Math. Commun. **31** (2026), 1–22
DOI: 10.1000/100

Mathematical Communications

Boundedness and global asymptotic stability for a parabolic-elliptic-ODE chemotaxis-haptotaxis model with the remodeling of a non-diffusible attractant

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Abstract. In this paper, we take into account the multifarious impacts arising from the intricate interplay among chemotaxis, haptotaxis, sub-logistic growth patterns, and remodeling mechanisms on the global boundedness of solutions within a mathematical model. Initially devised by Chaplain and Lolas (2006) [5], this model stands as a potent instrument, illuminating the complex dynamics that unfolds between cancer cells, matrix-degrading enzymes, and the host tissue during the invasive process of cancer cells into the extracellular matrix. The model, outlined as follows, encapsulates a vast array of biological phenomena:

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) - \xi\nabla\cdot(u\nabla w) + f(u,w), \\ 0 = \Delta v - v + u, \\ w_t = -vw + \eta w(1-u-w). \end{cases} \qquad \text{in } \Omega \subset \mathbb{R}^2.$$

Here, Ω represents a generic bounded domain with a smooth boundary, while $f(u,w)$ encapsulates the proliferation and death of cancer cells, processes that are intricately intertwined with competition for space involving the extracellular matrix. The constants $\chi > 0$, $\xi > 0$, and $\eta > 0$ reflect various biological processes with nuance. A pivotal aspect of our exploration focuses on cell kinetics, which is meticulously described by a versatile class of sub-logistic source functions. As illustrative examples, we consider source functions such as $u\left(\kappa - w - \frac{\mu u}{\ln^\gamma(u+1)}\right)$ with $\gamma \in (0,1)$ and $u\left(1 - w - \frac{u}{\ln(\ln(u+e))}\right)$, offering nuanced perspectives into the dynamic growth of cancer cells.

In the context of this system, we establish the existence and boundedness of nonnegative solutions to the system, thereby significantly broadening the horizons of previous findings reported in [29, 33, 37, 38, 39]. Regarding the qualitative behavior of solutions, our work uncovers an explicit smallness condition on w_0 that ensures the exponential decay of w in the long-time limit, while u and v persist in a specific sense. Our findings contribute immensely to the comprehension of the complex mechanisms that govern cancer cell invasion, offering invaluable insights that could potentially inform the development of targeted therapeutic interventions.

AMS subject classifications: 92C17, 35K55, 35K59, 35K20
Keywords: global asymptotic stability; boundedness; chemotaxis-haptotaxis; parabolic-elliptic-ODE; remodeling of non-diffusible attractant
Received February 19, 2026; accepted March 15, 2026

1. Introduction

The phenomenon where cells migrate towards high concentrations of chemical signals is called chemotaxis. Responding to this natural phenomenon, Keller and Segel developed a landmark mathematical model in the 1970s [19]. This model is presented in a concise and insightful form as follows:

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v), & x\in\Omega, t>0, \\ \tau v_t = \Delta v - v + u, & x\in\Omega, t>0. \end{cases} \tag{1}$$

In this context, the parameter χ is greater than 0, and τ belongs to the set $\{0,1\}$. Meanwhile, u and v represent cell density and chemical concentration, respectively. Here Ω, as a subset of $\mathbb{R}^N (N \geq 1)$, is a bounded domain with a smooth boundary $\partial\Omega$. Over the past five decades, numerous researchers have delved

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into exploring the existence of global solutions to this system within finite or infinite time frames, as well as the possible occurrence of singularity phenomena [13, 50]. A salient feature of KS-type models is the potential for solutions to blow up in finite or infinite time, which strongly depends on the spatial dimension. To gain a more comprehensive understanding of this field, we have also referred to numerous important sources in the literature [1, 9, 12].

In diverse application scenarios, the characteristics of biological environments necessitate a meticulous consideration of cellular proliferation and death processes. To address this need, an adapted version of (1) takes the following form:

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + f(u), & x \in \Omega, t > 0, \\ \tau v_t = \Delta v + u - v, & x \in \Omega, t > 0. \end{cases} \tag{2}$$

Within this framework, any reasonable and non-trivial choice of the function f will profoundly influence the dynamic evolution of the total population size $\int_\Omega u(\cdot, t)$ and, notably, significantly diminish the energy dissipation property exhibited by the original (1) in (2). On the other hand, from a biological perspective, most meaningful choices of the function f embody the increase in mortality rates at high cellular densities, a phenomenon that is particularly evident in the classical logistic growth model and exemplified by:

$$f(u) = ru - \mu u^2, \quad u \geq 0,$$

where $\mu > 0$, and typically $r \geq 0$. This model accurately captures the rise in cellular mortality due to resource limitations and competitive pressures as cell density increases. Actually, for $N \leq 2$, any $\mu > 0$ is sufficient to exclude any blow-up, as referenced in [26, 27, 43]. A recent detailed study from [56] further demonstrates that even sub-logarithmic sources, such as $au - \frac{bu^2}{\ln^\gamma(u+1)}$ or $au - \frac{bu^2}{\ln(\ln(u+e))}$ (where $a \in \mathbb{R}, b > 0, \gamma \in (0, 1)$), can prevent chemotactic aggregation. These results suggest that for $N \leq 2$, the blow-up phenomenon in (1) can be completely avoided as long as a logistic or sub-logistic source is present, and in such cases, the blow-up phenomenon inherent in (1) disappears entirely. For $N \geq 3$, preventing blow-up in (2) through logistic sources becomes increasingly intricate and has been qualitatively and quantitatively explored in a series of works [49]. In summary, it is currently known that only appropriately strong logistic damping in (2) can prevent blow-up driven by chemotactic cross-diffusion in (1). More specifically, in the parabolic-elliptic case when $\tau = 0$, logistic damping outweighs chemotactic aggregation when $b \geq \frac{(N-2)}{N}\chi$ [43]. In the fully parabolic case when $\tau = 1$, the issue becomes even more nuanced: for $N \geq 4$, sufficiently strong logistic damping can prevent blow-up [49], while for $N = 3$ or in convex domains, explicit smallness of $\frac{\chi}{\mu}$ is available for boundedness and convergence [49]. We would add that in three-dimensional bounded, smooth, and convex domains, even though logistic damping guarantees the global existence of weak solutions [21], weak damping sources may fail to suppress blow-up in (1). Indeed, for $N \geq 3$, radially symmetric blow-up has been observed in a parabolic-elliptic simplification of (2) under a suitable sub-quadratic damping source [52]. Regarding a pivotal variant of equation (1), researchers have made remarkable breakthroughs in exploring its global existence and boundedness by innovatively adopting the nonlinear diffusion term Δu^m of porous media type, where m spans a wide range of values greater than 1. These ground breaking research achievements have been comprehensively and profoundly elaborated in authoritative literature such as [6, 8, 9, 32, 43, 58], providing invaluable references and inspirations for research in related fields.

In addition to the above models, in order to describe the process of cancer cells invading surrounding healthy tissues, Chaplain and Lolas ([4, 5]) proposed an important extension to the classical chemotaxis model, applying it to more complex cellular migration mechanisms. The specific form of this model is as follows:

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) - \xi \nabla \cdot (u \nabla w) + f(u, w), & x \in \Omega, t > 0, \\ \tau v_t = \Delta v + u - v, & x \in \Omega, t > 0, \\ w_t = -vw + \eta w(1 - u - w), & x \in \Omega, t > 0, \end{cases} \tag{3}$$

where the unknown variables $u(x, t)$, $v(x, t)$, and $w(x, t)$ represent the density of cancer cells, the concentration of matrix-degrading enzymes (MDEs), and the density of the extracellular matrix (ECM), respectively. In the context of equation (3), setting $\tau = 0$ is a justified simplification rooted in the evident disparity between the diffusion rates: the MDE diffuses at a significantly higher pace than cancer cells, as evidenced in [5] (see also [15, 30]). Here, the parameters $\chi > 0$ and $\xi > 0$ denote the sensitivities of chemotaxis and haptotaxis, respectively. The term $f(u, w)$ details the proliferation or death process of cancer cells, including spatial competition with ECM. The expression $-v$ accounts for the decay of MDE, while $+u$ represents their spontaneous production, and $-vw$ depicts the degradation of the ECM. When $\tau = 0$, it can be justified here by the evidence that the diffusion rate of MDEs is much faster than that of cancer cells [5]. The model, denoted as (3), has undergone extensive scrutiny and examination over the preceding years, as evidenced by numerous studies, including but not limited to [3, 17, 22, 23, 33, 34, 37, 38, 40, 41, 46, 48, 53, 54, 55, 59]. For instance, in 2008, Tao and Wang [34] conducted a pioneering study on the global solvability of classical solutions to equation (1.2), specifically focusing on the $\eta = 0$ scenario across dimensions 1, 2, and 3. Following this, Cao [3] and Tao [33] further contributed by establishing the uniform boundedness of these solutions. Expanding the scope, Tao and Winkler [37, 38] successfully demonstrated the global existence, uniform boundedness, and stability of solutions in N-dimensional spaces. If $N \leq 3$ and $f(u, w) = \mu u(1 - u - w)$ with $\mu > 0$, Tao and Winkler [41] showed that the solution to (3) converges to the constant stationary solution $(1, 1, 0)$ uniformly and exponentially under the condition $\mu > \frac{\chi^2}{8}$, while if $N \geq 3$ and $f(u, w) = u(a - \mu u^{r-1} - \lambda w)$ with $a \in \mathbb{R}$, $\mu > 0$, $\lambda \geq 0$ and $r > 1$, Zheng and Ke [59] also established the global boundedness and stability of the solution when μ is appropriately large and under other technical assumptions.

After incorporating the intricate interplay of chemotaxis, haptotaxis, and the reconstruction mechanism of ECM components ($\eta \neq 0$) into the model, the complex interactions among these three factors undoubtedly add a deeper level of complexity to this research topic. Notably, when the specific parameter $\eta = 0$, we can ingeniously construct a one-way pointwise estimate, thereby establishing a close relationship between w and v. In contrast, when $\eta > 0$, the strong coupling effects among u, v, and w in model (3) pose a series of technical challenges, making related research findings particularly scarce. And therefore, in the current realm of knowledge, research on this chemotaxis-haptotaxis model that integrates the reconstruction mechanism is still in its nascent stage, with relatively limited achievements.

The parabolic-parabolic-ODE types (i.e., $\tau = 1$ in (3)). When $f(u, w) = \mu u(1 - u - w)$ with $\mu > 0$, the global existence and uniqueness of a classical solution to model (3) were established by Pang and Wang [28] for **large** μ on $N = 2$. In [18] (see also [16, 29]), we discarded the critical assumption made in [28] that the coefficient μ needs to be sufficiently large, thereby enabling a broader scope of discussion or analysis.

The parabolic-elliptic-ODE types (i.e., $\tau = 0$ in (3)). When $f(u, w) = \mu u(1 - u - w)$ with $\mu > 0$, Tao and Winkler [39] proved that system (3) admits a unique globally classical solution to $\mu > 0$ on $N = 2$. In [29], the authors not only successfully extended the core findings in [39] but also conducted further in-depth exploration under the same conditions, deriving a significant conclusion of uniform boundedness of the solution to system (3), thereby achieving remarkable expansion and deepening at the theoretical level.

This discovery prompts us to consider the following insightful and crucial question:

(Q) Given that the introduction of a logistic source in a two-dimensional environment is sufficient to prevent the tendency of explosion among chemotaxis-haptotaxis models, we are intrigued to know if the mere addition of a sub-logistic source would also suffice in preventing explosion phenomena in model (3). Notably, under the specific scenario where $\eta \equiv 0$, unlike the strictly required logistic source in (3), some implicit settings with a smaller initial mass $\int_\Omega u_0$ or weaker forms of sub-logistic sources, such as

$$\kappa u - \frac{\mu u^2}{\ln^\gamma(u+1)} \quad \text{with} \quad \gamma \in (0, 1)$$

or

$$u\left(1 - w - \frac{u}{\ln(\ln(u+e))}\right)$$

have been proven to ensure that the solution to (3) remains bounded in two-dimensional space (see Xiang-Zheng [57]). This finding not only broadens our understanding of model stability but also provides new perspectives and avenues for future research.

