{n} < \inf_{\partial B_\rho} I_b,$$

which implies that $u_n \in B_\rho$ for $n \in \mathbb{N}$ large enough.
On the one hand, for $w = u_n + tv$ with $t > 0$ and $v \in H$, we get

$$\frac{I_b(u_n) - I_b(u_n + tv)}{t} \leq \frac{1}{n}\|v\|$$

and then

$$-\langle I_b'(u_n), v\rangle \leq \frac{1}{n}\|v\|. \tag{13}$$

On the other hand, if we put $w = u_n - tv$, we obtain

$$\langle I_b'(u_n), v\rangle \le \frac{1}{n}\|v\|. \tag{14}$$

Combining (13) with (14), we can assert that

$$|\langle I_b'(u_n), v\rangle| \le \frac{1}{n}\|v\|, \qquad \forall v \in H,$$

and then

$$\sup_{\|v\|\le 1} |\langle I_b'(u_n), v\rangle| \le \frac{1}{n},$$

which implies that

$$\|I_b'(u_n)\|_{H^*} \longrightarrow 0, \qquad \text{as } n \longrightarrow +\infty.$$

Consequently, we have proved that

$$I_b(u_n) \longrightarrow c_b \quad \text{and} \quad I_b'(u_n) \longrightarrow 0 \text{ in } H^*. \tag{15}$$

Since $\{u_n\}$ is bounded in $\overline{B}_\rho$, which is a closed set, there exist a subsequence still denoted by $\{u_n\}$ and $u_0 \in \overline{B}_\rho \subset H$ such that

$$\begin{array}{lll} u_n \rightharpoonup u_0 & \text{weakly in} & H, \\ u_n' \rightharpoonup u_0' & \text{weakly in} & L^2(0,+\infty), \\ u_n(x) \to u_0(x) & & \text{a.e in } (0,+\infty). \end{array} \tag{16}$$

It follows from (16) and the continuity of f that

$$q(x)\Big[f\big(u_n(x)\big) - f\big(u_0(x)\big)\Big]\big(u_n(x) - u_0(x)\big) \longrightarrow 0, \qquad \text{as} \quad n \longrightarrow +\infty.$$

Moreover, by (f_1), (3) and the boundedness of $\{u_n\}$, one has

$$\begin{aligned} q(x)\Big|\big[f\big(u_n(x)\big) - f\big(u_0(x)\big)\big]\big(u_n(x) - u_0(x)\big)\Big| &\le q(x)\Big[\big|f\big(u_n(x)\big)\big| + \big|f\big(u_0(x)\big)\big|\Big]\big(\big|u_n(x)\big| + \big|u_0(x)\big|\big) \\ &\le \alpha_1^2 q(x)\big(\|u_n\|_\infty^{\theta-1} + \|u_0\|_\infty^{\theta-1}\big)\big(\|u_n\|_\infty + \|u_0\|_\infty\big) \\ &\le \alpha_1^2 \mu_\infty^\theta q(x)\big(\|u_n\|^{\theta-1} + \|u_0\|^{\theta-1}\big)\big(\|u_n\| + \|u_0\|\big) \\ &\le C q(x). \end{aligned}$$

As the function q is in $L^1(0,+\infty)$, by the Lebesgue dominated convergence theorem we get

$$\int_0^{+\infty} q(x)\big[f(u_n) - f(u_0)\big](u_n - u_0)\, dx \longrightarrow 0, \qquad \text{as} \quad n \longrightarrow +\infty. \tag{17}$$

On the other hand, since $I_b'(u_n) \longrightarrow 0$ in H^* and by (16), one has

$$\langle I_b'(u_n) - I_b'(u_0), u_n - u_0\rangle \longrightarrow 0, \qquad \text{as} \quad n \longrightarrow +\infty. \tag{18}$$

By a straightforward computation we get

$$\begin{aligned} \|u_n - u_0\|^2 = \langle I_b'(u_n) - I_b'(u_0), u_n - u_0\rangle - b\|u_n'\|_2^2 \int_0^{+\infty} u_n'(u_n' - u_0')dx + b\|u_0'\|_2^2 \int_0^{+\infty} u_0'(u_n' - u_0')dx \\ + \int_0^{+\infty} q(x)\Big[f(u_n) - f(u_0)\Big](u_n - u_0)dx, \end{aligned}$$

and taking into account (17), (18), Young's inequality and Hölder's inequality, we obtain

