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Generalized extrapolation methods for solving nonlinear Fredholm integral equations

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Abstract. In this paper we develop a class of generalized extrapolation methods for a numerical solution of nonlinear Fredholm integral equations of the second kind. The direct representation of this class allows us to simply discretize the nonlinear integral equations with smooth kernels. This approach enjoys several outstanding features of numerical methods such as: economized computational cost, high order accuracy, direct implementation, discretization on arbitrary nodes and applying the methods with positive weights. The comparison results demonstrate the superior results of the new class of methods versus the classical and recent approaches.

AMS subject classifications: 22E46, 53C35, 57S20

Key words: nonlinear integral equation, extrapolation method, high order method, arbitrary nodes

1. Introduction

Many physical phenomena are governed by nonlinear integral equations (NIE) of the form

$$u(x) - \lambda \int_{\Omega} K(x, y, u(y))dy = f(x), \quad x \in \Omega, \tag{1}$$

where we assume that the kernel K and source term f are smooth functions in the bounded region $\Omega \subseteq \mathbb{R}^k, k = 1, 2$, and λ is a parameter. The existence of a unique solution for this problem is guaranteed under some smoothness conditions like somewhat we mentioned above [12]. The general form (1) has many applications in modeling of problems in science and engineering or reformulations of other mathematical problems, e.g, see the study of Nee [25] on a nonlinear integral equation arising from the model of anisotropic multiband BCS Gap Equations of superconductivity. For general background on numerical methods for NIEs, the books of Atkinson [4] and Delves and Mohamed [14] are recommended. For a review of less recent methods for NIEs we refer to the survey by Atkinson [3]. There is a great deal of publication on the numerical solution of equation (1). Many researchers have studied numerical aspects of these equations. In the recent publications, different mathematical tools have been applied to solving and numerical implementations of

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(1). In part of these publications, some authors extensively uses the methods based on different kinds of wavelets [7, 8, 21, 23]. Several different variants of numerical or theoretical studies on (1), including the special classes like Hammerstein NIE, have been developed in the literature. For some examples, see papers [2, 15, 19, 22]. Polynomial approximation methods using different bases functions like the Chebyshev polynomials have been introduced; see for example [11, 27]. An approximation with Sinc functions has been developed in [24]. On the other hand, the iterated methods have been used extensively for integral equations [15, 16]. The product quadrature rule has been studied for such problems in [9]. Two-dimensional equations have been taken into account in several papers like [1]. The extrapolation methods have been discussed extensively in Sidi [26] and Brezinski and Zaglia [10] and for linear integral equations in [20]. With a glance on the extrapolation methods for linear or nonlinear integral equations we find two remarkable points. First, all of the schemes in the literature almost apply extrapolation methods in the sense of Romberg's algorithm, in which the steplength half repeatedly. Second, extrapolation is done in a recursive procedure that contains two major disadvantages. Firstly, for one extrapolation it is necessary to generate two grid point sets and then solve the corresponding nonlinear equations. Secondly, the first extrapolation is just evaluated on the courser grid and does not use the data from the finer grid points. This means that many calculated intermediate results on the finer grid are lost. In the present work, the mentioned problems have been addressed. Accordingly, it is not required to solve nonlinear equations several times for a single problem. The cost of this scheme is the same as applying a simple numerical quadrature formula for discretization of integrals. But, it is quite simple to increase the order of accuracy in numerical results with shrinking the steplength or increasing the quadrature nodes. We will see in examples that there is no need to use a courser grid to generate high order results and with a moderate number of nodes we can obtain reasonable results. The given algorithm is based on a nonrecursive procedure with a simple structure and it uses all generated results simultaneously. This requires to solve one system of nonlinear equations for a corresponding nonlinear integral equation. Different sequences for generation of extended nodes have been suggested by Romberg, Bauer, and Bulrish [13] for implementation of extrapolation methods. We generalize the choice of sequences and discuss their numerical behavior in numerical quadratures. Recently, extensive amount of work has been done on the numerical solution of nonlinear integral equations.

This paper has been organized as follows. In Section 2, we review the basic results of extrapolation methods and their convergence properties. The calculation of the nodes and weights discussed in Section 3 for different and arbitrary sequences. In Section 4, we demonstrate the numerical behavior of the weights and density and distribution of nodes. This theory helps us to compare the behavior of different nodes from several points of view such as accuracy, distribution of nodes and the effect of the nodes on the accuracy, etc. In Section 5, the nonlinear integral equation has been discretized to obtain a nonlinear system of equations. The convergence of the given methods has been investigated in Section 6. In Section 7, with several numerical tests we discuss some aspects of the method.