In addition to the achievements and papers explicitly listed above, [16, 29, 31, 60] and the references cited therein contain richer results and studies for readers to further explore and refer to. It is particularly noteworthy that in [60] Zheng and Ke investigated a high-dimensional haptotaxis system describing oncolytic virotherapy, focusing on the boundedness and long-time behavior of solutions in a bounded domain with zero-flux boundary conditions. Under relatively mild conditions on the parameters, they established for the first time the existence and uniform boundedness of global classical solutions to the system in three-dimensional space. Furthermore, they studied the asymptotic decay behavior of solutions when the coefficient of the cell proliferation term is zero, and proved that the solutions converge to the zero equilibrium at an algebraic rate.

Motivated by the above papers, the primary objective of this endeavor is to establish the intricate interplay between chemotaxis, haptotaxis, sub-logistic growth patterns, and remodeling mechanisms, and successfully establish the chemotaxis-haptotaxis system (3) that incorporates the remodeling characteristics of the ECM. Thus, we embark on analyzing the system given by

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) - \xi \nabla \cdot (u \nabla w) + f(u, w), & x \in \Omega, t > 0, \\ 0 = \Delta v + u - v, & x \in \Omega, t > 0, \\ w_t = -vw + \eta w(1 - u - w), & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} - \chi u \frac{\partial v}{\partial \nu} - \xi \frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x), w(x, 0) = w_0(x), & x \in \Omega, \end{cases} \tag{4}$$

where $\Omega \subset \mathbb{R}^2$ represents the physical domain assumed to be bounded and possess a smooth boundary, χ as well as ξ are predefined positive parameters, and where $\frac{\partial}{\partial \nu}$ represents the outward normal derivative on the boundary $\partial\Omega$. Additionally, for the sake of convenience, throughout this paper we assume that the initial data (u_0, w_0) satisfy

$$\begin{cases} u_0 \in C^{2+\vartheta}(\bar{\Omega}) \text{ with } u_0 > 0 \text{ in } \Omega \text{ and } \dfrac{\partial u_0}{\partial \nu} = 0 \text{ on } \partial\Omega, \\ w_0 \in C^{2+\vartheta}(\bar{\Omega}) \text{ with } w_0 > 0 \text{ in } \Omega \text{ and } \dfrac{\partial w_0}{\partial \nu} = 0 \text{ on } \partial\Omega, \end{cases} \tag{5}$$

with some $\vartheta \in (0, 1)$.

In the aforementioned frameworks, we can ascertain the existence of globally bounded classical solutions to system (4), accompanied by the initial data (5), provided that the initial mass $\|u_0\|_{L^1(\Omega)}$ or χ is below a certain prescribed threshold. For clarity and to reinforce our argument, let us revisit the Gagliardo-Nirenberg inequality in the two-dimensional setting:

$$\|\nabla \phi\|^4_{L^4(\Omega)} \le C^4_{GN} \left(\|\Delta \phi\|^2_{L^2(\Omega)} \|\phi\|^2_{L^\infty(\Omega)} + \|\phi\|^4_{L^\infty(\Omega)} \right), \quad \text{for all } \phi \in W^{2,2}(\Omega) \cap L^\infty(\Omega) \tag{6}$$

and

$$\|\phi\|^4_{L^4(\Omega)} \le C_{GN,*} \|\nabla \phi\|^2_{L^2(\Omega)} \|\phi\|^2_{L^2(\Omega)} + C_{GN,**} \|\phi\|^4_{L^2(\Omega)}, \quad \text{for all } \phi \in W^{1,2}(\Omega),$$

where C_{GN} as well as $C_{GN,*}$ and $C_{GN,**}$ are some positive constants depending only on Ω.

Suppose $g \in L^{\frac{4}{3}}((0, T); L^{\frac{4}{3}}(\Omega))$, where g is a function defined on the spatio-temporal domain $\Omega \times (0, T)$. We consider v to be a solution to the following initial boundary value problem:

$$\begin{cases} -\Delta v + v = g, & (x, t) \in \Omega \times (0, T), \\ \frac{\partial v}{\partial \nu} = 0, & (x, t) \in \partial\Omega \times (0, T). \end{cases}$$

Drawing upon the embedding property $W^{2,\frac{4}{3}}(\Omega) \hookrightarrow W^{1,4}(\Omega)$ in two spatial dimensions and leveraging the elliptic L^p estimates (refer, for instance, to Section 19 of Part 1 in [7]), we can derive the following inequality with positive constants α_1 and $\alpha_2 > 0$:

$$\|\nabla v\|^2_{L^4(\Omega)} \le \alpha_1 \|v\|^2_{W^{2,\frac{4}{3}}(\Omega)} \le \alpha_2 \|g\|^2_{L^{\frac{4}{3}}(\Omega)}. \tag{7}$$

This inequality bounds the L^4-norm of the gradient of v in terms of the norm of g in $L^{\frac{4}{3}}(\Omega)$ through the intermediate norm of v in $W^{2,\frac{4}{3}}(\Omega)$.

Within the framework of these presuppositions, our primary findings affirm the global existence and boundedness of solutions within a specified domain, in the following nuanced manner:

Theorem 1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a smooth boundary, let χ and ξ be positive constants, and let the initial data (u_0, w_0) be such that hypotheses (5) hold. Besides, assume that the locally bounded source function f is nonnegative at the origin for any w, and that it further adheres to*

$$\text{either } f \equiv 0 \text{ or } \left\{\exists\, r \in \mathbb{R}, b > 0 \text{ s.t. } f(s,w) \le r - bs \text{ on } (0,+\infty) \times (0, \max_{x\in\bar{\Omega}} w_0(x))\right\}. \tag{8}$$

More importantly, we introduce the notion of an extended asymptotic damping rate μ defined as

$$\mu = \liminf_{s\to+\infty} \left\{ \inf_{0\le w\le \max_{x\in\bar{\Omega}} w_0(x)} \left\{ -f(s,w)\frac{\ln s}{s^2} \right\} \right\}, \tag{9}$$

and we also suppose that the following conditions:

$$\begin{cases} \mu > \frac{\chi^2\alpha_2}{2} e^{\xi \max\{1, \|w_0\|_{L^\infty(\Omega)}\}} M_1 + \xi\eta e^{2\xi \max\{1,\|w_0\|_{L^\infty(\Omega)}\}} & \text{if } f \equiv 0, \\ \frac{1}{2C_{GN}M_1} > \frac{\chi^2\alpha_2}{2} e^{\xi \max\{1,\|w_0\|_{L^\infty(\Omega)}\}} M_1 + \xi\eta e^{2\xi \max\{1,\|w_0\|_{L^\infty(\Omega)}\}} & \text{if } f \not\equiv 0, \end{cases} \tag{10}$$

are valid with M_1 being a finite quantity defined as

$$M_1 = \|u_0\|_{L^1(\Omega)} + |\Omega| \begin{cases} 0 & \text{if } f \equiv 0, \\ \inf_{\varepsilon\in(0,b]} \frac{\sup\{f(s,w)+\varepsilon s:\ (s,w)\in(0,\infty)\times(0,\max_{x\in\bar{\Omega}} w_0(x))\}}{\varepsilon} & \text{if } f \not\equiv 0, \end{cases} \tag{11}$$

where C_{GN} denotes the Gagliardo-Nirenberg constant (see (6)), while α_2 is specified in (7). Under these assumptions, there exists a unique nonnegative solution triple $(u, v, w) \in (C^{2,1}(\bar{\Omega} \times (0,\infty)))^3$ that solves system (3) in the classical sense. Furthermore, the functions u, v, and w exhibit uniform boundedness in the precise manner that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|w(\cdot,t)\|_{L^\infty(\Omega)} \le C, \quad \text{for all } t > 0,$$

where $C = C(u_0, w_0, |\Omega|, f, \chi, \xi) > 0$ is a constant that remains independent of the time variable t, ensuring a uniform bound across the entire temporal domain.

Ahead of the discussion, it is pertinent to make a few preliminary observations or remarks.

Remark 1. (i) In the case where $\eta \equiv 0$, it becomes readily apparent that under the premises stipulated in Theorem 1, the solutions for $N = 2$ are uniformly bounded, thereby enriching and extending the insights of Osaki et al. ([26]).

(ii) Our research endeavors have comprehensively addressed and resolved the lingering queries raised in [1] and [16], offering robust evidence and profound insights crucial for surmounting pivotal challenges within this domain.

(iii) When $w \equiv 0$, simplifying the PDE system (4) into a chemotaxis-centric model, actually given the conditions outlined in Theorem 1, the solutions remain uniformly bounded in two dimensions. This observation harmoniously aligns with the findings reported by Winkler ([49]).

(iv) For logistic and super-logistic source terms exemplified by $f(u,w) = u(a - bu^\theta - w)$, where $a \in \mathbb{R}, b > 0$, and $\theta \ge 2$, alongside sub-logistic sources formulated as $f(u,w) = u\left(a - w - \frac{bu}{\ln^\gamma(s+1)}\right)$ with $a \in \mathbb{R}, b > 0, \gamma \in (0,1)$, a straightforward application of (9) unveils that $\mu = +\infty$, trivially fulfilling (10). This pivotal discovery ensures the global boundedness of (4) in two-dimensional settings for all physically pertinent initial data, underscoring how Theorem 1 augments existing knowledge by broadening the realm of 2D global existence and boundedness from logistic to encompass sub-logistic sources, surpassing previous endeavors (cf. [33, 34, 35, 37, 38, 41]).

(v) In the exclusive chemotaxis realm, where $w \equiv 0$, Theorem 1 elevates the outcomes presented in [56, Theorem 1.1], reinforcing their significance.

(vi) Harnessing Theorem 1, we deduce that for any $\eta = 0$ and a sufficiently substantial $\int_\Omega u_0(x)$, the solutions to model (4) exhibit both global existence and boundedness, resonating with the conclusions drawn by [56].

(vii) To the best of our knowledge, these revelations constitute the inaugural results pertaining to the boundedness of the system, marking a significant milestone in the field.

(viii) It is imperative to emphasize that the research methodologies adopted in [3, 49] do not cater to the intricacies of the current model, necessitating an urgent pursuit of innovative research pathways to unravel and comprehend the model's intricate characteristics and behaviors.

(ix) Shifting focus to the haptotaxis-only scenario with $\chi = 0$, (10) automatically holds true, ensuring the boundedness of classical solutions to (4), irrespective of growth source presence, for sufficiently ample initial data. This achievement transcends previous global existence findings reported in [24, 42, 45].

Large time behavior of the chemotaxis-haptotaxis model by remodeling a non-diffusible attractant. The second core objective aims to establish a rational hypothesis framework to delve into the dynamic behavioral characteristics of these solutions over long time scales. Given that the Keller-Segel-growth system (2) has exhibited extremely complex dynamic evolution patterns through numerical simulations (see [10]) and equilibrium set-related analyses (see [20]), and that we do not anticipate the introduction of chemotaxis interactions and re-establishment of ECM components to bring about any form of regularization effects, we will specifically focus our research on the scenario where $f(u) = ru - \mu u^2$ with $\mu > 0$ and its value is sufficiently large. This decision is grounded on solid previous research, which indicates that when the value of μ increases significantly, the non-trivial equilibrium states of system (2) without chemotaxis exhibit unique uniqueness and global attractivity (see [51]). This discovery provides invaluable guidance and profound insights for our exploration. Furthermore, it is worth mentioning that in [1], Tao and Winkler proposed an important assertion: under the condition $N \le 3$, if (u, v, w) is a bounded global classical solution to equation (4) with $f(u, w) = \mu u(1 - u - w)$, then when $\mu > \frac{\chi^2}{8}$, (u, v, w) will asymptotically converge to the stable state $(1, 1, 0)$ in an exponential manner. Another noteworthy research achievement comes from [51], which states that in the chemotaxis subsystem of equation (2), if the ratio of $\frac{\mu}{\chi}$ is sufficiently large and the domain of investigation is convex, all solutions will approach the specific state $(1, 1)$ at an exponential rate. This conclusion also applies to models without self-repair terms, and relevant details can be found in references [47, 59]. However, it must be emphasized that due to the presence of re-establishment of ECM components, acquiring the initial spatial regularity, final regularity, and even uniform temporal regularity with respect to the ∇w of the solution component w in the third equation becomes exceptionally challenging. This directly leads to the establishment of spatially and temporally uniform boundedness of the solution component w in models with self-repair terms becoming highly demanding, which is crucial in analyzing the long-time asymptotic behavior of solutions. To the best of our knowledge, research on the long-time asymptotic behavior of such models is still in its infancy. Therefore, our research findings fill an important knowledge gap in this field.