$$\begin{aligned} o_n(1) &= \|u_n - u_0\|^2 + b\|u_n'\|_2^2 \int_0^{+\infty} u_n'(u_n' - u_0')dx - b\|u_0'\|_2^2 \int_0^{+\infty} u_0'(u_n' - u_0')dx \\ &= \|u_n - u_0\|^2 + b\Big(\|u_n'\|_2^4 - \|u_n'\|_2^2 \int_0^{+\infty} u_n' u_0' dx - \|u_0'\|_2^2 \int_0^{+\infty} u_n' u_0' dx + \|u_0'\|_2^4\Big) \\ &\geq \|u_n - u_0\|^2 + b\Big(\|u_n'\|_2^4 - \frac{1}{2}\|u_n'\|_2^4 - \frac{1}{2}\|u_n'\|_2^2\|u_0'\|_2^2 - \frac{1}{2}\|u_0'\|_2^2\|u_n'\|_2^2 - \frac{1}{2}\|u_0'\|_2^4 + \|u_0'\|_2^4\Big) \\ &= \|u_n - u_0\|^2 + b\Big(\frac{1}{2}\|u_n'\|_2^4 - \|u_n'\|_2^2\|u_0'\|_2^2 + \frac{1}{2}\|u_0'\|_2^4\Big) \\ &= \|u_n - u_0\|^2 + \frac{b}{2}\Big(\|u_n'\|_2^2 - \|u_0'\|_2^2\Big)^2. \end{aligned} \tag{19}$$

By passing to the limit in (19) as $n \longrightarrow +\infty$, one has

$$u_n \longrightarrow u_0 \text{ strongly in } H \text{ and } \|u_n'\|_2 \longrightarrow \|u_0'\|_2. \tag{20}$$

From (16) and (20) we deduce that

$$\lim_{n\longrightarrow+\infty} \Big(\int_0^{+\infty} |u_n'|^2 dx\Big) \int_0^{+\infty} u_n' v' dx = \Big(\int_0^{+\infty} |u_0'|^2 dx\Big) \int_0^{+\infty} u_0' v' dx.$$

Consequently, by passing to the limit in $\langle I_b'(u_n), v\rangle$ as $n \longrightarrow +\infty$ and by using once again the dominated convergence theorem, we obtain for all $v \in H$

$$\lim_{n\longrightarrow+\infty} \langle I_b'(u_n), v\rangle = (u_0, v) + b\Big(\int_0^{+\infty} |u_0'|^2 dx\Big) \int_0^{+\infty} u_0' v' dx - \int_0^{+\infty} q(x) f(u_0) v dx,$$

and from (15) we deduce that

$$(u_0, v) + b\Big(\int_0^{+\infty} |u_0'|^2 dx\Big) \int_0^{+\infty} u_0' v' dx - \int_0^{+\infty} q(x) f(u_0) v dx = 0, \qquad \forall v \in H.$$

Finally, from (15), (20) and the fact that $I \in C^1(H, \mathbb{R})$, we deduce that

$$I_b'(u_0) = 0 \qquad \text{and} \qquad I_b(u_0) = c_b,$$

which means that u_0 is a critical point of I_b and then a weak solution of problem $(\mathcal{P}_b)$. The proof is complete. □

3.3. The asymptotic behavior of solutions

In this subsection, we investigate the asymptotic behavior of solutions obtained in Theorem 1 with respect to the parameter b.

Proof of Theorem 2. Noticing that $b = 0$ is allowed in the proof of Theorem 1. Therefore, there exists a solution $v_* \in H$ to problem $(\mathcal{P}_0)$ such that

$$I_0(v_*) = c_0 \qquad \text{and} \qquad I_0'(v_*) = 0,$$

where $c_0 = \inf_{u\in\overline{B}_\rho} I_0(u)$.

For any sequence $\{b_n\} \subset (0, +\infty)$ with $b_n \longrightarrow 0$ as $n \longrightarrow +\infty$, let $u_n := u_{b_n} \in \overline{B}_\rho \subset H$ be a solution of problem $(\mathcal{P}_b)$ obtained by Theorem 1. Then we have

$$I_{b_n}(u_n) = c_{b_n} < 0 \qquad \text{and} \qquad I_{b_n}'(u_n) = 0 \text{ in } H^* \tag{21}$$

with $c_{b_n} = \inf_{u \in \overline{B}_\rho} I_{b_n}(u)$. Therefore

$$(u_n, v) + b_n\Big(\int_0^{+\infty} |u_n'|^2 dx\Big) \int_0^{+\infty} u_n' v' \, dx - \int_0^{+\infty} q(x) f(u_n) v \, dx = 0, \quad \forall v \in H. \tag{22}$$