2. Extrapolation methods

Let $T(h)$ be a complex valued approximate value of $T(0)$ for $h \in (0, h_0]$. We suppose that $T(h)$ has the following asymptotic expansion

$$I - T(h) = c_1 h^2 + c_2 h^4 + \dots \tag{2}$$

The accuracy of $T(h)$ can be improved using the approximated values of $T(h)$ at $h_0 > h_1 > \cdots > h_n$. According to the Euler-Maclorian formula, it is possible to improve the accuracy of the quantity $T(h)$ that has the following asymtotic error formula

$$I - T(h) = c_1 h^2 + c_2 h^4 + \dots \tag{3}$$

In our study $T(0)$ is $\int_a^b f(x)\,dx$. We use (3) for an arbitrary decreasing sequence of steplengths

$$h_0 > h_1 > \cdots > h_n \tag{4}$$

and we consider a linear combination of intermediate approximated values $T(h_k)$ and then require that the final approximation

$$T_0^{(n)} = \sum_{k=0}^{n} c_k^{(n)} T(h_k) \tag{5}$$

satisfies

$$T_0^{(n)} = T(0) + O(h_0^{2n+2}), \quad h_0 \to 0^+. \tag{6}$$

The evaluation of $c_k^{(n)}$ is closely related to the problem of polynomial interpolation. We can interpret it as an interpolation of the values $T(h)$ at $h = h_k, k = n, \dots, n+m$ in terms of h^2. In other words, let $p_n^{(m)}$ be the polynomial interpolation of degree m; then $T_n^{(m)} = p_n^{(m)}(0)$ is the desired approximation. According to the Neville's algorithm, we can compute $T_n^{(m)}$ with the following algorithm

$$T_n^{(0)} = T(h_n) \tag{7}$$

$$T_n^{(m)} = T_{n+1}^{(m-1)} + \frac{T_{n+1}^{(m-1)} - T_n^{(m-1)}}{(h_n/h_{n+m})^2 - 1}, \tag{8}$$

where $h_i = (b-a)/n_i$ and $\{n_i\}$ is an arbitrary increasing sequence of natural numbers. An alterative way is to use the rational interpolation instead of polynomial interpolation. The latter is a nonlinear procedure and the presented theory is not straightforward in this case. We can demonstrate the $\{T_n^{(m)}\}$ sequence in a triangular array. As we proceed in the columns or rows of this array we can obtain more accurate results. The theory of convergence for the sequence $\{T_n^{(m)}\}$ exists under simple unrestricted conditions. We review the results here to provide the basics for direct representation of extrapolation methods and then use the results in discretization of nonlinear integral equations.

The following theorem states the convergence issue of the $\{T_n^{(m)}\}$ sequence.

Theorem 1 (see [17]). *The necessary and sufficient condition for the function $T(h)$, continuous from the right in $h = 0$ to have $\lim_{n\to\infty} T_0^{(n)} = T(0)$ is that*

$$\alpha = \sup_{n\geq 0} \frac{h_{n+1}}{h_n} < 1. \tag{9}$$

An important issue in practical algorithms is the rate of convergence of the given method. For extrapolation methods the properties of which we discuss the following theorem states their order of convergence.

Theorem 2 (see [17]). *Suppose $T(h)$ has asymptotic expansion (1) and let $\sup_{n\geq 0} \frac{h_{n+1}}{h_n} \leq \alpha < 1$. Then, as $n \to \infty$,*

$$T_n^{(m)} - T(0) = (-1)^m e_{m+1}(h_n \dots h_{n+m})^2 + o((h_n \dots h_{n+m})^2). \tag{10}$$

If, in addition, $0 < \beta \leq \inf_{n\geq 0} \frac{h_{n+1}}{h_n}$, then there exist constants E_m such that for any $m \geq 0$ we have

$$|T_0^{(m)} - T(0)| \leq E_{m+1}(h_n \dots h_{n+m})^2, \quad n \geq 0. \tag{11}$$

Relation (10) states that each column converges to $T(0)$ faster than the previous column and (11) shows that the principle diagonal converges faster than any. According the above theorems, it is required that the sequence of divisions $\{n_i\}$ satisfies the given conditions to obtain a convergent method.

3. Calculation of nodes and weights

In this section, we provide an explicit algorithm for evaluation of generalized extrapolation methods. The following theorem enables us to represent $T_j^{(k)}$ explicitly as a linear combination of the values of integrand function $f(x)$

$$T_j^{(k)} = \sum_{i=0}^{n} \omega_i^{(k,j)} f(x_i^{(j)}) = \omega^{(k,j)^T}.f^{(j)}. \tag{12}$$

In particular, $\omega_i^{(0,j)}$ are the weights of the trapezoidal rule. The proof of the following theorem is a direct application of the recursive formula (8), therefore we omit it.