Theorem 2. *Let $\chi > 0, \xi > 0$ and $f(u, w) = u(r - \mu u - w)$ with $\mu > 0$ being the same as Theorem 1. Moreover, suppose that the initial data (u_0, w_0) satisfy (5) and*

$$\eta \in (0, C_*),$$

where

$$\begin{aligned} C_* =& r - (1 + \xi\eta) e^{\xi \|w_0\|_{L^\infty(\Omega)}} \|w_0\|_{L^\infty(\Omega)} - \frac{3r^2}{2\mu^2} \left[\frac{\chi^2}{8} e^{\xi \|w_0\|_{L^\infty(\Omega)}} + 1 \right. \\ & \left. + \|w_0\|^2_{L^\infty(\Omega)} \left(\frac{\xi^2}{4} e^{2\xi \|w_0\|_{L^\infty(\Omega)}} + \xi\eta \right) + \frac{\mu\xi\eta}{r} e^{\xi \|w_0\|_{L^\infty(\Omega)}} (\|w_0\|_{L^\infty(\Omega)} + 1) \right]. \end{aligned} \tag{12}$$

Then the corresponding global-in-time solution to (4) has the properties

$$\limsup_{t\to\infty} \int_\Omega u(x,t)dx > 0 \quad \text{and} \quad \limsup_{t\to\infty} \inf_{x\in\Omega} v(x,t) > 0,$$

as well as

$$\|w(\cdot,t)\|_{L^\infty(\Omega)} \le C e^{-\gamma t}, \quad \text{for all } t > 0,$$

with some positive constants γ and C.

Before delving deeply into this topic, it is imperative to present a series of closely related and insightful preliminary opinions and profound comments, which will undoubtedly lay a solid foundation for our discussion.

Remark 2. (i) As far as we know, our findings constitute the inaugural investigation into the large-time dynamics within a chemotaxis-haptotaxis model that incorporates the remodeling of non-diffusible attractants.

(ii) The meticulous conditions stipulated in Theorem 2, wherein $w_0$is required to be small and r to be large, play a pivotal role in maintaining w at a relatively subdued level, under a specific interpretation. Through a thorough examination of the functional $\int_\Omega e^{\xi w} \ln a$ within the crucial Lemma 6, we demonstrate that hypothesis (12) not only furnishes a desirable lower bound for $\int_{t_2}^t \int_\Omega e^{2\xi w} a(\cdot, s)ds$, but this lower bound serves as a cornerstone for subsequent analytical endeavors. Moreover, it enables the derivation of the temporal decay characteristic of the term $e^{-\int_0^t v(x,s)ds}$, which, coupled with the explicit formulations of w (refer to (50)), directly signifies the asymptotic extinction of w under the conditions outlined in Theorem 2.

2. Preliminaries

Prior to establishing our core findings, we commence by revisiting several fundamental lemmas that serve as the cornerstone in our subsequent analyses throughout this paper. Notably, the Gagliardo-Nirenberg inequality emerges as a pivotal instrument, anticipated to be invoked frequently in the upcoming proofs. Readers desiring a detailed derivation of this lemma are referred to the seminal work by Nirenberg and associates [25], with additional insights also available in [44].

Lemma 1. *Let $\theta \in (0, p)$. There exists a positive constant C_{GN} such that for all $u \in W^{1,2}(\Omega) \cap L^\theta(\Omega)$; then*

$$\|u\|_{L^p(\Omega)} \le C_{GN} \left(\|\nabla u\|^a_{L^2(\Omega)} \|u\|^{1-a}_{L^\theta(\Omega)} + \|u\|_{L^\theta(\Omega)} \right)$$

is valid with $a = \frac{\frac{N}{\theta} - \frac{N}{p}}{1 - \frac{N}{2} + \frac{N}{\theta}} \in (0, 1)$.

In certain aspects of our subsequent analysis, we introduce a significant variable transformation (as referenced in the works of Tao et al. [35, 36, 39] and Pang-Wang [28]), given by:

$$a = ue^{-\xi w},$$

which alters the form of (4) by transforming it into the following comprehensive system of partial differential equations:

$$\begin{cases} a_t = e^{-\xi w} \nabla \cdot (e^{\xi w} \nabla a) - \chi e^{-\xi w} \nabla \cdot (e^{\xi w} a \nabla v) + \xi a v w & \\ \quad -a\xi\eta w(1 - e^{\xi w} a - w) + f(ae^{\xi w}, w), & x \in \Omega, t > 0, \\ 0 = \Delta v + ae^{\xi w} - v, & x \in \Omega, t > 0, \\ w_t = -vw + \eta w(1 - ae^{\xi w} - w), & x \in \Omega, t > 0, \\ \frac{\partial a}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ a(x, 0) := a_0(x) = u_0(x) e^{-\xi w_0(x)}, w(x, 0) = w_0(x), & x \in \Omega, \end{cases} \tag{13}$$

which elegantly encapsulates the intricate, dynamic interplay among the variables a, v, and w within the spatial domain Ω across time t, all while rigorously adhering to the stipulated boundary and initial conditions.

Afterward, the next lemma concerns the theme of the local-in-time existence and uniqueness of a classical solution pertaining to problem (4), a subject that has been extensively expounded upon in esteemed literature, including [28, 29, 39].

Lemma 2. *Assume that the nonnegative initial data u_0 and w_0 satisfy the conditions stated in (5) for some $\vartheta \in (0, 1)$. Then there exist a maximal existence time $T_{max} \in (0, \infty]$ and a uniquely determined triple of nonnegative functions*

$$\begin{cases} a \in C^0(\bar{\Omega} \times [0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max})), \\ v \in C^0(\bar{\Omega} \times [0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max})), \\ w \in C^{2,1}(\bar{\Omega} \times (0, T_{max})), \end{cases}$$

which solves system (13) *classically and satisfies*

$$0 \le w \le \rho := \max\{1, \|w_0\|_{L^\infty(\Omega)}\} \quad \text{in } \Omega \times (0, T_{max}).$$

Moreover, if $T_{max} < +\infty$, *then*

$$\|a(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|\nabla w(\cdot,t)\|_{L^4(\Omega)} \to \infty \quad \text{as } t \nearrow T_{max}. \tag{14}$$

In our subsequent analysis, we will harness the following auxiliary statement related to the boundedness property within an ordinary differential inequality, thereby enriching our understanding and facilitating a more comprehensive examination.

Lemma 3. *Let* $T > 0$, $\iota \in (0,T)$, $A > 0$ *as well as* $\alpha > 0$ *and* $B > 0$, *and suppose that* $y : [0,T) \to [0,\infty)$ *is absolutely continuous such that*

$$y'(t) + Ay^\alpha(t) \le h(t), \quad \text{for all } t \in (0,T),$$

with some nonnegative function $h \in L^1_{loc}([0,T))$ *satisfying*

$$\int_t^{t+\iota} h(s)ds \le B, \quad \text{for all } t \in (0, T-\tau).$$

Then

$$y(t) \le \max\left\{y_0 + B, \frac{1}{\iota}\left(\frac{B}{A}\right)^{\frac{1}{\alpha}} + 2B\right\}, \quad \text{for all } t \in (0,T).$$

3. Existence and boundedness

This section is devoted to the meticulous and exhaustive verification of Theorem 1. Unless otherwise specifically noted, we will operate under the presumption that all the prerequisites, as stipulated in Lemma 2 and Theorem 1, have been duly fulfilled. In keeping with the established conventions, we commence our analysis by scrutinizing the L^1-norm of u, thereby navigating towards the subsequent, pivotal lemma, which stands as a cornerstone in our exposition.

Lemma 4. *Let* $\chi > 0, \xi > 0$ *as well as* $\eta > 0$. *Then the solution to system* (13) *conforms to*

$$u > 0, \; v > 0 \quad \text{and} \quad 0 \le w(x,t) \le \rho := \max\{1, \|w_0\|_{L^\infty(\Omega)}\}, \quad \text{for all } (x,t) \in \Omega \times (0, T_{max}).$$

More importantly, the first component u *of solutions fulfills the properties that for all* $t \in (0, T_{max})$,

$$\|u(\cdot,t)\|_{L^1(\Omega)} \begin{cases} = M_1 & \text{if } f \equiv 0, \\ \le M_1 & \text{if } f \not\equiv 0, \end{cases} \tag{15}$$

where M_1 *is the same as* (11).

Proof. By reason of $w_0 > 0$, from the explicit solution representation or, alternatively, two straightforward ODE comparison arguments we obtain that

$$0 \le w(x,t) \le \rho = \max\{1, \|w_0\|_{L^\infty(\Omega)}\}, \quad \text{for all } (x,t) \in \Omega \times (0, T_{max}),$$

whereas in light of the parabolic strong maximum principles, it is not difficult to check

$$u > 0 \quad \text{and} \quad v > 0, \quad \text{for all } (x,t) \in \Omega \times (0, T_{max}).$$

In terms of claim (15), we only need to prove it for the setting $f \not\equiv 0$, since given the circumstance that $f \equiv 0$, the confirmation of the conclusion appears to be more intuitive and simpler. To this end, through a direct integration for both sides of the first equation in (3), we take advantage of the homogeneous Neumann boundary conditions to derive a Gronwall inequality, namely, for any $\varepsilon > 0$,

$$\frac{d}{dt}\int_\Omega u = \int_\Omega f(u,w) \le -\varepsilon \int_\Omega u + M_\varepsilon |\Omega|,$$

evidently giving rise to

$$\int_\Omega u \le \int_\Omega u_0 + \frac{M_\varepsilon}{\varepsilon}|\Omega|,$$

which upon taking the infimum over $\varepsilon \in (0,1)$, and recalling the definition of M in (11), implies the L^1-bound of u as stated in (15). Here, due to (8),

$$M_\varepsilon = \sup\Big\{ f(s,w) + \varepsilon s : \ (s,w) \in (0,\infty) \times (0, \max_{x\in\bar\Omega} w_0(x))\Big\} < \infty.$$

Actually, the definition of μ provided by (9) results in

$$\exists\, \hat{s} \gg 1 \ \text{ s.t. } \ f(s,w) \le -\hat{\mu}\frac{s^2}{\ln s}, \quad \text{for all } s \ge \hat{s} \text{ and } w \in (0, \max_{x\in\bar\Omega} w_0(x)),$$

where $\hat{\mu} = \min\{2, \frac{\mu}{2}\}$. This entails

$$f(s,w) + \varepsilon s \le -\hat{\mu}\frac{s^2}{\ln s} + \varepsilon s < 0, \quad \text{for all } s \ge \hat{s} \text{ and } w \in (0, \max_{x\in\bar\Omega} w_0(x)),$$

whence, in conjunction with the evident fact that f is bounded on any finite interval, we would like to infer that M_ε is finite, and moreover, the assertion is carried out. □

As a direct consequence of Lemma 4, we now proceed to establish two crucial estimates that are pertinent to u.