Since $u_n \in \overline{B}_\rho$, it follows that the sequence $\{u_n\}$ is bounded in H, and then there exists $u_* \in H$ such that up to a subsequence

$$\begin{array}{lll} u_n \rightharpoonup u_* & \text{weakly in} & H, \\ u_n' \rightharpoonup u_*' & \text{weakly in} & L^2(0, +\infty), \\ u_n(x) \to u_*(x) & & \text{a.e in } (0, +\infty). \end{array} \tag{23}$$

By using (21), (23), the dominated convergence theorem and the fact that $b_n \longrightarrow 0$, it is easy to show that

$$\langle I_{b_n}'(u_n) - I_{b_n}'(u_*), u_n - u_* \rangle \longrightarrow 0, \qquad \text{as} \qquad n \longrightarrow +\infty.$$

By similar arguments as in (19), we prove that

$$u_n \longrightarrow u_* \text{ strongly in } H \qquad \text{and} \qquad \|u_n'\|_2 \longrightarrow \|u_*'\|_2. \tag{24}$$

By passing to the limit in (22) as $n \longrightarrow \infty$, one has

$$(u_*, v) - \int_0^{+\infty} q(x) f(u_*) v \, dx = 0, \quad \forall v \in H,$$

which means that u_* is a weak solution of the following equation:

$$\begin{cases} -au'' + p(x)u = q(x) f(u), & x \in (0, +\infty), \\ u(0) = 0. \end{cases}$$

Next, we prove that $I_0(u_*) = c_0$. By using once again the dominated convergence theorem, and combining (23) and (24) with the fact that $b_n \longrightarrow 0$, it holds that

$$c_{b_n} \longrightarrow I_0(u_*).$$

In view of the definition of c_0, we assert that

$$I_0(u_*) \geq c_0. \tag{25}$$

Thus, it follows from (4) and (25)

$$c_0 = I_0(v_*) = I_{b_n}(v_*) - \frac{b_n}{4} ||v_*'||_2^4 \geq c_{b_n} - \frac{b_n}{4} ||v_*'||_2^4,$$

which yields

$$\lim_{n \longrightarrow +\infty} c_{b_n} \leq c_0,$$

or else

$$I_0(u_*) \leq c_0. \tag{26}$$

Hence, by (25) and (26) we deduce that

$$I_0(u_*) = c_0.$$

The proof is complete. □

3.4. Example

Let $f(x) = x^{\frac{1}{5}}$, $p(x) = e^x$ and $q(x) = \dfrac{1}{1 + x^2}$. It can be seen that (P), (Q), (f_1) and (f_2) are satisfied.

Then, by Theorem 1, the problem

$$\begin{cases} -\Big(a + b \displaystyle\int_0^{+\infty} |u'| dx\Big) u'' + e^x u = \dfrac{1}{1 + x^2} u^{\frac{1}{5}}, & x \in (0, +\infty); \\ u(0) = 0 \end{cases}$$

has at least one nontrivial solution.

4. The superlinear case

In this section, we analyze problem ($\mathcal{P}_b$) in the case where the nonlinear term f exhibits superlinear growth. The main results of this section are stated in the following theorem.

Theorem 4. *Assume that (P), (Q) hold and f satisfies*

(F_1) *$f \in C(\mathbb{R})$ and there exist $\theta > 2$, $\alpha_2, \beta > 0$ such that*

$$|f(s)| \leq \alpha_2 + \beta|s|^{\theta-1} \qquad \forall s \in \mathbb{R};$$

(F_2) $\lim\limits_{s\to 0} \dfrac{f(s)}{s} = 0;$

(F_3) $\lim\limits_{s\to\infty} \dfrac{F(s)}{s^2} = +\infty$, *where* $F(s) = \int_0^s f(t)dt;$

(F_4) *there exists $L > 0$ such that*

$$4F(s) \leq f(s)s, \qquad \forall |s| \geq L.$$

Then, problem ($\mathcal{P}_b$) has at least one nontrivial solution. Moreover, for every vanishing sequence $\{b_n\}$, let u_{b_n} be a solution of problem ($\mathcal{P}_b$). Then, the sequence $\{u_{b_n}\}$ converges to u_0 in H, where u_0 is a solution of the problem

$$\begin{cases} -au'' + p(x)u = q(x)f(u), & x \in (0, +\infty), \\ u(0) = 0. \end{cases} \tag{$\mathcal{P}_0$}$$

Remark 3. Since the energy functional associated to ($\mathcal{P}_b$) involves a 4-order homogeneous term (i.e., $\left(\int_0^{+\infty} |u'|^2 dx\right)^2$), it is natural to impose the well-known Ambrosetti–Rabinowitz condition (see [2]), namely,

$$\text{there exists } \mu > 4 \text{ such that } 0 < \mu F(u) \leq u f(u) \text{ for all } u \in \mathbb{R}. \tag{AR}$$

This condition has two crucial uses. The first one is to check the mountain pass geometry for the energy functional I_b and the second one is to guarantee the Palais-Smale compactness condition. Our assumptions (F_3) and (F_4) are very relaxed compared with (AR)-condition. To see this, consider the function $F(u) = u^4 \ln(1 + u^2)$. It is easy to check that F and its derivative f satisfy (F_3)-(F_4) but not (AR).