Theorem 3. *The weights of the numerical approximation $T_j^{(k)}$ is given with the following recursive relation*

$$\omega_i^{(k,j)} = \omega_i^{(k-1,j+1)} + \frac{\omega_i^{(k-1,j+1)} - \omega_i^{(k-1,j)}}{(h_j/h_{j+k})^2 - 1}. \tag{13}$$

3.1. Positivity and bounds on weights

It is of main interest to introduce the quadratures with positive weights. Positive weights have useful properties in the computation of approximate values of integrals

with lower roundoff errors and without losing the accuracy. In extrapolation methods we can search for the quadratures that have positive weights. In the literature the quadratures are given by division sequences $\{n_i\}$. In general, this sequence of positive integers is arbitrary. However, for convergence of the numerical quadrature it is required to be chosen to satisfy condition (9). Some authors investigated and suggested typical sequences for the generation of steplengths in extrapolation method. Three common sequences are Romberg, Bauer and Bulirsch sequences [13], and they are listed as follows:

Romberg: $n_i = 2^i,\ i = 0, 1, 2, \ldots$

Bauer: $n_{2j} = 3^j,\ n_{2j+1} = 2(3^j),\ j = 0, 1, 2, \ldots$

Bulirsch: $n_0 = 1,\ n_{2j-1} = 2^j,\ n_{2j} = 3(2^{j-1}),\ j = 1, 2, \ldots.$

Division sequences are not restricted to these three families. Any sequence that satisfies condition (9) is acceptable to be used in the extrapolation process. By the result given in [13], it is proven that the weights of quadratures based on Romberg divisors are positive. More precisely, let $n = 2^{j+k} = \frac{1}{h}$ and $\omega_i^{(k,j)} = h\tilde{\omega}_i^{(k,j)}$. Then the coefficients $\tilde{\omega}_i^{(k,j)}$ satisfy the inequalities

$$0.48 < \tilde{\omega}_i^{(k,j)} < 1.46.$$

However, for Bauer and Bulirsch divisors the situation is not straightforward. For Bauer divisors the subsequences that end with an even number have positive weights. According to the definition of this sequence, the quadratures with positive weights correspond to the divisor sequences of type $\{n_i\}_{i=0}^{2j+1}, j = 0, 1, 2, \ldots$. On the other hand, for Bulirsch divisors there is no subsequentce with positive weights.

The distribution of nodes and growth and decay of corresponding weights are noticeable points in the solution of some class of problems. Romberg divisors generate uniform nodes distributed in the given interval. However, in a variety of problems we need to distribute extra nodes in the boundaries or in special locations. In such situations we are interested to invoke other types of divisors. In the next section, by a bar graph we will illustrate numerical properties of weights and the corresponding node distribution.

3.2. Decay rate of weights on arbitrary grids

There are important facts in the choice of the division sequence. In this section, we propose the general distribution of nodes introduced by the given division sequences and their corresponding rate of changes in the weights. We do the comparison of three sequences in more detail with demonstration of graphs of weights for the given typical sequences. The practical comparative results will be given in numerical results.

Figure 1 illustrates the distribution of nodes and normalized weights for Bauer, Romberg and Bulirsch sequences with approximately for same number of overall nodes. In this view, the Bulirsch sequence generates more extrapolations with the

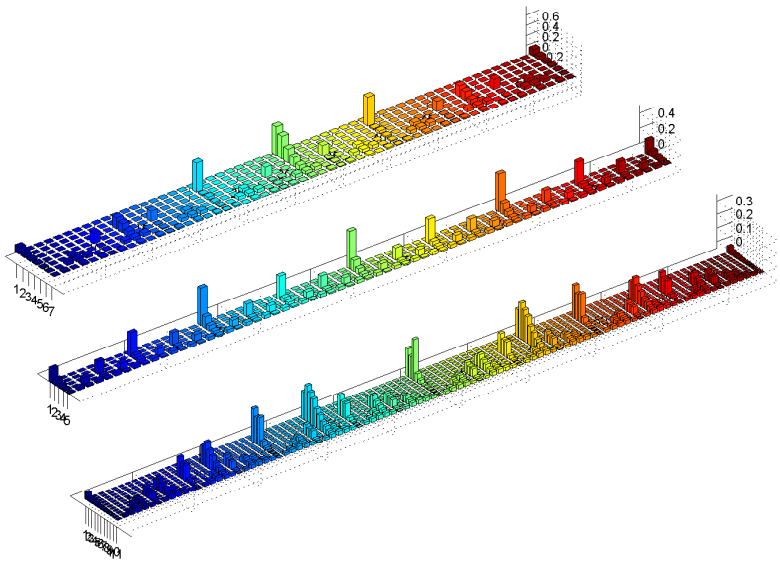

Figure 1: *Extrapolations nodes and weights for Bauer's sequence (top), Romberg's sequence (mid) and Bulirsch's sequence (bottom)*