Lemma 5. *Under the assumptions of Lemma 2, one may introduce a positive constant β_1 such that the solution to system (13) satisfies*

$$\int_\Omega v^{l_0} + \int_\Omega |\nabla v|^l \le \beta_1, \quad \textit{for all } t \in (0, T_{max}), \tag{16}$$

where $l_0 \in [1, +\infty)$ and $l \in [1,2)$.

Proof. According to (15), there exists a certain suitable constant $\alpha_0 > 0$ such that

$$\int_\Omega v(\cdot,t) \le \alpha_0, \quad \text{for all } t \in (0, T_{max}),$$

whereupon invoking the classical results by Brézis and Strauss ([2]) and the Minkowski inequality, this ascertains the existence of several positive constants C_1, C_2, and C_3, with the properties that for all $l \in [1,2)$,

$$\|v(\cdot,t)\|_{W^{1,l}(\Omega)} \le C_1\|\Delta v(\cdot,t) - v(\cdot,t)\|_{L^1(\Omega)} \le C_2\|u\|_{L^1(\Omega)} \le C_3, \quad \text{for all } t \in (0,T_{max}), \tag{17}$$

and then, gathering the above as well as the Sobolev embedding theorem together, we henceforth arrive at

$$\|v(\cdot,t)\|_{L^{l_0}(\Omega)} \le C_4 \quad \text{for all } t \in (0, T_{max}) \text{ and } l_0 \in (1, +\infty), \tag{18}$$

with C_4 being another positive constant, whereas (16) becomes an instant conclusion of (17)-(18). □

Drawing upon Lemmas 4 through 5, our initial conclusion yields the subsequent time-invariant bounds for a.

Lemma 6. *Let*

$$\begin{cases}\mu > \frac{\chi^2\alpha_2}{2}e^{\xi\rho}M_1 + \xi\eta e^{2\xi\rho} & \text{if } f\equiv 0,\\ \frac{1}{2C_{GN}M_1} > \frac{\chi^2\alpha_2}{2}e^{\xi\rho}M_1 + \xi\eta e^{2\xi\rho} & \text{if } f\not\equiv 0,\end{cases}$$

where M_1 originates from (11). Then a positive constant C can be found such that for any initial configurations satisfying (5), the corresponding classical solution (a, v, w) to system (4) complies with

$$\int_\Omega a(\cdot,t)|\ln a(\cdot,t)| \le C, \quad \text{for all } t\in(0,T_{max}). \tag{19}$$

Proof. Given the first equation of (13) implies that

$$(ae^{\xi w})_t = \nabla\cdot(e^{\xi w}\nabla a) - \chi\nabla\cdot(e^{\xi w}a\nabla v) + f(ae^{\xi w}, w),$$

it further follows from the integration by parts several times that

$$\begin{aligned}\frac{d}{dt}\int_\Omega e^{\xi w}a\ln a+\int_\Omega e^{\xi w}\frac{|\nabla a|^2}{a} &= \int_\Omega (e^{\xi w}a)_t\ln a + \int_\Omega e^{\xi w}a_t + \int_\Omega e^{\xi w}\frac{|\nabla a|^2}{a}\\ &=\chi\int_\Omega e^{\xi w}\nabla a\cdot\nabla v + \int_\Omega e^{\xi w}\left[f(ae^{\xi w},w) - \xi\eta aw(1-w-ae^{\xi w})\right]\\ &\quad+\int_\Omega e^{\xi w}\left[\ln a f(ae^{\xi w},w) + \xi avw\right]\\ &\le\frac{1}{2}\int_\Omega e^{\xi w}\frac{|\nabla a|^2}{a} + \frac{\chi^2}{2}\int_\Omega e^{\xi w}a|\nabla v|^2\\ &\quad+\int_\Omega e^{\xi w}[\ln a+1]f(ae^{\xi w},w) + \xi\eta\int_\Omega\left(a^2we^{2\xi w} + aw^2e^{\xi w}\right)\\ &\quad+\chi\int_\Omega ae^{\xi w}\xi vw,\end{aligned} \tag{20}$$

for all $t\in(0,T_{max})$, whence we next pay attention to absorbing the terms on the right-hand side of (20). In the integrals to be considered herein, observing by the Hölder inequality that

$$\frac{\chi^2}{2}\int_\Omega e^{\xi w}a|\nabla v|^2 \le \frac{\chi^2}{2}e^{\xi\rho}\|a\|_{L^2(\Omega)}\|\nabla v\|^2_{L^4(\Omega)}, \quad \text{for all } t\in(0,T_{max}), \tag{21}$$

(7) applies so as to allow for a choice of $\alpha_2 > 0$ such that

$$\|\nabla v\|^2_{L^4(\Omega)} \le \alpha_2\|u\|^2_{L^{\frac{4}{3}}(\Omega)}, \quad \text{for all } t\in(0,T_{max}), \tag{22}$$

which together with the interpolation inequality in L^p space domain contributes to

$$\|u\|^2_{L^{\frac{4}{3}}(\Omega)} \le \|u\|_{L^2(\Omega)}\|u\|_{L^1(\Omega)} \le \|u\|_{L^2(\Omega)}M_1, \quad \text{for all } t\in(0,T_{max}),$$

for M_1 chosen as in Lemma 4, and which upon being inserted into (22) yields that on the basis of (21) and 5, we obtain

$$\begin{aligned}\frac{\chi^2}{2}\int_\Omega e^{\xi w}a|\nabla v|^2 \le&\frac{\chi^2}{2}e^{\xi\rho}\|a\|_{L^2(\Omega)}\|\nabla v\|^2_{L^4(\Omega)}\\ \le&\frac{\chi^2\alpha_2}{2}e^{\xi\rho}\|a\|^2_{L^2(\Omega)}M_1, \quad \text{for all } t\in(0,T_{max}).\end{aligned} \tag{23}$$

A combination of (20) and (23) yields

$$\begin{aligned}\frac{d}{dt}\int_\Omega e^{\xi w}a\ln a + \frac{1}{2}\int_\Omega e^{\xi w}\frac{|\nabla a|^2}{a} \le& \int_\Omega e^{\xi w}[\ln a+1]f(ae^{\xi w},w) + \frac{\chi^2\alpha_2}{2}e^{\xi\rho}\|a\|^2_{L^2(\Omega)}M_1\\ &+\xi\eta\int_\Omega\left(a^2e^{2\xi w} + aw^2e^{\xi w}\right) + \chi\int_\Omega ae^{\xi w}\xi vw,\\ &\text{for all } t\in(0,T_{max}),\end{aligned} \tag{24}$$

where we intend to estimate the leftmost summand on (24) by virtue of Lemma 4 and (9). Actually, in accordance with the definition of μ as introduced by (9), for each small $\varepsilon \in (0, \mu - \frac{\chi^2}{\delta_1})$ and $\mu > \chi$, we may select a constant $s_\varepsilon > 1$ adhering to

$$e^{\xi w}[\ln s + 1] f(se^{\xi w}, w) \le -(\mu - \varepsilon)s^2, \quad \text{for all } (s, w) \in (s_\varepsilon, \infty) \times (0, \rho),$$

consequently promoting

$$\begin{aligned}
&\frac{\chi^2\alpha_2}{2} e^{\xi\rho} \|a\|^2_{L^2(\Omega)} M_1 + \xi\eta \int_\Omega a^2 e^{2\xi w} + \int_\Omega \left[e^{\xi w}[\ln a + 1] f(ae^{\xi w}, w) \right] \\
&\quad\le \frac{\chi^2\alpha_2}{2} e^{\xi\rho} \|a\|^2_{L^2(\Omega)} M_1 + \xi\eta e^{2\xi\rho} \int_\Omega a^2 + \int_\Omega \left[e^{\xi w}[\ln a + 1] f(ae^{\xi w}, w) \right] \\
&\quad= \int_{\{a \le s_\varepsilon\}} \left[\left(\frac{\chi^2\alpha_2}{2} e^{\xi\rho} M_1 + \xi\eta e^{2\xi\rho} \right) a^2 + e^{\xi w}[\ln a + 1] f(ae^{\xi w}, w) \right] \\
&\qquad+ \int_{\{a > s_\varepsilon\}} \left[\left(\frac{\chi^2\alpha_2}{2} e^{\xi\rho} M_1 + \xi\eta e^{2\xi\rho} \right) a^2 + e^{\xi w}[\ln a + 1] f(ae^{\xi w}, w) \right] \\
&\quad\le \sup_{0 < s < s_\varepsilon} \sup_{0 < w < \max_{x \in \bar\Omega} w_0(x)} \left[\left(\frac{\chi^2\alpha_2}{2} e^{\xi\rho} M_1 + \xi\eta e^{2\xi\rho} \right) a^2 + e^{\xi w}[\ln a + 1] f(ae^{\xi w}, w) \right] |\Omega| \\
&\qquad+ \left[\frac{\chi^2\alpha_2}{2} e^{\xi\rho} M_1 + \xi\eta e^{2\xi\rho} - (\mu - \varepsilon) \right] \int_\Omega a^2.
\end{aligned} \tag{25}$$

Putting (25) into (24) and fully utilizing of the Young inequality accompanied by Lemma 5, we can write

$$\frac{d}{dt} \int_\Omega e^{\xi w} a \ln a + \frac{1}{2} \int_\Omega e^{\xi w} \frac{|\nabla a|^2}{a} \le \frac{1}{2} \left[\frac{\chi^2\alpha_2}{2} e^{\xi\rho} M_1 + \xi\eta e^{2\xi\rho} - (\mu - \varepsilon) \right] \int_\Omega a^2 + C_1, \quad \text{for all } t \in (0, T_{max}). \tag{26}$$

Moving forward, we define the function $\varphi : [0, \infty) \to \mathbb{R}$ by letting

$$\varphi(z) := \begin{cases} z \ln z - \frac{1}{2} \left[\frac{\chi^2\alpha_2}{2} e^{\xi\rho} M_1 + \xi\eta e^{2\xi\rho} - (\mu - \varepsilon) \right] z^2 & \text{if } z > 0, \\ 0 & \text{if } z = 0 \end{cases}$$

satisfy

$$\frac{\varphi(z)}{z^2} \to -\frac{1}{2} \left[\frac{\chi^2\alpha_2}{2} e^{\xi\rho} M_1 + \xi\eta e^{2\xi\rho} - (\mu - \varepsilon) \right], \quad \text{as } z \to \infty,$$

so that for some $z_0 > 0$ we have $\varphi < 0$ on (z_0, ∞). Since φ apparently admits the continuous property on $[0, \infty)$, there exists a constant $C_2 > 0$ such that

$$-\frac{1}{2} \left[\frac{\chi^2\alpha_2}{2} e^{\xi\rho} M_1 + \xi\eta e^{2\xi\rho} - (\mu - \varepsilon) \right] \|a\|^2_{L^2(\Omega)} \ge \int_\Omega a \ln a - C_2. \tag{27}$$

Set

$$y(t) := \int_\Omega a \ln a, \quad \text{for all } t \in [0, T_{\max}),$$

whence going back to (26)-(27), this means that

$$y'(t) + y(t) \le C_1 + C_2, \quad \text{for all } t \in (0, T_{max}), \tag{28}$$

and thus, taking into account the standard ODE comparison argument, one may pick some positive constant C_3 obeying

$$\int_\Omega a(\cdot, t) \ln a(\cdot, t) \le C_3, \quad \text{for all } t \in (0, T_{max}). \tag{29}$$