In order to prove Theorem 4, we will need the following definition and theorem.

Definition 2. A functional $I \in C^1(X, \mathbb{R})$ satisfies the Palais-Smale condition at level $c \in \mathbb{R}$, denoted by $(PS)_c$ if every sequence $\{u_n\} \subset X$ satisfies

$$I(u_n) \longrightarrow c \qquad \text{and} \qquad I'(u_n) \longrightarrow 0, \;\; n \longrightarrow +\infty, \tag{27}$$

possesses a strongly convergent subsequence.

Remark 4. If I satisfies the $(PS)_c$ condition for every $c \in \mathbb{R}$, then we say that I satisfies the (PS) condition.

Theorem 5. *([25, Theorem 1.15], mountain pass theorem) Let X be a Banach space, $I \in C^1(X, \mathbb{R})$ satisfies the (PS) condition, $I(0) = 0$ and*

1. *There exist $\rho, \alpha > 0$ such that $I(v) \geq \alpha$ whenever $\|v\| = \rho$.*
2. *There exists $e \in X$ with $\|e\| > \rho$ such that $I(e) \leq 0$.*

Then, I has at least a critical value $c \geq \alpha$, which is characterized by

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I\big(\gamma(t)\big),$$

where

$$\Gamma = \big\{\gamma \in C([0, 1], X) : \gamma(0) = 0, \gamma(1) = e\big\}.$$

4.1. Some useful lemmas

In this section, we introduce some technical lemmas which will be used to prove our main result.

Lemma 4. *Assume that* (F_1) *and* (F_2) *hold. Then there exist* $\rho_*, \alpha_* > 0$ *such that* $I_b(u) \geq \alpha_*$ *whenever* $\|u\| = \rho_*$.

Proof. From (F_1) and (F_2), for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f(u)| \leq \varepsilon|u| + C_\varepsilon|u|^{\theta-1} \qquad \text{and} \qquad |F(u)| \leq \frac{\varepsilon}{2}|u|^2 + \frac{C_\varepsilon}{\theta}|u|^\theta, \quad \forall u \in \mathbb{R}. \tag{28}$$

Hence, from (3), (28) and Hölder's inequality we obtain

$$\begin{aligned} I_b(u) &= \frac{1}{2}\|u\|^2 + \frac{b}{4}\Big(\int_0^{+\infty} |u'|^2 dx\Big)^2 - \int_0^{+\infty} q(x)F(u)dx \\ &\geq \frac{1}{2}\|u\|^2 - \int_0^{+\infty} q(x)F(u)dx \\ &\geq \frac{1}{2}\|u\|^2 - \int_0^{+\infty} q(x)\left(\frac{\varepsilon}{2}|u|^2 + \frac{C_\varepsilon}{\theta}|u|^\theta\right)dx \\ &\geq \frac{1}{2}\left(1 - \varepsilon\mu_\infty^2\|q\|_1\right)\|u\|^2 - \frac{C_\varepsilon}{\theta}\mu_\infty^\theta\|q\|_1\|u\|^\theta. \end{aligned}$$

By taking $0 < \varepsilon \leq \dfrac{1}{2\mu_\infty^2\|q\|_1}$, one has

$$I_b(u) \geq \frac{1}{4}\|u\|^2 - \frac{C_\varepsilon}{\theta}\mu_\infty^\theta\|q\|_1\|u\|^\theta, \qquad \forall u \in H.$$

Taking $\rho_* = \left[\dfrac{\theta}{8C_\varepsilon\mu_\infty^\theta\|q\|_1}\right]^{\frac{1}{\theta-2}}$, then for all $u \in H$ with $\|u\| = \rho_*$ we get

$$I_b(u) \geq \frac{1}{4}\left(1 - \frac{4C_\varepsilon}{\theta}\mu_\infty^\theta\|q\|_1\rho_*^{\theta-2}\right)\rho_*^2 = \frac{1}{8}\rho_*^2 := \alpha_* > 0,$$

and this completes the proof. □

Lemma 5. *Assume that* (F_1) *and* (F_3) *hold. Then there exists a function* $e \in H$ *with* $\|e\| > \rho_*$ *such that* $I_b(e) \leq 0$.