same number of nodes and then Bauer and Romberg sequences with fewer extrapolations, respectively. The growth of nodes in the Romberg sequence is rather uniform but very severe compared to two others. Also, it turns out from the behavior of weights and distribution of nodes that the choice of these different nodes will produce different numerical results for especially nonlinear equations with high oscillatory solutions. However, the possibility of arbitrary node distribution of the new generalized extrapolation scheme given in this paper overcomes this difficulty.

4. Approximation with extrapolation methods

4.1. Discrete form in one dimension

Using representation (12) on an arbitrary set of collocation points we can approximate solution of equation (1) for requested order of accuracy. Setting $x = x_i$ in (1) and using extrapolation rule (12) we obtain

$$u_i - \sum_{j=0}^{N} \omega_i K(x_i, y_j, u_j) = f_i, \quad i = 0, 1, \ldots, N, \tag{14}$$

where u_j is the approximate value of the exact solution in the node x_i and $f_i = f(x_i)$. As we noted in the previous section, it is important to highlight again that the weights in this approximation are evaluated once and the main advantage is the arbitrary choice of the node in the discretization of the problem. Only under a slight

condition on the division sequence we can ensure that the quadrature is convergent. In the next section, we will prove the convergence of the above approximation for general kernels that satisfy the Lipschitz condition. The nonlinear system of equations can be written in the following vector form

$$U - K(U)W = F, \tag{15}$$

where

$$U = \begin{pmatrix} u(x_0) \\ u(x_1) \\ \vdots \\ u(x_N) \end{pmatrix}, \quad F = \begin{pmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_N) \end{pmatrix}, \quad W = \begin{pmatrix} \omega_0 \\ \omega_1 \\ \vdots \\ \omega_N \end{pmatrix}. \tag{16}$$

Also, the matrix $K(U)$ is a square matrix whose elements are $K(U)_{ij} = K(x_i, y_j, U_j)$. Note that in the case $K(x, y, u(y)) = K(x, y)u^m(y)$ we can evaluate the exact jacobian for running the steps of the iterative nonlinear solver such as the Newton method. For a nonlinear system of equation

$$G(U) = U - K(U)W - F = 0$$

we obtain

$$DG(U) = I - mB,$$

where $K(U)_{ij} = K_{ij}U_j^m$ and $B_{ij} = K_{ij}U_j^{m-1}$.

4.2. Discrete form in two dimensions

In two-dimensional case the discretization process is similar to the one-dimensional case. Consider the nonlinear integral equation of the form

$$u(x, y) - \lambda \int_0^1 \int_0^1 K(x, y, s, t, u(s, t))dsdt = f(x, y), \quad x, y \in [0, 1]. \tag{17}$$

Applying the explicit approximation (12) we obtain

$$u_{ij} - \lambda \sum_{m=1}^{N} \sum_{n=1}^{N} K(x_i, x_j, x_m, x_n, u_{mn})\omega_m\omega_n = f_{ij}, \quad i, j = 1, 2, \ldots, N. \tag{18}$$

In the case of the linear integral equation $K(x, y, s, t, u(s, t)) = k(x, y, s, t)u(s, t)$ we get the following linear system

$$(I - \lambda K)U = F,$$

where, K is a $N^2 \times N^2$ matrix with entries $K_{ijmn} = k(x_i, x_j, x_m, x_n)\omega_m\omega_n$ and U and F are $N^2 \times 1$ vectors:

$$K = \begin{bmatrix} K_{1111} & \dots & K_{111N} & K_{1121} & \dots & K_{112N} & \dots & K_{11N1} & \dots & K_{11NN} \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ K_{1N11} & \dots & K_{1N1N} & K_{1N21} & \dots & K_{1N2N} & \dots & K_{1NN1} & \dots & K_{1NNN} \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ K_{N111} & \dots & K_{N11N} & K_{N121} & \dots & K_{N12N} & \dots & K_{N1N1} & \dots & K_{N1NN} \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ K_{NN11} & \dots & K_{NN1N} & K_{NN21} & \dots & K_{NN2N} & \dots & K_{NNN1} & \dots & K_{NNNN} \end{bmatrix}, \quad (19)$$

$$U = \begin{bmatrix} u_{11} & \dots & u_{1N} & u_{21} & \dots & u_{2N} & \dots & u_{N1} & \dots & u_{NN} \end{bmatrix}^T,$$

and

$$F = \begin{bmatrix} f_{11} & \dots & f_{1N} & f_{21} & \dots & f_{2N} & \dots & f_{N1} & \dots & f_{NN} \end{bmatrix}^T.$$