Apart from that, the case when $f \equiv 0$ can be proven in a very similar and straightforward manner. Indeed, we deduce through (24) connected with $f \equiv 0$ that

$$\frac{d}{dt}\int_\Omega e^{\xi w} a\ln a + \frac{1}{2}\int_\Omega e^{\xi w}\frac{|\nabla a|^2}{a} \le \frac{\chi^2\alpha_2}{2}e^{\xi\rho}\|a\|^2_{L^2(\Omega)}M_1 + \xi\eta\int_\Omega\left(a^2e^{2\xi w} + aw^2e^{\xi w}\right) + \chi\int_\Omega ae^{\xi w}\xi vw \quad \text{for any } t\in(0,T_{max}), \tag{30}$$

whereby depending on Lemma 5, as well as the definition of a, the Gagliardo-Nirenberg inequality (see Lemma 1) becomes applicable to enable us to present a constant $C_{GN} > 0$ that fulfills

$$\begin{aligned}\int_\Omega a^2 =& \|a^{\frac12}\|^4_{L^4(\Omega)}\\ \le& C_{GN}\|\nabla a^{\frac12}\|^2_{L^2(\Omega)}\|a^{\frac12}\|^2_{L^2(\Omega)} + C_{GN}\|a^{\frac12}\|^4_{L^2(\Omega)}\\ =& C_{GN}\|\nabla a^{\frac12}\|^2_{L^2(\Omega)}\|a_0\|_{L^1(\Omega)} + C_{GN}\|a_0\|^2_{L^1(\Omega)}\\ \le& C_{GN}\|\nabla a^{\frac12}\|^2_{L^2(\Omega)}M_1 + C_{GN}M_1^2, \quad \text{for all } t\in(0,T_{max}),\end{aligned}$$

being particularly turned into

$$\frac12\int_\Omega e^{\xi w}\frac{|\nabla a|^2}{a} \ge \frac{1}{2C_{GN}M_1}\left(\int_\Omega a^2 - C_{GN}M_1^2\right), \quad \text{for all } t\in(0,T_{max}),$$

which in consideration of (30) implies

$$\begin{aligned}&\frac{d}{dt}\int_\Omega e^{\xi w}a\ln a + \frac{1}{2C_{GN}M_1}\left(\int_\Omega a^2 - C_{GN}M_1^2\right)\\ &\le \frac{\chi^2\alpha_2}{2}e^{\xi\rho}\|a\|^2_{L^2(\Omega)}M_1 + \xi\eta\int_\Omega\left(a^2e^{2\xi w} + aw^2e^{\xi w}\right) + \chi\int_\Omega ae^{\xi w}\xi vw,\\ &\quad \text{for all } t\in(0,T_{max}),\end{aligned}$$

and to proceed further, amalgamated with Lemma 5 and the Young inequality, simultaneously leads to

$$\begin{aligned}&\frac{d}{dt}\int_\Omega e^{\xi w}a\ln a + \left(\frac{1}{2C_{GN}M_1} - \frac{\chi^2\alpha_2}{2}e^{\xi\rho}M_1 - \xi\eta e^{2\xi\rho}\right)\int_\Omega a^2\\ &\le \xi\eta\rho^2e^{\xi\rho}\int_\Omega a + \chi e^{\xi\rho}\xi\rho\int_\Omega av + \frac12 M_1\\ &\le \xi\eta\rho^2e^{\xi\rho}\int_\Omega a + \frac12\left(\frac{1}{2C_{GN}M_1} - \frac{\chi^2\alpha_2}{2}e^{\xi\rho}M_1 - \xi\eta e^{2\xi\rho}\right)\int_\Omega a^2 + C_4\int_\Omega v^2 + \frac12 M_1\\ &\le \frac12\left(\frac{1}{2C_{GN}M_1} - \frac{\chi^2\alpha_2}{2}e^{\xi\rho}M_1 - \xi\eta e^{2\xi\rho}\right)\int_\Omega a^2 + C_5, \quad \text{for all } t\in(0,T_{max}).\end{aligned} \tag{31}$$

At this point, extending the analogous statements shown in (28) to the current content, it holds from the results of (27) and (31) that

$$\int_\Omega a(\cdot,t)\ln a(\cdot,t) \le C_6, \qquad \text{for all } t\in(0,T_{max}), \tag{32}$$

with some constant $C_6 > 0$. As the basic inequality $a\ln a \ge -e^{-1}$ for all $a > 0$ culminates in

$$\begin{aligned}\int_\Omega a(\cdot,t)|\ln a(\cdot,t)| &= \int_\Omega a(\cdot,t)\ln a(\cdot,t) - 2\int_{a<1} a(\cdot,t)\ln a(\cdot,t)\\ &\le \int_\Omega a(\cdot,t)\ln a(\cdot,t) + 2|\Omega|,\end{aligned}$$

thereby (19) follows readily upon (29) and (32). □

Utilizing the aforementioned $L\ln L(\Omega)$ estimation of a, we derive the following:

Corollary 1. *There exists some $C>0$ such that for any (u_0,w_0) fulfilling (5), the corresponding classical solution (a,v,w) to system (4) satisfies*

$$\|v(\cdot,t)\|_{W^{1,2}(\Omega)}\le C,\quad \text{for all } t\in(0,T_{max}).$$

Proof. Recalling Lemma 6 and the definition of u, there exists a constant $C_1>0$ such that

$$\int_\Omega u(\cdot,t)|\ln u(\cdot,t)|\le C_1,\quad \text{for all } t\in(0,T_{max}),$$

whereupon arguing in the reasoning quite in the same lines as Lemma A.4 of [39], we conclude that

$$\|v(\cdot,t)\|_{W^{1,2}(\Omega)}\le C_2,\quad \text{for all } t\in(0,T_{max}),$$

for some positive constant C_2. □

Subsequently, we shall fully utilize the estimates acquired from Lemma 4, along with Lemma 6 and Corollary 1, to construct the bounds for both a and ∇v within the space domain $L^\infty(\Omega)$.

Lemma 7. *Let (a,v,w) be the classical solution to system (4) in the bounded domain $\Omega\times(0,T_{max})$. Then one may figure out a certain constant $C>0$ satisfying*

$$\|a(\cdot,t)\|_{L^\infty(\Omega)}\le C,\quad \text{for all } t\in(0,T_{max})$$

and

$$\|v(\cdot,t)\|_{W^{1,\infty}(\Omega)}\le C,\quad \text{for all } t\in(0,T_{max}).$$

In particular, for all $T>0$, there exists $C(T)>0$ such that

$$\|\nabla w(\cdot,t)\|_{L^4(\Omega)}\le C,\quad \text{for all } t\in(0,T_0),$$

where we have set $T_0:=\min\{T,T_{max}\}$.

Proof. As a starting point of the arguments, Lemma 7 becomes applicable so as to ascertain the existence of a positive constant C_1 such that

$$\|u(\cdot,t)\|_{L^6(\Omega)}\le C_1,\quad \text{for all } t\in(0,T_{max}),$$

whence on the basis of the standard L^p-theory for elliptic equations, in combination with the Sobolev embedding theorem, this means that several positive constants C_2, C_3 and C_4 can be picked to fulfill

$$\begin{aligned}\|\nabla v(\cdot,t)\|_{L^\infty(\Omega)}\le& C_2\|v(\cdot,s)\|_{W^{2,6}(\Omega)}\\ \le& C_3\|u(\cdot,s)\|_{L^6(\Omega)}\\ \le& C_4,\quad \text{for all } t\in(0,T_{max}),\end{aligned}\tag{33}$$

and whence if we invoke [28, Lemma 3.6] or [39, Lemma 2.3], then there exists $C_5>0$ such that

$$\|a(\cdot,t)\|_{L^\infty(\Omega)}\le C_5,\quad \text{for all } t\in(0,T_{max}).\tag{34}$$

On the other hand, for any $T>0$, in light of [39, Corollary 3.15], we would also find a positive constant C_6 conforming to

$$\|\nabla w(\cdot,t)\|_{L^5(\Omega)}\le C_6,\quad \text{for all } t\in(0,\min\{T,T_{max}\}),$$

which alongside (33)-(34) confirms Lemma 7. □

Consolidating the preceding three lemmas seamlessly culminates in our primary finding concerning the global existence and boundedness.

Proof of Theorem 1. Relying on the extensibility criterion (14) in Lemma 2, this means that the validity of Theorem 1 becomes a straightforward deduction from Lemma 7. □

4. Asymptotic behavior

In order to derive estimates for w, given the third equation in (4), it seems favorable to investigate lower bounds for v. As a preparatory step, we state a pointwise estimate from below for the Neumann heat semigroup $(e^{\sigma\Delta})_{\sigma\geq 0}$ in Ω.

Lemma 8. *There exists a positive constant Γ_0 such that for all nonnegative $z \in C^0(\bar{\Omega})$ we have*

$$(e^{\tau\Delta}z)(x) \geq \Gamma_0 \int_\Omega z, \quad \textit{for all } x \in \Omega \textit{ and } \tau \geq 1.$$

Proof. By adopting precisely the same methodology outlined in Lemma 3.1 of [11], we are able to definitively confirm the conclusion. To avoid repetition of details, the specific proofs are hereby omitted. Instead, readers are recommended to consult the pertinent conclusions presented in the equally insightful literature [47] for further elaboration and verification. □

Recalling the second equation and given the support of Lemma 8, we are now able to derive a lower bound estimate for the L^1-norm of the solution component v over time t.

Lemma 9. *One may choose $\Gamma > 0$ and $C > 0$ such that*

$$\int_0^t v(x,s)ds \geq \Gamma \int_0^t \int_\Omega u(y,s)dyds - C, \quad \textit{for all } x \in \Omega \textit{ and } t \geq 0.$$

Proof. Drawing inspiration from [11], the verification process of Lemma 9 is meticulously divided into two primary steps.

Step 1: There exists a constant $\Gamma > 0$ such that

$$\int_0^t v(x,s)ds \geq \Gamma \cdot \int_0^{t-2} \int_\Omega u(y,s)dyds, \quad \textbf{for all } x \in \Omega \textbf{ and } t \geq 2. \tag{35}$$

To this end, according to the associated variation-of-constants formula we represent v in the form

$$v(\cdot,t) = e^{-t}e^{t\Delta}v_0 + \int_0^t e^{-(t-s)}e^{(t-s)\Delta}u(\cdot,s)ds, \quad \text{for all } t \geq 0,$$

with $e^{-t}e^{t\Delta}v_0 \geq 0$ in Ω due to $v_0 \geq 0$. Given that Lemma 8 implies that for some $\Gamma_0 > 0$,

$$e^{(t-s)\Delta}u(\cdot,s) \geq \Gamma_0 \cdot \int_\Omega u(\cdot,s) \quad \text{in } \Omega \text{ whenever } t-s \geq 1,$$

thus for $t \geq 1$ it is not difficult to reach

$$\begin{aligned} v(\cdot,t) \geq & \int_0^{t-1} e^{-(t-s)}e^{(t-s)\Delta}u(\cdot,s)ds \\ \geq & \Gamma_0 \cdot \int_0^{t-1} e^{-(t-s)} \cdot \int_\Omega u(\cdot,s)ds, \end{aligned} \tag{36}$$

whereby according to a straightforward time integration of (36), this means that

$$\int_1^t v(x,s)ds \geq \Gamma_0 \int_1^t \int_0^{s-1} e^{-(s-\sigma)} \int_\Omega u(\cdot,\sigma)d\sigma ds, \quad \text{for all } x \in \Omega \text{ and } t \geq 1, \tag{37}$$

which upon being solved by the Fubini theorem contributes to

$$\begin{aligned} \Gamma_0 \cdot \int_1^t \int_0^{s-1} e^{-(s-\sigma)} \cdot \int_\Omega u(\cdot,\sigma)d\sigma ds = & \Gamma_0 \cdot \int_0^{t-1} \left(\int_{\sigma+1}^t e^{-(s-\sigma)}ds \right) \cdot \int_\Omega u(\cdot,\sigma)d\sigma \\ = & \Gamma_0 \cdot \int_0^{t-1} (e^{-1} - e^{-(t-\sigma)}) \cdot \int_\Omega u(\cdot,\sigma)d\sigma, \quad \text{for all } t \geq 1. \end{aligned} \tag{38}$$

Now, if $t-\sigma \geq 2$, then $e^{-1}-e^{-(t-\sigma)} \geq e^{-1}-e^{-2}$. Therefore, for $t \geq 2$, we arrive at

$$\Gamma_0 \cdot \int_0^{t-1} (e^{-1}-e^{-(t-\sigma)}) \cdot \int_\Omega u(\cdot,\sigma)d\sigma \geq \Gamma_0 \cdot (e^{-1}-e^{-2}) \cdot \int_0^{t-2}\int_\Omega u(x,\sigma)dxd\sigma, \quad \text{for all } t \geq 1, \tag{39}$$

and moreover, we infer from (37)-(39) that (35) holds if we set $\Gamma := \Gamma_0 \cdot (e^{-1}-e^{-2})$.