Proof. From (F_1) and (F_3), it follows that for all $M > 0$ there exists a constant $C_M > 0$ such that

$$F(u) \geq M|u|^2 - C_M|u|, \qquad \forall u \in \mathbb{R}. \tag{29}$$

Let $R > 1$ and we consider the function φ_R defined on $[0, +\infty)$ by

$$\varphi_R = \frac{2\rho_*}{\|\psi_R\|}\psi_R, \tag{30}$$

where ψ_R is introduced in (11).
It is clear that $\varphi_R \in H$ and by a straightforward computation we get

$$\|\varphi_R\| = 2\rho_* \qquad \text{and} \qquad \|\varphi_R'\|_2^2 = \frac{8{\rho_*}^2}{\|\psi_R\|^2R^3}, \tag{31}$$

which implies that $\varphi_R \in H\backslash\overline{B}_{\rho_*}$ for all $R > 1$.
Moreover, by (11), (12), (29), (30) and (31), one has

$$\begin{aligned} I_b(\varphi_R) &= \frac{1}{2}\|\varphi_R\|^2 + \frac{b}{4}\|\varphi_R'\|_2^4 - \int_0^{+\infty} q(x)F(\varphi_R)dx \\ &\leq 2\rho_*^2 + \frac{16b{\rho_*}^4}{\|\psi_R\|^4R^6} - M\int_0^{+\infty} q(x)\varphi_R^2(x)dx + C_M\int_0^{+\infty} q(x)\varphi_R(x)dx \\ &\leq 2\rho_*^2 + \frac{4b}{a^2}{\rho_*}^4 - M\int_0^{+\infty} q(x)\varphi_R^2(x)dx + 2C_M\|q\|_1\mu_\infty\rho_*, \end{aligned}$$

and then

$$I_b(\varphi_R) \leq h(\rho_*) - M \int_0^{+\infty} q(x)\varphi_R^2(x)dx,$$

where $h(\rho_*) = 2\rho_*^2 + \dfrac{4b}{a^2}\rho_*{}^4 + 2C_M\|q\|_1\mu_\infty\rho_* > 0$.

By choosing $M > \dfrac{h(\rho_*)}{\int_0^{+\infty} q(x)\varphi_R^2(x)dx}$, we obtain

$$I_b(\varphi_R) < 0.$$

Thus, we complete the proof by taking $e = \varphi_R \in H\backslash\overline{B}_\rho$. □

Lemma 6. *Assume that (F_1)-(F_3) hold. Then the functional I_b satisfies the (PS) condition.*

Proof. Let $\{u_n\} \subset H$ be a Palais-Smale sequence at level $c \in \mathbb{R}$, namely satisfying (27). We easily see that there exists $C_1 > 0$ such that

$$|I_b(u_n)| \leq C_1 \qquad \text{and} \qquad \left|\langle I_b'(u_n), u_n\rangle\right| \leq C_1\|u_n\|, \qquad \forall n \in \mathbb{N}. \tag{32}$$

We divide the proof into two steps.

Step 1. We shall prove that $\{u_n\}$ is bounded in H.
Reasoning by contradiction, assume that the sequence $\{u_n\}$ is unbounded in H, that is

$$\|u_n\| \longrightarrow +\infty, \qquad \text{as} \qquad n \longrightarrow +\infty, \tag{33}$$

and set

$$\Omega_n = \left\{x \in (0, +\infty) : \ |u_n(x)| \leq L\right\} \qquad \text{and} \qquad \Omega_n' = (0, +\infty)\backslash\Omega_n.$$

From (F_4) and (32), through a direct computation we obtain

$$\begin{aligned}
C_1\left(1 + \frac{1}{4}\|u_n\|\right) &\geq I_b(u_n) - \frac{1}{4}\langle I_b'(u_n), u_n\rangle \\
&= \frac{1}{2}\|u_n\|^2 + \frac{b}{4}\|u_n'\|_2^4 - \int_0^{+\infty} q(x)F(u_n)dx - \frac{1}{4}\left(\|u_n\|^2 + b\|u_n'\|_2^4 - \int_0^{+\infty} q(x)f(u_n)u_n dx\right) \\
&= \frac{1}{4}\|u_n\|^2 + \frac{1}{4}\int_{\Omega_n} q(x)\left[f(u_n)u_n - 4F(u_n)\right]dx + \frac{1}{4}\int_{\Omega_n'} q(x)\left[f(u_n)u_n - 4F(u_n)\right]dx \\
&\geq \frac{1}{4}\|u_n\|^2 + \frac{1}{4}\int_{\Omega_n} q(x)\left[f(u_n)u_n - 4F(u_n)\right]dx,
\end{aligned}$$

which yields

$$C_1\left(\frac{1}{\|u_n\|^2} + \frac{1}{4\|u_n\|}\right) \geq \frac{1}{4} + \frac{1}{4\|u_n\|^2}\int_{\Omega_n} q(x)\left[f(u_n)u_n - 4F(u_n)\right]dx. \tag{34}$$