In the general case, we have a nonlinear system of equations

$$G(U) = U - \lambda K(U) - F = 0, \quad (20)$$

where $K_{ijmn} = K(x_i, x_j, x_m, x_n, u_{mn})\omega_m\omega_n$. In some special cases we may compute the exact jacobian matrix to make iterations of a nonlinear solver. For example, suppose that the kernel is in the form $K(x, y, s, t, u(s, t)) = k(x, y, s, t)u(s, t)^m$. Then the jacobian of (20) reads

$$DG(U) = I - mKU^{m-1},$$

where K is introduced in (19).

5. Convergence of the method

In this section, we investigate the convergence of the method that we have developed in previous sections. A general convergence proof is given in [14] for the nonlinear Voltera equations which is discretized with a convergent numerical method. It is not straightforward to extend the same proof for the case of nonlinear fredholm equation. However, we give a different proof for the convergence. We remark that the numerical quadrature

$$\int_a^b K(x_i, y, u(y))\, dy = \sum_{j=0}^{N} w_{ij} K(x_i, y_j, u(y_j)) + E_{i,y}(K(x_i, y, u(y)), \quad (21)$$

is convergent if

$$\lim_{h_i \to 0} E_{i,y}(K(x_i, y, u(y)) = 0. \quad (22)$$

Consistency condition (22) is satisfied for extrapolation methods under the conditions given in Theorem 1.

Theorem 4. *If $\omega_i \geq 0$ and $\{h_i\}$ satisfies (9) and $K(x, y, z)$ is a lipschitz function with respect to z with constant $L < 1$, then the extrapolation method for the solution of nonlinear integral equation (1) is convergent.*

Proof. At $x = x_i$, $i = 0, 1, \ldots, N$, we obtain by inserting (21) in (1)

$$u(x_i) = f(x_i) + \sum_{j=0}^{N} w_j K(x_i, y_j, u(y_j)) + E_{i,y}(K(x_i, y, u(y)). \tag{23}$$

Similarly, the approximate equations are

$$u_i = f(x_i) + \sum_{j=0}^{N} w_j K(x_i, y_j, u_j), \quad i = 0, 1, \ldots, N. \tag{24}$$

On subtracting equations (23) and (24) we obtain

$$\begin{aligned} e_i := u(x_i) - u_i = \sum_{j=0}^{N} w_j \Big[K(x_i, y_j, u(y_j)) - K(x_i, y_j, u_j)\Big] \\ + E_{i,y}(K(x_i, y_j, u(y_j)), \quad i = 0, \ldots, N. \end{aligned} \tag{25}$$

The Lipschitz property of the kernel function K reads

$$|K(x_i, y_j, u(y_j)) - K(x_i, y_j, u_j)| \leq L|u(y_j) - u_j|.$$

Inserting this result in (25), we obtain

$$|e_i| \leq \sum_{j=0}^{N} w_j L|e_j| + |E_{i,y_j}(K(x_i, y_j, u(y_j))|, \quad i = 0, \ldots, N. \tag{26}$$

Let $e_l = \max_j |e_j|$. We set $i = l$ in (26) and using $\sum_{j=0}^{N} w_{ij} = b - a$ we obtain

$$|e_l| \leq (b - a)L|e_l| + |E_{l,y_j}(K(x_l, y_j, u(y_j))|, \tag{27}$$

or

$$|e_l| \leq \frac{|E_{l,y_j}(K(x_l, y_j, u(y_j))|}{1 - (b - a)L}, \tag{28}$$

the quadrature is convergent, so in the limit as $h_j \to 0$ we obtain the result. The proof is complete. □

6. Numerical results

In this section, we verify the implementation results of the novel method. The comparative results are given to justify the superior behavior of the method.