Step 2: There exists $C > 0$ such that

$$\int_0^t v(x,s)ds \geq \Gamma \cdot \int_0^t \int_\Omega u(y,s)dyds - C, \quad \textbf{for all } x \in \Omega \textbf{ and } t \geq 0.$$

In view of Lemma 4, we have $\int_\Omega u(\cdot,t) \leq C_1$ for all $t \geq 0$ and some $C_1 > 0$. When $t \geq 2$, using Step 1, one obtains

$$\Gamma \int_0^t \int_\Omega u \leq \int_0^t v(x,s)ds + \Gamma \int_{t-2}^t \int_\Omega u \leq \int_0^t v(x,s)ds + 2C_1\Gamma, \quad \text{for all } x \in \Omega,$$

while in the case of $t < 2$, we trivially have $\int_0^t v(x,s)ds \geq 0$ for all $x \in \Omega$, and hence

$$\Gamma \int_0^t \int_\Omega u \leq C_1\Gamma t \leq \int_0^t v(x,s)ds + 2C_1\Gamma, \quad \text{for all } t < 2,$$

which enforces the claim upon choosing $C := 2C_1\Gamma$ for instance. □

After Lemma 9, let us invoke the following simple statement regarding the eventual validity of the appropriate bounds for both $\int_\Omega u(\cdot,t)$ and $\int_\Omega v(\cdot,t)$.

Lemma 10. *Let (u,v,w) be a global classical solution to system (4), and let $f(u,w) = u(r-\mu u - w)$ with some $\mu > 0$. Then, we have*

$$(i)\ \limsup_{t\to\infty} \|u(\cdot,t)\|_{L^1(\Omega)} \leq \frac{r|\Omega|}{\mu} \tag{40}$$

and

$$(ii)\ \limsup_{t\to\infty} \|v(\cdot,t)\|_{L^1(\Omega)} \leq \frac{r|\Omega|}{\mu}. \tag{41}$$

Proof. For the first conclusion (i), by integrating the first equation in (4) with respect to $x \in \Omega$, and using $f(u,w) = u(r-\mu u - w)$, it follows from the nonnegativity of solution $w \geq 0$ that

$$\frac{d}{dt}\int_\Omega u(x,t) \leq r\int_\Omega u(x,t) - \mu \int_\Omega u^2(x,t), \tag{42}$$

whence as a consequence of the Cauchy-Schwarz inequality and a series of elementary calculus, we effectively see

$$\frac{d}{dt}\int_\Omega u(x,t) + r\int_\Omega u(x,t) \leq \frac{r^2|\Omega|}{\mu},$$

which, upon an integration of both sides with respect to t, implies that

$$\|u(\cdot,t)\|_{L^1(\Omega)} \leq \int_\Omega u_0[e^{-rt}] + \frac{r|\Omega|}{\mu}(1-e^{-rt}), \quad \text{for all } t > 0,$$

henceforth giving rise to (40).

In the aftermath of the above, regarding the second claim (ii), noticing that (i) allows for any choice of $\varepsilon > 0$ such that there is $t_1 := t_1(\dot{\varepsilon})$ (the dot over ε is removed for clarity) fulfilling the property that for all $t \geq t_1$

$$\|u(\cdot,t)\|_{L^1(\Omega)} < \frac{r|\Omega|}{\mu} + \varepsilon,$$

then followed by (4), we deduce

$$\int_\Omega v(\cdot,t)dx = \int_\Omega u(\cdot,s)dx < \frac{r|\Omega|}{\mu} + \varepsilon, \tag{43}$$

which, along with the arbitrariness of ε, implies that (41) holds. □

Drawing upon Lemmas 4 through 5, our initial findings lead to the subsequent derivation of time-invariant bounds for a.

Lemma 11. *Let (u, v, w) be a global classical solution to the corresponding issue (4). Provided that*

$$C_* = r - \frac{3r^2}{2\mu^2}\left[\frac{\chi^2}{8}e^{\xi\|w_0\|_{L^\infty(\Omega)}} + 1 + \|w_0\|^2_{L^\infty(\Omega)}\left(\frac{\xi^2}{4}e^{2\xi\|w_0\|_{L^\infty(\Omega)}} + \xi\eta\right)\right.$$
$$\left. + \frac{\mu\xi\eta}{r}e^{\xi\|w_0\|_{L^\infty(\Omega)}}(\|w_0\|_{L^\infty(\Omega)} + 1)\right] - (1+\xi\eta)e^{\xi\|w_0\|_{L^\infty(\Omega)}}\|w_0\|_{L^\infty(\Omega)}$$

holds, then there exist $\beta > 0$ and $C_1^* > 0$ such that

$$\int_0^t\int_\Omega u(\cdot, s)ds \geq \beta t - C_1^*, \quad \text{for all } t > 0. \tag{44}$$

Proof. In consideration of the strong maximum principle and the premise $u_0 > 0$ outlined in (5), we confirm that u remains positive across $\Omega \times (0, \infty)$, where another application of the integration by parts over the domain Ω results in

$$\begin{aligned}
\frac{d}{dt}\int_\Omega e^{\xi w}\ln a =& \int_\Omega e^{\xi w}\frac{1}{a}a_t + \int_\Omega e^{\xi w}\ln a\xi w_t \\
=& \int_\Omega \frac{1}{a}\left[\nabla\cdot(e^{\xi w}\nabla a) - \chi\nabla\cdot(e^{\xi w}a\nabla v) + \xi avwe^{\xi w} - a\xi\eta e^{\xi w}w(1 - e^{\xi w}a - w) + e^{\xi w}f(ae^{\xi w}, w)\right] \\
=& \int_\Omega e^{\xi w}\frac{|\nabla a|^2}{a^2} - \chi\int_\Omega \frac{e^{\xi w}}{a}\nabla v\cdot\nabla a + \xi\int_\Omega vwe^{\xi w} - \xi\eta\int_\Omega e^{\xi w}w(1 - e^{\xi w}a - w) \\
&+ \int_\Omega e^{\xi w}(r - \mu e^{\xi w}a - w) + \xi\int_\Omega e^{\xi w}\ln a[-vw + \eta w(1 - ae^{\xi w} - w)] \\
\geq& -\frac{\chi^2}{4}\int_\Omega e^{\xi w}|\nabla v|^2 - \mu\int_\Omega e^{2\xi w}a - \xi\eta\int_\Omega e^{\xi w}w - \int_\Omega e^{\xi w}w + r\int_\Omega e^{\xi w} \\
&+ \xi\int_{a>1} e^{\xi w}\ln a[-vw - \eta w(ae^{\xi w} + w)] + \xi\int_{0<a<1} e^{\xi w}\ln a\eta w \\
\geq& -\frac{\chi^2}{4}\int_\Omega e^{\xi w}|\nabla v|^2 - \mu\int_\Omega e^{2\xi w}a - \xi\eta\int_\Omega e^{\xi w}w - \int_\Omega e^{\xi w}w + r|\Omega| \\
&+ \xi\int_{a>1} e^{\xi w}\ln a[-vw - \eta w(ae^{\xi w} + w)] + \xi\eta\int_{0<a<1} e^{\xi w}\ln aw.
\end{aligned} \tag{45}$$

Let

$$C_* = r - \frac{3r^2}{2\mu^2}\left[\frac{\chi^2}{8}e^{\xi\|w_0\|_{L^\infty(\Omega)}} + 1 + \|w_0\|^2_{L^\infty(\Omega)}\left(\frac{\xi^2}{4}e^{2\xi\|w_0\|_{L^\infty(\Omega)}} + \xi\eta\right)\right.$$
$$\left. + \frac{\mu\xi\eta}{r}e^{\xi\|w_0\|_{L^\infty(\Omega)}}(\|w_0\|_{L^\infty(\Omega)} + 1)\right] - (1+\xi\eta)e^{\xi\|w_0\|_{L^\infty(\Omega)}}\|w_0\|_{L^\infty(\Omega)}.$$

As pointed out in Lemma 10, there exists $t_2 > 0$ such that for all $t \geq t_2$

$$\int_\Omega u(\cdot, t) \leq \frac{3r|\Omega|}{2\mu} \quad \text{and} \quad \int_\Omega v(\cdot, t) \leq \frac{3r|\Omega|}{2\mu}, \tag{46}$$

and thus (42) gives

$$\begin{aligned}
\int_{t_2}^t\int_\Omega u^2(\cdot, s)ds \leq& \frac{r}{\mu}\int_{t_2}^t\int_\Omega u(\cdot, s)ds + \frac{1}{\mu}\int_\Omega u(\cdot, t_2) \\
\leq& \frac{3r^2|\Omega|}{2\mu^2}(t - t_2) + \frac{3r|\Omega|}{2\mu^2}.
\end{aligned}$$

Moreover, we multiply the second equation of (4) by v and apply the Young inequality to estimate

$$2\int_\Omega |\nabla v(\cdot, t)|^2 + \int_\Omega v^2(\cdot, t) \leq \int_\Omega u^2(\cdot, t)$$

and to proceed further,

$$\begin{aligned}\int_{t_2}^{t}\int_\Omega |\nabla v(\cdot,s)|^2 ds \leq & \frac{1}{2}\int_{t_2}^{t}\int_\Omega u^2(\cdot,s)ds\\ \leq & \frac{1}{2}\left\{\frac{3r^2|\Omega|}{2\mu^2}(t-t_2)+\frac{3r|\Omega|}{2\mu^2}\right\}\end{aligned}$$

and

$$\begin{aligned}\int_{t_2}^{t}\int_\Omega v^2(\cdot,t) \leq & \int_{t_2}^{t}\int_\Omega u^2(\cdot,s)ds\\ \leq & \frac{3r^2|\Omega|}{2\mu^2}(t-t_2)+\frac{3r|\Omega|}{2\mu^2}.\end{aligned} \tag{47}$$