On the other hand, for $x \in \Omega_n$, by (F_1) and (33), it follows that

$$\begin{aligned}
q(x)\left|f(u_n)u_n - 4F(u_n)\right| &\leq q(x)\left(\alpha_2|u_n| + \beta|u|^\theta + 4\alpha_2|u_n| + \frac{4\beta}{\theta}|u_n|^\theta\right) \\
&\leq \left(5\alpha_2 L + \frac{(\theta+4)\beta}{\theta}L^\theta\right)q(x),
\end{aligned}$$

and then

$$\begin{aligned}
\frac{1}{\|u_n\|^2}\left|\int_{\Omega_n} q(x)f(u_n)u_n - 4F(u_n)dx\right| &\leq \frac{1}{\|u_n\|^2}\left(5\alpha_2 L + \frac{(\theta+4)\beta}{\theta}L^\theta\right)\int_{\Omega_n} q(x)dx \\
&\leq \frac{1}{\|u_n\|^2}\left(5\alpha_2 L + \frac{(\theta+4)\beta}{\theta}L^\theta\right)\|q\|_1 \longrightarrow 0,
\end{aligned}$$

as $n \longrightarrow +\infty$. Hence,

$$\frac{1}{\|u_n\|^2}\int_{\Omega_n} q(x)\,[f(u_n)u_n - 4F(u_n)]\,dx \longrightarrow 0. \tag{35}$$

Taking into account (33) and (35), by passing to the limit in (34) as $n \longrightarrow +\infty$, we obtain a contradiction. Consequently, $\{u_n\}$ is bounded in H and that what needs to be demonstrated.

Step 2. We will prove that $\{u_n\}$ converges strongly in H.
In Step 1, it can be seen that the sequence $\{u_n\}$ is bounded in H; then we may assume for a subsequence that

$$\begin{aligned} &u_n \rightharpoonup u_0 \;\text{ weakly in }\; H,\\ &u_n(x) \to u_0(x) \qquad \text{a.e in } (0,+\infty). \end{aligned} \tag{36}$$

An easy computation shows that

$$\begin{aligned} \|u_n-u_0\|^2 &= \langle I_b'(u_n) - I_b'(u_0), u_n-u_0\rangle - b\|u_n'\|_2^2\int_0^{+\infty} u_n'(u_n'-u_0')dx + b\|u_0'\|_2^2\int_0^{+\infty} u_0'(u_n'-u_0')dx\\ &\quad + \int_0^{+\infty} q(x)\Big(f(u_n)-f(u_0)\Big)(u_n-u_0)dx. \end{aligned} \tag{37}$$

By (36) and the continuity of f, it clear that for almost every $x \in (0,+\infty)$

$$\int_0^{+\infty} q(x)\Big(f(u_n(x)) - f(u_0(x))\Big)\big(u_n(x)-u_0(x)\big)\,dx \longrightarrow 0, \qquad \text{as}\quad n\longrightarrow +\infty.$$

Moreover, from (F_1), (3) and the boundedness of $\{u_n\}$ one has

$$\begin{aligned} \Big|q(x)\Big(f(u_n)-f(u_0)\Big)(u_n-u_0)\Big| &\le \left[2\alpha_2 + \beta|u_n|^{\theta-1} + \beta|u_0|^{\theta-1}\right](|u_n|+|u_0|)\,q(x)\\ &\le \left[2\alpha_2 + \beta\mu_\infty^{\theta-1}\Big(C^{\theta-1} + \|u_0\|^{\theta-1}\Big)\right](\mu_\infty C + \mu_\infty\|u_0\|)\,q(x)\\ &\le Cq(x) \in L^1(0,+\infty). \end{aligned}$$

By the dominated convergence theorem we get

$$\int_0^{+\infty} q(x)\Big(f(u_n)-f(u_0)\Big)(u_n-u_0)\,dx \longrightarrow 0, \qquad \text{as}\quad n\longrightarrow +\infty. \tag{38}$$

Taking into account (32) and the fact that $u_n \rightharpoonup u_0$ in H, we get

$$\langle I'(u_n) - I'(u_0), u_n-u_0\rangle \longrightarrow 0, \qquad \text{as } n\longrightarrow +\infty. \tag{39}$$