Example 1. *The first example is a nonlinear problem with a smooth kernel which is also a lipschitz function*

$$u(x) + \int_{-1}^{1} \frac{u(y)}{x^2 + u(y)^2}\, dy = \mathrm{e}^{-x} + \left(-\tan^{-1}\left(\frac{\mathrm{e}^{1}}{x}\right) + \tan^{-1}\left(\frac{\mathrm{e}^{-1}}{x}\right)\right) x^{-1}.$$

We consider $11, 6$ *and* 8 *elements of the Bulirsch, Romberg and Bauer sequences, respectively. The corresponding overall number of nodes is* $65, 65$ *and* 55*, respectively. The initial guess is* $u(x) = 1$ *which is far from the true solution* $u(x) = e^{-x}$*.*

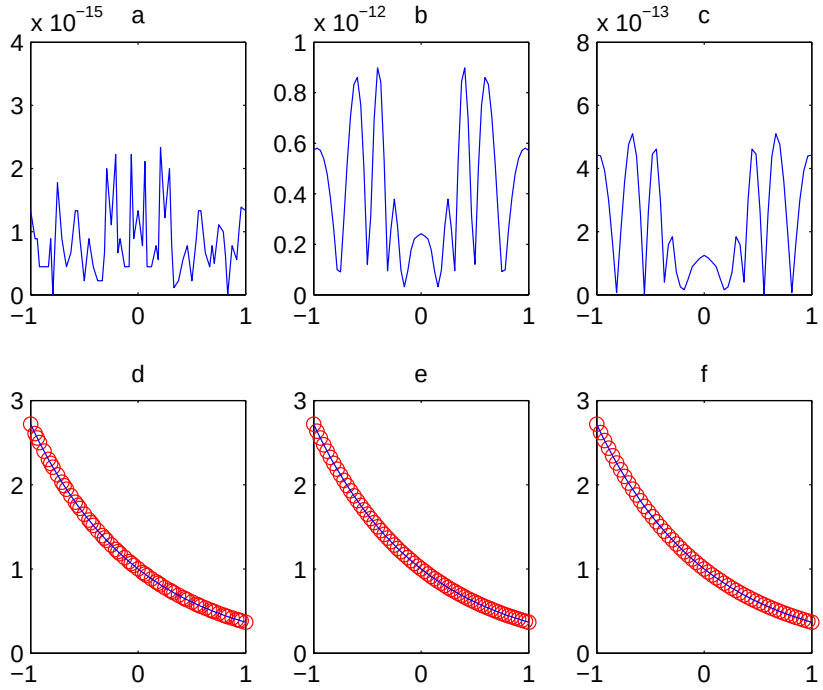

Figure 2: *The plots* a, b *and* c *are the errors and the plots* d, e *and* f *are numerical and true solutions of Example 1 using the three sequences (Bulirsch, Romberg and Bauer, respectively)*

Example 2. *This test problem demonstrates the results of application of three sequences for an oscillatory problem. Let*

$$u(x) + \int_{-\pi}^{\pi} \sin(xu(y))\, dy = f(x).$$

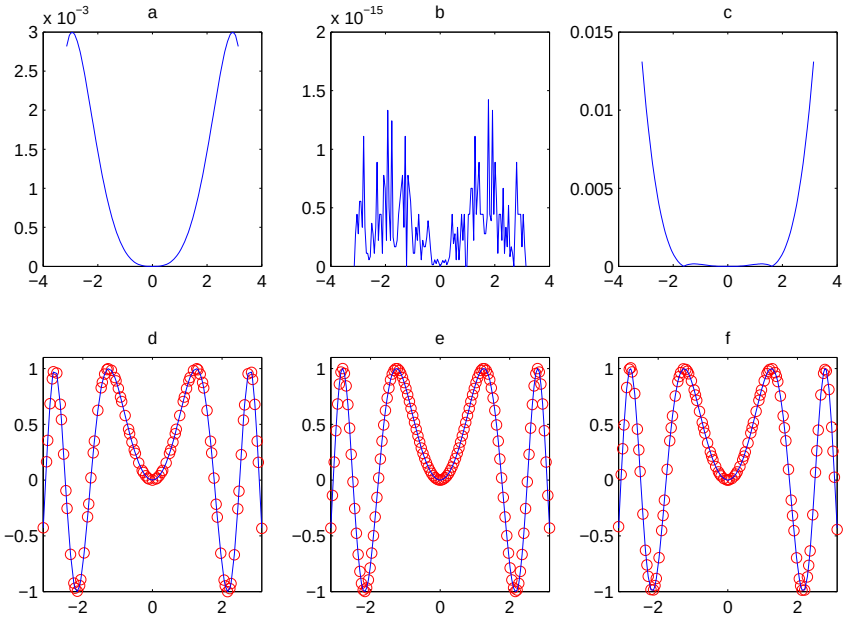

Figure 3: *The plots a, b and c are the errors and the plots d, e and f are numerical and true solutions of Example 2 using the three sequences (Bulirsch, Romberg and Bauer, respectively)*

We evaluate the function $f(x)$ in a way that the exact solution is $u(x) = \sin(x^m)$, for even integer m. Numerical results with error plots are presented in Figure 3.