By integrating (45) over (t_2,t), in tandem with identities (46) and (47), it shows that

$$\begin{aligned}&\int_\Omega e^{\xi w(\cdot,t)}\ln a(\cdot,t)-\int_\Omega e^{\xi w(\cdot,t_2)}\ln a(\cdot,t)(\cdot,t_2)+\mu\int_{t_2}^{t}\int_\Omega e^{2\xi w}a(\cdot,s)ds\\ &\geq r|\Omega|(t-t_2)-\frac{\chi^2}{4}\int_{t_2}^{t}\int_\Omega e^{\xi w}|\nabla v|^2-(1+\xi\eta)|\Omega|e^{\xi\|w_0\|_{L^\infty(\Omega)}}\|w_0\|_{L^\infty(\Omega)}(t-t_2)-J_1-J_2,\\ &\quad\text{for all } t>t_2,\end{aligned} \tag{48}$$

where

$$J_1=\xi\int_{t_2}^{t}\int_{a>1}e^{\xi w}\ln a[vw+\eta w(ae^{\xi w}+w)] \quad\text{and}\quad J_2=\xi\eta\int_{t_2}^{t}\int_{0<a<1}e^{\xi w}|\ln a|w,$$

whence we next improve our knowledge of handling the last two factors on the right-hand side of (48). Within this aim in mind, observe by the evident fact $\ln a\leq a$ that

$$\begin{aligned}J_1=&\xi\int_{t_2}^{t}\int_{a>1}e^{\xi w}\ln a[vw+\eta w(ae^{\xi w}+w)]\\ \leq&\int_{t_2}^{t}\int_{a>1}v^2+\frac{\xi^2}{4}\|w_0\|^2_{L^\infty(\Omega)}e^{2\xi\|w_0\|_{L^\infty(\Omega)}}\int_{t_2}^{t}\int_\Omega \ln^2 a\\ &+\xi\eta\|w_0\|_{L^\infty(\Omega)}\int_{t_2}^{t}\int_{a>1}ae^{2\xi w}+\xi\eta\|w_0\|^2_{L^\infty(\Omega)}\int_{t_2}^{t}\int_{a>1}a^2e^{\xi w}\\ \leq&\int_{t_2}^{t}\int_{a>1}v^2+\frac{\xi^2}{4}\|w_0\|^2_{L^\infty(\Omega)}e^{2\xi\|w_0\|_{L^\infty(\Omega)}}\int_{t_2}^{t}\int_\Omega a^2\\ &+\xi\eta\|w_0\|_{L^\infty(\Omega)}e^{\xi\|w_0\|_{L^\infty(\Omega)}}\int_{t_2}^{t}\int_\Omega u+\xi\eta\|w_0\|^2_{L^\infty(\Omega)}\int_{t_2}^{t}\int_\Omega u^2\\ \leq&\int_{t_2}^{t}\int_\Omega v^2+\frac{\xi^2}{4}\|w_0\|^2_{L^\infty(\Omega)}e^{2\xi\|w_0\|_{L^\infty(\Omega)}}\int_{t_2}^{t}\int_\Omega u^2\\ &+\xi\eta\|w_0\|_{L^\infty(\Omega)}e^{\xi\|w_0\|_{L^\infty(\Omega)}}\int_{t_2}^{t}\int_\Omega u+\xi\eta\|w_0\|^2_{L^\infty(\Omega)}\int_{t_2}^{t}\int_\Omega u^2\end{aligned}$$

and

$$\begin{aligned}J_2=&\xi\eta\int_{t_2}^{t}\int_{0<a<1}e^{\xi w}|\ln a|w\\ \leq&\xi\eta e^{\xi\|w_0\|_{L^\infty(\Omega)}}\int_{t_2}^{t}\int_\Omega e^{\xi w}a\\ =&\xi\eta e^{\xi\|w_0\|_{L^\infty(\Omega)}}\int_{t_2}^{t}\int_\Omega u,\end{aligned}$$

where we have also taken into account $a \le u$ and $e^{\xi w} \le e^{\xi \|w_0\|_{L^\infty(\Omega)}}$ for all $t > 0$, a substitution of the above two inequalities into (48), we benefit from (45)-(47) as well as the definition of C_* that

$$\begin{aligned}
&\int_\Omega e^{\xi w(\cdot,t)} \ln a(\cdot,t) - \int_\Omega e^{\xi w(\cdot,t_2)} \ln a(\cdot,t)(\cdot,t_2) + \mu \int_{t_2}^t \int_\Omega e^{2\xi w} a(\cdot,s)ds \\
&\quad \ge r|\Omega|(t-t_2) - \frac{\chi^2}{4} e^{\xi \|w_0\|_{L^\infty(\Omega)}} \int_{t_2}^t \int_\Omega |\nabla v|^2 - (1+\xi\eta)|\Omega| e^{\xi \|w_0\|_{L^\infty(\Omega)}} \|w_0\|_{L^\infty(\Omega)} (t-t_2) \\
&\qquad - \int_{t_2}^t \int_\Omega v^2 - \left(\xi\eta \|w_0\|_{L^\infty(\Omega)} e^{\xi \|w_0\|_{L^\infty(\Omega)}} + \xi\eta e^{\xi \|w_0\|_{L^\infty(\Omega)}}\right) \int_{t_2}^t \int_\Omega u \\
&\qquad - \left(\frac{\xi^2}{4} \|w_0\|^2_{L^\infty(\Omega)} e^{2\xi \|w_0\|_{L^\infty(\Omega)}} + \xi\eta \|w_0\|^2_{L^\infty(\Omega)}\right) \int_{t_2}^t \int_\Omega u^2 \\
&\quad \ge r|\Omega|(t-t_2) - \frac{\chi^2}{4} e^{\xi \|w_0\|_{L^\infty(\Omega)}} \int_{t_2}^t \int_\Omega |\nabla v|^2 - (1+\xi\eta)|\Omega| e^{\xi \|w_0\|_{L^\infty(\Omega)}} \|w_0\|_{L^\infty(\Omega)} (t-t_2) \\
&\qquad - \int_{t_2}^t \int_\Omega v^2 - \xi\eta e^{\xi \|w_0\|_{L^\infty(\Omega)}} (\|w_0\|_{L^\infty(\Omega)} + 1) \int_{t_2}^t \int_\Omega u \\
&\qquad - \left(\frac{\xi^2}{4} \|w_0\|^2_{L^\infty(\Omega)} e^{2\xi \|w_0\|_{L^\infty(\Omega)}} + \xi\eta \|w_0\|^2_{L^\infty(\Omega)}\right) \int_{t_2}^t \int_\Omega u^2 \\
&\quad \ge r|\Omega|(t-t_2) - \frac{\chi^2}{8} e^{\xi \|w_0\|_{L^\infty(\Omega)}} \left\{ \frac{3r^2|\Omega|}{2\mu^2}(t-t_2) + \frac{3r|\Omega|}{2\mu^2} \right\} \\
&\qquad - (1+\xi\eta)|\Omega| e^{\xi \|w_0\|_{L^\infty(\Omega)}} \|w_0\|_{L^\infty(\Omega)} (t-t_2) \\
&\qquad - \left\{ \frac{3r^2|\Omega|}{2\mu^2}(t-t_2) + \frac{3r|\Omega|}{2\mu^2} \right\} - \xi\eta e^{\xi \|w_0\|_{L^\infty(\Omega)}} (\|w_0\|_{L^\infty(\Omega)} + 1) \left(\frac{3r|\Omega|}{2\mu}\right)(t-t_2) \\
&\qquad - \left(\frac{\xi^2}{4} \|w_0\|^2_{L^\infty(\Omega)} e^{2\xi \|w_0\|_{L^\infty(\Omega)}} + \xi\eta \|w_0\|^2_{L^\infty(\Omega)}\right) \left\{ \frac{3r^2|\Omega|}{2\mu^2}(t-t_2) + \frac{3r|\Omega|}{2\mu^2} \right\} \\
&\quad = C_*|\Omega|(t-t_2) - C_{1,*}
\end{aligned}$$

with

$$C_{1,*} = \frac{\chi^2}{8} e^{\xi \|w_0\|_{L^\infty(\Omega)}} \frac{3r|\Omega|}{2\mu^2} + \frac{3r|\Omega|}{2\mu^2} + \left(\frac{\xi^2}{4} \|w_0\|^2_{L^\infty(\Omega)} e^{2\xi \|w_0\|_{L^\infty(\Omega)}} + \xi\eta \|w_0\|^2_{L^\infty(\Omega)}\right) \frac{3r|\Omega|}{2\mu^2}.$$

Depending on the nonnegative property of solution component u, we check

$$\begin{aligned}
\mu \int_{t_2}^t \int_\Omega e^{2\xi w} a(\cdot,s)ds &\ge |\Omega| C_*(t-t_2) - C_{1,*} + \int_\Omega \ln u(\cdot,t_2) - \int_\Omega \ln u(\cdot,t) \\
&\ge |\Omega| C_*(t-t_2) - C_{1,*} + \int_\Omega \ln u(\cdot,t_2) - \int_\Omega u(\cdot,t) \\
&\ge |\Omega| C_*(t-t_2) - C_{1,*} + \int_\Omega \ln u(\cdot,t_2) - \frac{3r|\Omega|}{2\mu} \\
&\ge |\Omega| C_* t - C_{2,*},
\end{aligned}$$

for $C_{2,*} = C_{1,*} + \frac{3r|\Omega|}{2\mu}$, whereas given the case when $t \le t_2$, once again invoking the nonnegativity of u leads to

$$\mu \int_0^t \int_\Omega e^{2\xi w} a(\cdot,s)ds \ge 0 \ge |\Omega| C_* t - |\Omega| C_* t_2.$$

This in particular allows us to deduce from (44) that (43) holds by choosing

$$\beta := \frac{1}{e^{\xi \|w_0\|_{L^\infty(\Omega)}}} \frac{|\Omega| C_*}{\mu}$$

and

$$C_1^* := \frac{1}{e^{\xi \|w_0\|_{L^\infty(\Omega)}}} \max\left\{\beta t_2, \frac{C_{2,*}}{\mu}\right\},$$

for instance. □

To establish an appropriate lower bound for v, we apply the inherent positivity of Green's function associated with the Helmholtz operator $-\Delta + 1$, as detailed below.

Lemma 12. *Let $\chi > 0, \xi > 0$. Then there exists $C_G > 0$ with the properties that the pointwise inequality*

$$v(x,t) \geq C_G \int_\Omega u(y,t)dy, \quad \textit{for all } x \in \Omega \textit{ and } t > 0, \tag{49}$$

holds.

Proof. We define G as the Green's function associated with $-\Delta + 1$ under homogeneous Neumann boundary conditions within the domain Ω. According to Theorem 18.2 (i) in [14], it is well-established that $G(x,y) \geq C_G$ holds for all distinct points $(x,y) \in \Omega^2$, with a positive constant $C_G > 0$. Given the second equation in (4), we have

$$v(x,t) = \int_\Omega G(x,y)u(y,t)\,\mathrm{d}y, \quad x \in \Omega, t > 0.$$

Given the nonnegativity of u, the inequality stated in (49) arises as a direct consequence of this formulation. □

By integrating the aforementioned information with Lemma 11 and Lemma 12, we can readily derive an exponential decay pattern for w.

Lemma 13. *Let $\chi > 0$ and $\xi > 0$. In addition, presume that the initial data u_0 and w_0 satisfy (5) as well as the smallness condition (12). One can fix certain constants $C > 0$ and $\gamma > 0$ such that the solution to system (4) fulfils*

$$\|w(\cdot,t)\|_{L^\infty(\Omega)} \leq Ce^{-\gamma t}, \quad \textit{for all } t > 0.$$

Proof. Noticing $w \neq 0$ in the identity as obtained from dividing the third equation of system (4) by w^2, and subsequently multiplying by an integrating factor as well as rearranging the terms, we rewrite the third equation in the form asserting

$$\frac{d}{ds}\left(\frac{1}{w(x,s)} e^{-\int_0^s [v(x,r)+\eta u(x,r)-\eta]dr}\right) = \eta e^{-\int_0^s [v(x,r)+\eta u(x,r)-\eta]dr},$$

whence integrating the above from $s = 0$ to t, we take advantage of performing a series of routine manipulations to trivially arrive at

$$w(x,t) = \frac{w_0(x)e^{-\int_0^t [v(x,r)+\eta u(x,r)-\eta]dr}}{1 + \eta w_0(x)\int_0^t e^{-\int_0^s [v(x,r)+\eta u(x,r)-\eta]dr} ds}. \tag{50}$$

Furthermore, Lemma 9 and Lemma 11 become applicable so as to guarantee

$$\int_0^t v(x,s)ds \geq \Gamma \cdot \int_0^t \int_\Omega u(y,s)dyds - C \geq \Gamma\beta t - \Gamma C_1^* - C, \quad \text{for all } x \in \Omega,\ t > 0,$$

which by virtue of (50) and the fact $\eta > 0$ culminates in

$$w(x,t) \leq w_0(x)e^{-\int_0^t [v(x,r)+\eta u(x,r)-\eta]dr} \leq w_0(x)e^{-(\eta-\Gamma\beta)t+\Gamma C_1^*+C}.$$

From this, we may conclude as intended. □

Indeed, the verification of our ultimate conclusion essentially boils down to an amalgamation of the preceding three assertions.

Proof of Theorem 2. The stipulated lower bounds for u and v stem from Lemmas 11 and 12, whereas Lemma 13 affirms the proclaimed decay characteristic of w. □

Acknowledgements

The author is very grateful to the anonymous reviewer for the careful reading and valuable suggestions that have greatly improved this work.