Combining (38) and (39) with (37), by the same reasoning as in (19), we prove that

$$\begin{aligned} o_n(1) &= \|u_n-u_0\|^2 + b\|u_n'\|_2^2\int_0^{+\infty} u_n'(u_n'-u_0')dx - b\|u_0'\|_2^2\int_0^{+\infty} u_0'(u_n'-u_0')dx\\ &\ge \|u_n-u_0\|^2 + \frac{b}{2}\Big(\|u_n'\|_2^2 - \|u_0'\|_2^2\Big)^2, \end{aligned} \tag{40}$$

which yields

$$u_n \longrightarrow u_0 \text{ strongly in } H,$$

and this proves that I_b satisfies the (PS) condition at any level $c\in\mathbb{R}$. □

4.2. Proof of the main result

In this subsection, we will give the proof of Theorem 4 which is divided into two steps. The first step refers to the existence of solutions of $(\mathcal{P}_b)$, and the second one to the study of the asymptotic behavior of solutions by considering b as a parameter.

Proof of Theorem 4. **Step 1.** We have $I_b \in C^1(H, \mathbb{R})$ and $I_b(0) = 0$. By Lemmas 4 and 5, the functional I_b satisfies the geometric property of the mountain pass theorem. Lemma 6 implies that the functional I_b satisfies the (PS) condition. Therefore, applying the mountain pass theorem, we deduce that there exists $v_0 \in H$ such that

$$I_b(v_0) = c \geq \alpha_* > 0 \text{ and } I_b'(v_0) = 0,$$

which means that v_0 is a weak solution of $(\mathcal{P}_b)$, and this completes Step 1.

Step 2. Let $\{b_n\} \subset (0, +\infty)$ be a sequence such that

$$b_n \longrightarrow 0, \qquad \text{as} \quad n \longrightarrow +\infty, \tag{41}$$

and let $u_n := u_{b_n} \in H$ be a solution of $(\mathcal{P}_b)$. Then

$$(u_n, v) + b_n \|u_n'\|_2^2 \int_0^{+\infty} u_n' v' \, dx - \int_0^{+\infty} q(x) f(u_n) v \, dx = 0, \qquad \forall v \in H. \tag{42}$$

In the same way as in Step 1 in the proof of Lemma 6, we can prove that $\{u_n\}$ is bounded in H, and then there exists $u_0 \in H$ such that up to a subsequence

$$\begin{array}{lll} u_n \rightharpoonup u_0 & \text{weakly in} & H, \\ u_n' \rightharpoonup u_0' & \text{weakly in} & L^2(0, +\infty), \\ u_n(x) \rightarrow u_0(x) & & \text{a.e in } (0, +\infty). \end{array} \tag{43}$$

Similarly to (40), we show that

$$u_n \longrightarrow u_0 \ \text{ in } \ H. \tag{44}$$

Hence, from (F_1), (41), (43), (44), and by the dominated convergence theorem we get

$$(u_n, v) \longrightarrow (u_0, v), \ \ b_n \|u_n'\|_2^2 \int_0^{+\infty} u_n' v' \, dx \longrightarrow 0$$

and

$$\int_0^{+\infty} q(x) f(u_n) v \, dx \longrightarrow \int_0^{+\infty} q(x) f(u_0) v \, dx,$$

as $n \longrightarrow +\infty$. Consequently, by passing to the limit in (42) as $n \longrightarrow +\infty$, we obtain

$$(u_0, v) - \int_0^{+\infty} q(x) f(u_0) v \, dx = 0, \qquad \forall v \in H,$$

which means that u_0 is a weak solution of the problem $(\mathcal{P}_0)$, and this completes the proof. □

4.3. Example

Let $f(u) = u^3 \ln(1 + u^2) + \frac{u^5}{2(1+u^2)}$, $p(x) = \ln(1 + x^2) + 1$ and $q(x) = e^{-x}$. Is is easy to check that (P), (Q) and (F_1)-(F_4) are satisfied.

Then, by Theorem 4, the problem

$$\begin{cases} -\left(a + b \displaystyle\int_0^{+\infty} |u'| dx\right) u'' + \left(\ln(1 + x^2) + 1\right) u = e^{-x} \left(u^3 \ln(1 + u^2) + \dfrac{u^5}{2(1 + u^2)}\right), & x \in (0, +\infty); \\ u(0) = 0. \end{cases}$$

has at least one nontrivial solution.

Acknowledgements

The authors would like to thank the handling editor and the anonymous referee for careful reading the manuscript and suggesting many valuable comments.

References

[1] C. Alves and F. Correa, On existence of solutions for a class of problem involving a nonlinear operator, *Comm. Appl. Nonlinear Anal.* **8** (2001), no. 2, 43–56.