The solution of this test problem is an oscillatory function with rapidly variant gradients. Among the three used sequences the Romberg sequence is closest to the uniform grid. The comparison of the Bulrisch's and Bauer's sequences discloses that clustering of nodes in the Romberg's sequence is appropriate for such problems. In fact, the distribution of the Romberg's nodes is rather uniform while the distribution of the other two sequences is not close to uniform. In other words, the nodes are densely distributed in the interval $(-2, 2)$ and Figure 3 demonstrates that the error increases outside of this interval. The given algorithms provide the opportunity to study and apply more general clustering nodes in the applications.

Example 3. *It is worth to study the comparison of the current method with the Galerkin method. To this end, we use the following test problem*

$$u(x) = f(x) + \int_0^1 \frac{1}{x + y + u(y)}\, dy, \quad 0 \le x \le 1$$

where $f(x)$ is evaluated such that for an arbitrary constant α we have the exact solution

$$u(x) = \frac{1}{x + \alpha}.$$

This problem has been solved in [5, 3] with the Galerkin method. The numerical results is given in Table 1, Table 2 and Table 3 with $N = 9, 9, 7$ nodes, respectively (see

Figure 4). The comparison of the algorithms and results of the current method with the Galerkin method justifies considerable accuracy and simplicity of this method.

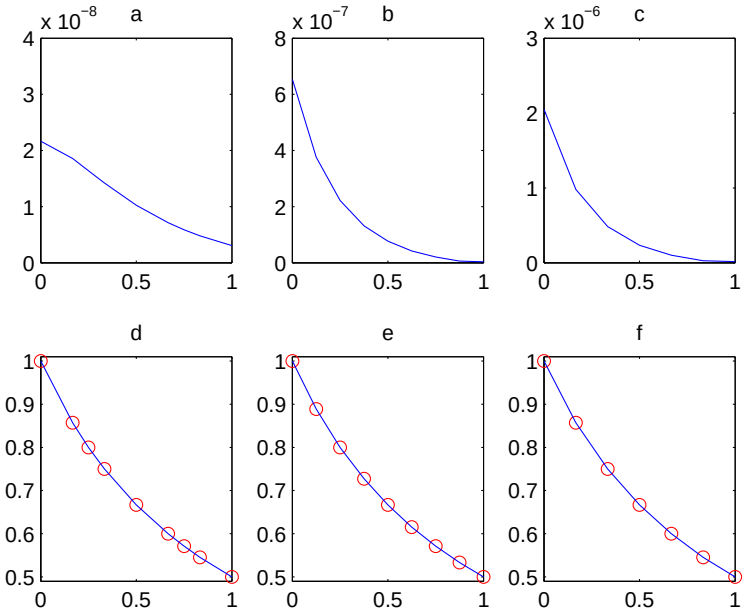

Figure 4: *The plots a, b and c are the errors and the plots d, e and f are numerical and true solutions of Example 3 using the three sequences (Bulirsch, Romberg and Bauer, respectively)*

Example 4 (see [18])**.** *In this example, we demonstrate numerical results for a two-dimensional linear integral equation*

$$u(x,y) - \int_0^1 \int_0^1 k(x,y,t,s)u(t,s)\,dtds = f(x,y), \quad 0 \le x,y \le 1,$$

where

$$k(x,y,t,s) = \exp((\frac{x}{5})^5 t) - 1,$$

and we choose $f(x,y)$ such that the exact solution is $u(x,y) = xy$. The error graph is shown in Figure 5. We have used 7×7 mesh grids of the Bauer sequence.

Example 5 (see [6])**.** *Consider the nonlinear integral equation*

$$u(x,y) - \int_0^1 \int_0^1 (t\sin s + 1)u^3(t,s)\,dtds = f(x,y), \quad 0 \le x,y \le 1,$$

where

$$f(x,y) = x\cos y + \frac{1}{20}(\cos^4 1 - 1) - \frac{1}{12}\sin 1(\cos^2 1 + 2),$$

and the exact solution is $u(x,y) = x\cos y$. The error graph is shown in Figure 6. We have used 25×25 mesh grids of the Bulirsch sequence.

x	Exact solution	Numerical solution
0.00	1.000000000	1.000000022
0.17	0.857142857	0.857142876
0.25	0.800000000	0.800000016
0.33	0.750000000	0.750000014
0.50	0.666666667	0.666666677
0.67	0.600000000	0.600000007
0.75	0.571428571	0.571428577
0.83	0.545454545	0.545454550
1.00	0.500000000	0.500000003