References

[1] N. Bellomo, A. Belloquid, Y. Tao, and M. Winkler, Toward a mathematical theory of Keller-Segel models of pattern formation in biological tissues, *Math. Models Methods Appl. Sci.* **25** (2015), no. 9, 1663–1763.

[2] H. Brézis and W. A. Strauss, Semi-linear second-order elliptic equations in L^1, *J. Math. Soc. Japan* **25** (1973), no. 4, 565–590.

[3] X. Cao, Boundedness in a three-dimensional chemotaxis-haptotaxis model, *Z. Angew. Math. Phys.* **67** (2016), no. 4, 1–13.

[4] M. Chaplain and G. Lolas, Mathematical modelling of cancer invasion of tissue: the role of the urokinase plasminogen activation system, *Math. Models Methods Appl. Sci.* **15** (2005), no. 11, 1685–1734.

[5] M. Chaplain and G. Lolas, Mathematical modelling of cancer invasion of tissue: dynamic heterogeneity, *Netw. Heterog. Media* **1** (2006), no. 3, 399–439.

[6] T. Cieślak and M. Winkler, Finite-time blow-up in a quasilinear system of chemotaxis, *Nonlinearity* **21** (2008), no. 5, 1057–1076.

[7] A. Friedman, *Partial Differential Equations*, Holt, Rinehart and Winston, Inc., New York-Montreal, Que.-London, 1969.

[8] T. Hillen and K. Painter, Global existence for a parabolic chemotaxis model with prevention of overcrowding, *Adv. Appl. Math.* **26** (2001), no. 4, 281–301.

[9] T. Hillen and K. Painter, A user's guide to PDE models for chemotaxis, *J. Math. Biol.* **58** (2009), no. 1-2, 183–217.

[10] T. Hillen and K. Painter, Spatio-temporal chaos in a chemotaxis model, *Phys. D* **240** (2011), no. 4-5, 363–375.

[11] T. Hillen, K. Painter, and M. Winkler, Convergence of a cancer invasion model to a logistic chemotaxis model, *Math. Models Methods Appl. Sci.* **23** (2013), no. 1, 165–198.

[12] D. Horstmann, From 1970 until present: the Keller-Segel model in chemotaxis and its consequences, *Jahresbericht der Deutschen Math.-Ver.* **105** (2003), no. 3, 103–165.

[13] D. Horstmann and M. Winkler, Boundedness vs. blow-up in a chemotaxis system, *J. Diff. Eqns.* **215** (2005), no. 1, 52–107.

[14] S. Ito, *Diffusion Equations*, *Translations of Mathematical Monographs* no. 114, Amer. Math. Soc., Providence, RI, 1992.

[15] W. Jäger and S. Luckhaus, On explosions of solutions to a system of partial differential equations modelling chemotaxis, *Trans. Amer. Math. Soc.* **329** (1992), no. 2, 819–824.

[16] C. Jin, Global classical solution and boundedness to a chemotaxis-haptotaxis model with re-establishment mechanisms, *Bull. Lond. Math. Soc.* **50** (2018), no. 4, 598–618.

[17] H. Jin and T. Xiang, Negligibility of haptotaxis effect in a chemotaxis-haptotaxis model, *Math. Models Methods Appl. Sci.* **31** (2021), no. 7, 1373–1417.

[18] Y. Ke and J. Zheng, A note for global existence of a two-dimensional chemotaxis-haptotaxis model with remodeling of non-diffusible attractant, *Nonlinearity* **31** (2018), no. 10, 4602–4620.

[19] E. Keller and L. Segel, Initiation of slime mold aggregation viewed as an instability, *J. Theoret. Biol.* **26** (1970), no. 3, 399–415.

[20] K. Kuto, K. Osaki, T. Sakurai, and T. Tsujikawa, Spatial pattern formation in a chemotaxis-diffusion growth model, *Phys. D* **241** (2012), no. 19, 1629–1639.

[21] J. Lankeit, Eventual smoothness and asymptotics in a three-dimensional chemotaxis system with logistic source, *J. Diff. Eqns.* **258** (2015), no. 4, 1158–1191.

[22] L. Liu, Boundedness and global existence in a higher-dimensional parabolic-elliptic-ODE chemotaxis-haptotaxis model with remodeling of non-diffusible attractant, *J. Math. Anal. Appl.* **549** (2025), no. 1, 129473.

[23] L. Liu, A note on the global existence and boundedness of an n-dimensional parabolic-elliptic predator-prey system with indirect pursuit-evasion interaction, *Open Math.* **23** (2025), no. 1, 20240122.

[24] C. Morales-Rodrigo, Local existence and uniqueness of regular solutions in a model of tissue invasion by solid tumours, *Math. Comput. Model.* **47** (2008), no. 5-6, 604–613.

[25] L. Nirenberg, On elliptic partial differential equations, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **13** (1959), no. 3, 115–162.

[26] K. Osaki, T. Tsujikawa, A. Yagi, and M. Mimura, Exponential attractor for a chemotaxis growth system of equations, *Nonlinear Anal.* **51** (2002), no. 1, 119–144.

[27] K. Osaki and A. Yagi, Finite dimensional attractors for one-dimensional Keller-Segel equations, *Funkcial. Ekvac.* **44** (2001), no. 3, 441–469.

[28] P. Pang and Y. Wang, Global existence of a two-dimensional chemotaxis-haptotaxis model with remodeling of non-diffusible attractant, *J. Diff. Eqns.* **263** (2017), no. 2, 1269–1292.

[29] P. Pang and Y. Wang, Global boundedness of solutions to a chemotaxis-haptotaxis model with tissue remodeling, *Math. Models Methods Appl. Sci.* **28** (2018), no. 11, 2211–2235.

[30] B. Perthame, *Transport Equations in Biology*, Birkhäuser Verlag, Basel, Switzerland, 2007.

[31] C. Stinner, C. Surulescu, and M. Winkler, Global weak solutions in a PDE-ODE system modeling multiscale cancer cell invasion, *SIAM J. Math. Anal.* **46** (2014), no. 3, 1969–2007.

[32] Y. Sugiyama, Time global existence and asymptotic behavior of solutions to degenerate quasilinear parabolic systems of chemotaxis, *Diff. Integral Eqns.* **20** (2007), no. 2, 133–180.

[33] Y. Tao, Boundedness in a two-dimensional chemotaxis-haptotaxis system, *J. Oceanogr.* **70** (2014), no. 2, 165–174.

[34] Y. Tao and M. Wang, Global solution for a chemotactic-haptotactic model of cancer invasion, *Nonlinearity* **21** (2008), no. 10, 2221–2238.

[35] Y. Tao and M. Wang, A combined chemotaxis-haptotaxis system: the role of logistic source, *SIAM J. Math. Anal.* **41** (2009), no. 4, 1533–1558.

[36] Y. Tao and M. Winkler, A chemotaxis-haptotaxis model: the roles of nonlinear diffusion and logistic source, *SIAM J. Math. Anal.* **43** (2011), no. 2, 685–704.

[37] Y. Tao and M. Winkler, Boundedness and stabilization in a multi-dimensional chemotaxis-haptotaxis model, *Proc. Roy. Soc. Edinburgh Sect. A* **144** (2014), no. 5, 1067–1084.

[38] Y. Tao and M. Winkler, Dominance of chemotaxis in a chemotaxis-haptotaxis model, *Nonlinearity* **27** (2014), no. 6, 1225–1239.

[39] Y. Tao and M. Winkler, Energy-type estimates and global solvability in a two-dimensional chemotaxis-haptotaxis model with remodeling of non-diffusible attractant, *J. Diff. Eqns.* **257** (2014), no. 3, 784–815.

[40] Y. Tao and M. Winkler, Boundedness and decay enforced by quadratic degradation in a three-dimensional chemotaxis-fluid system, *Z. Angew. Math. Phys.* **66** (2015), no. 5, 2555–2573.

[41] Y. Tao and M. Winkler, Large time behavior in a multidimensional chemotaxis-haptotaxis model with slow signal diffusion, *SIAM J. Math. Anal.* **47** (2015), no. 6, 4229–4250.

[42] Y. Tao and G. Zhu, Global solution to a model of tumor invasion, *Appl. Math. Sci.* **1** (2007), no. 48, 2385–2398.

[43] J. I. Tello and M. Winkler, A chemotaxis system with logistic source, *Comm. Part. Diff. Eqns.* **32** (2007), no. 6, 849–877.

[44] R. Temam, *Navier-Stokes Equations: Theory and Numerical Analysis*, *Stud. Math. Appl.* **2**, North-Holland, Amsterdam, 1977.

[45] C. Walker and G. Webb, Global existence of classical solutions for a haptotaxis model, *SIAM J. Math. Anal.* **38** (2007), no. 5, 1694–1713.

[46] Y. Wang, Boundedness in the higher-dimensional chemotaxis-haptotaxis model with non-linear diffusion, *J. Diff. Eqns.* **260** (2016), no. 2, 1975–1989.

[47] Y. Wang and Y. Ke, Large time behavior of solution to a fully parabolic chemotaxis-haptotaxis model in higher dimensions, *J. Diff. Eqns.* **260** (2016), no. 9, 6960–6988.

[48] M. Winkler, Aggregation vs. global diffusive behavior in the higher-dimensional Keller-Segel model, *J. Diff. Eqns.* **248** (2010), no. 12, 2889–2905.

[49] M. Winkler, Boundedness in the higher-dimensional parabolic-parabolic chemotaxis system with logistic source, *Comm. Part. Diff. Eqns.* **35** (2010), no. 8, 1516–1537.

[50] M. Winkler, Finite-time blow-up in the higher-dimensional parabolic-parabolic Keller-Segel system, *J. Math. Pures Appl.* **100** (2013), no. 5, 748–767.

[51] M. Winkler, Global asymptotic stability of constant equilibria in a fully parabolic chemotaxis system with strong logistic dampening, *J. Diff. Eqns.* **257** (2014), no. 4, 1056–1077.

[52] M. Winkler, Finite-time blow-up in low-dimensional Keller-Segel systems with logistic-type superlinear degradation, *Z. Angew. Math. Phys.* **69** (2018), no. 2, 40.

[53] T. Xiang, Boundedness and global existence in the higher-dimensional parabolic-parabolic chemotaxis system with/without growth source, *J. Diff. Eqns.* **258** (2015), no. 12, 4275–4323.

[54] T. Xiang, Chemotactic aggregation versus logistic damping on boundedness in the 3D minimal Keller-Segel model, *SIAM J. Appl. Math.* **78** (2018), no. 5, 2420–2438.

[55] T. Xiang, How strong a logistic damping can prevent blow-up for the minimal Keller-Segel chemotaxis system?, *J. Math. Anal. Appl.* **459** (2018), no. 2, 1172–1200.

[56] T. Xiang, Sub-logistic source can prevent blow-up in the 2D minimal Keller-Segel chemotaxis system, *J. Math. Phys.* **59** (2018), no. 11, 111502.

[57] T. Xiang and J. Zheng, A new result for 2D boundedness of solutions to a chemotaxis-haptotaxis model with/without sub-logistic source, *Nonlinearity* **32** (2019), no. 12, 4890–4911.

[58] J. Zheng, Boundedness of solutions to a quasilinear parabolic-elliptic Keller-Segel system with logistic source, *J. Diff. Eqns.* **259** (2015), no. 1, 120–140.

[59] J. Zheng and Y. Ke, Large time behavior of solutions to a fully parabolic chemotaxis-haptotaxis model in N dimensions, *J. Diff. Eqns.* **266** (2019), no. 4, 1969–2018.

[60] J. Zheng and Y. Ke, Boundedness and large time behavior of solutions of a higher-dimensional haptotactic system modeling oncolytic virotherapy, *Math. Models Methods Appl. Sci.* **33** (2023), no. 9, 1875–1907.