[2] A. Ambrosetti and P. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Analysis* **14** (1973), 349–381.

[3] D. Arcoya, J. Carmona, and P. Martínez-Aparicio, Multiplicity of solutions for an elliptic Kirchhoff equation, *Milan J. Math.* **90** (2022), no. 2, 679–689.

[4] A. Arosio and S. Panizzi, On the well-posedness of Kirchhoff string, *Trans. Amer. Math. Soc.* **348** (1996), 305–330.

[5] M. Bełdziński and M. Galewski, On the existence and uniqueness of Dirichlet problems on a positive half-line, **4** (2019), no. 1, 55–69.

[6] D. Bouafia, T. Moussaoui, and D. O'Regan, Existence of solutions for a second order problem on the half-line via Ekeland's variational principle, *Discuss. Math. Differ. Incl. Control Optim.* **36** (2016), 131–140.

[7] H. Boulaiki, T. Moussaoui, and R. Precup, Multiple positive solutions for a second order boundary value problem on the half-line, *J. Nonlinear Funct. Anal.* (2017), 1–25.

[8] H. Boulaiki, T. Moussaoui, and R. Precup, Positive solutions for second-order differential equations of Kirchhoff type on the half-line, *Carpathian J. Math.* **37** (2021), no. 2, 325–338.

[9] B. Cheng, G. Li, and X. Tang, Nehari-type ground state solutions for Kirchhoff type problems in $\mathbb{R}^n$, *Appl. Anal.* **98** (2019), no. 7, 1255–1266.

[10] I. Ekeland, On the variational principle, *J. Math. Anal. Appl.* **47** (1974), 324–353.

[11] M. Furtado, E. Silva, and U. Severo, Kirchhoff elliptic problems with asymptotically linear or superlinear nonlinearities, *Z. Angew. Math. Phys.* **71** (2020), no. 186, 1–18.

[12] M. Galewski, T. Moussaoui, and I. Soufi, On the existence of solutions for a boundary value problem on the half-line, *Electron. J. Qual. Theory Differ. Equ.* **12** (2018), 1–12.

[13] S. Heidarkhani, G. Afrouzi, and S. Moradi, Existence results for a Kirchhoff-type second-order differential equation on the half-line with impulses, *Asymptot. Anal.* **105** (2017), no. 3-4, 137–158.

[14] N. Ikoma, Existence of ground state solutions to the nonlinear Kirchhoff type equations with potentials, *Discrete Contin. Dyn. Syst.* **35** (2015), no. 3, 943–966.

[15] S. Khoutir, Existence of ground state solution for a class of one-dimensional Kirchhoff-type equations with asymptotically cubic nonlinearities, *J. Elliptic Parabol. Equ.* **9** (2023), no. 1, 155–169.

[16] G. Kirchhoff, *Vorlesungen über mathematische Physik: Mechanik*, 3. Auflage ed., B.G. Teubner, Leipzig, 1883 (in German).

[17] J. Lions, On some questions in boundary value problems of mathematical physics, **30** (1978), 284–346.

[18] X. Meng and X. Zeng, Existence and asymptotic behavior of minimizers for the Kirchhoff functional with periodic potentials, *J. Math. Anal. Appl.* **507** (2022), no. 1, 125727.

[19] A. Pankov, Gap solitons in almost periodic one-dimensional structures, *Calc. Var. Partial Differential Equations* **54** (2015), 1963–1984.

[20] R. Precup and A. Stan, Stationary Kirchhoff equations and systems with reaction terms, **7** (2022), no. 8, 15258–15281.

[21] W. Rzymowski, One-dimensional Kirchhoff equation, *Nonlinear Anal.* **48** (2002), 209–221.

[22] Y. Sato, The existence and non-existence of solutions for the nonlinear Schrödinger equation in one dimension, *Nonlinear Anal. Real World Applications* **43** (2018), 477–494.

[23] E. Silva and J. Silva, Ground state solutions for asymptotically periodic nonlinearities for Kirchhoff problems, *Commun. Pure Appl. Anal.* **21** (2022), no. 9, 2933–2963.

[24] J. Sun and T. Wu, On the Kirchhoff type equations in $\mathbb{R}^n$, *Adv. Differential Equations* **27** (2022), no. 3-4, 97–146.

[25] M. Willem, *Minimax Theorems*, **24**, Birkhäuser, 1996.

[26] Y. Wu, Y. Huang, and Z. Liu, On a Kirchhoff type problem in $\mathbb{R}^N$, *J. Math. Anal. Appl.* **425** (2015), 548–564.