Table 1: *Example 3: Solution with 5 elements of Bulirsch sequence ($N = 9$)*

x	Exact solution	Numerical solution
0.00	1.000000000	1.000000653
0.13	0.888888889	0.888889265
0.25	0.800000000	0.800000222
0.38	0.727272727	0.727272859
0.50	0.666666667	0.666666743
0.63	0.615384615	0.615384658
0.75	0.571428571	0.571428592
0.88	0.533333333	0.533333339
1.00	0.500000000	0.499999996

Table 2: *Example 3: Solution with 3 elements of Romberg sequence ($N = 9$)*

x	Exact solution	Numerical solution
0.00	1.000000000	1.000002050
0.17	0.857142857	0.857143836
0.33	0.750000000	0.750000482
0.50	0.666666667	0.666666900
0.67	0.600000000	0.600000100
0.83	0.545454545	0.545454572
1.00	0.500000000	0.499999985

Table 3: *Example 3: Solution with 4 elements of Bauer sequence ($N = 7$)*

7. Discussion and future work

In this paper, we have presented a direct representation of the generalized extrapolation methods. The simplicity of evaluating the nodes and weights of the quadrature rules has been demonstrated. The low cost and higher order of accuracy are major features of the derived class of methods. It is also possible to evaluate the corresponding weights for an arbitrary set of nodes and this procedure is done one time for a given set of nodes or corresponding dividing sequences. The results are presented for equations in one and two dimensions. The extension of this class is

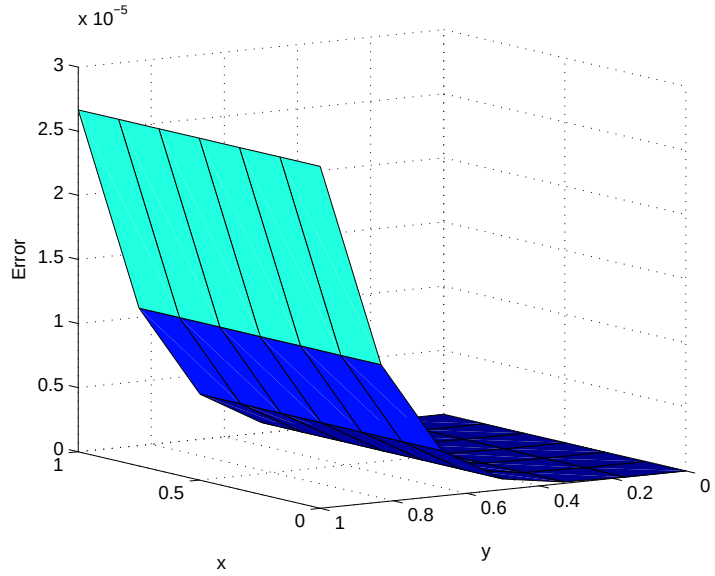

Figure 5: *Two-dimensional equation of example 4 with* 7×7 *mesh grids*

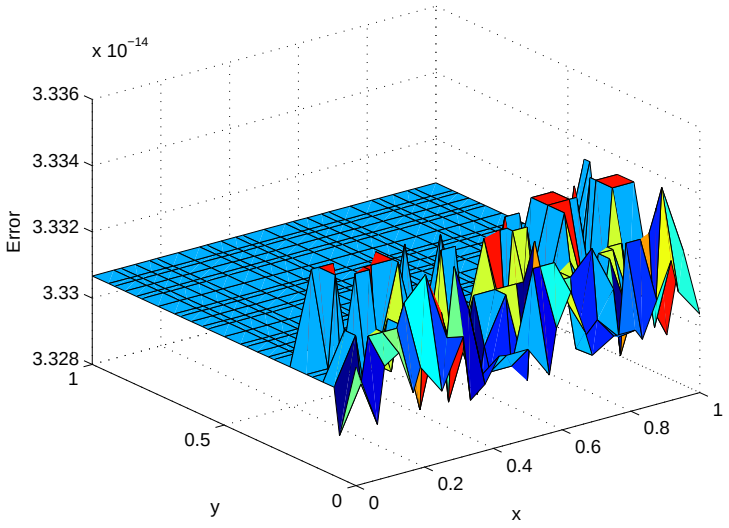

Figure 6: *Two-dimensional equation of example 5 with* 25×25 *mesh grids*

straightforward for higher dimensions. We have studied the quality of different node distributions in the solution domain. With a high oscillatory function we have illustrated the role of a different distribution of nodes. To generate an automatic code with adaptive error control we will study the possibility of error estimation in the solution of nonlinear integral equations. This is the property of the T table that as

we proceed in the columns of this table we obtain better results. Therefore, we can provide error estimates from successive approximations.

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