

Some inequalities for polynomials and transcendental entire functions of exponential type

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Abstract. Let p be a polynomial of degree n such that $|p(z)| \leq M$ ($|z| = 1$). The Bernstein's inequality for polynomials states that $|p'(z)| \leq Mn$ ($|z| = 1$). A polynomial p of degree n that satisfies the condition $p(z) \equiv z^n p(1/\bar{z})$ is called a self-reciprocal polynomial. If p is a self-reciprocal polynomial, then $f(z) = p(e^{iz})$ is an entire function of exponential type n such that $f(z) = e^{inz} f(-z)$. Thus the class of entire functions of exponential type τ whose elements satisfy the condition $f(z) = e^{i\tau z} f(-z)$ is a natural generalization of the class of self-reciprocal polynomials. In this paper we present some Bernstein's type inequalities for self-reciprocal polynomials and related entire functions of exponential type under certain restrictions on the location of their zeros.

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1. Introduction and statement of results

1.1. Bernstein's inequality for polynomials

Let $\mathcal{P}_n$ denote the class of all polynomials of degree at most n and let $f \in \mathcal{P}_n$. An inequality for polynomials in $\mathcal{P}_n$, known as Bernstein's inequality, gives an estimate for $|f'(z)|$ on the unit circle in terms of the maximum of $|f(z)|$ on the same circle. It states (see [15], p. 508) that

$$\max_{|z|=1} |f'(z)| \leq n \max_{|z|=1} |f(z)|, \quad f \in \mathcal{P}_n, \quad (1)$$

where the equality holds for polynomials of the form cz^n , $c \neq 0$.

It is known [13] that if f is as above and $f^*(z) := z^n \overline{f(1/\bar{z})}$, then on $|z| = 1$

$$|f'(z)| + |f^{*'}(z)| \leq n \max_{|z|=1} |f(z)|, \quad f \in \mathcal{P}_n. \quad (2)$$

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Let $\mathcal{P}_n^\sim$ be the subclass of $\mathcal{P}_n$ consisting of all polynomials f which satisfy the condition $f(z) \equiv f^*(z)$. It follows from (2) that

$$\max_{|z|=1} |f'(z)| \leq \frac{n}{2} \max_{|z|=1} |f(z)|, \quad f \in \mathcal{P}_n^\sim. \quad (3)$$

Let $f \in \mathcal{P}_n$ and z_0 a point on the unit circle such that $|f(z_0)| = \max_{|z|=1} |f(z)|$. Clearly, $|f^{*'}(z_0)| = |nf(z_0) - z_0 f'(z_0)| \geq n|f(z_0)| - |f'(z_0)|$. Hence, if $f \in \mathcal{P}_n^\sim$, then

$$\max_{|z|=1} |f'(z)| \geq \frac{n}{2} |f'(z_0)| = \frac{n}{2} \max_{|z|=1} |f(z)|$$

and so, in (3), the inequality sign “ $\leq$ ” may be replaced by “ $=$ ”. Thus, we have

$$\max_{|z|=1} |f'(z)| = \frac{n}{2} \max_{|z|=1} |f(z)|, \quad f \in \mathcal{P}_n^\sim. \quad (4)$$

The subclass $\mathcal{P}_n^\sim$ of $\mathcal{P}_n$ is of considerable importance. There is another subclass of $\mathcal{P}_n$ which has proved itself to be equally significant, if not more. It consists of those polynomials f in $\mathcal{P}_n$ which satisfy the condition $f(z) \equiv z^n f(1/z)$. Let us denote it by $\mathcal{P}_n^\vee$. The condition defining the subclass $\mathcal{P}_n^\vee$ looks very similar to the one defining $\mathcal{P}_n^\sim$. As regards the distribution of their zeros, polynomials in $\mathcal{P}_n^\sim$ and those in $\mathcal{P}_n^\vee$, they all have at least half of their zeros outside the open unit disk (here it is understood that a polynomial f belonging to $\mathcal{P}_n$ but of degree $m < n$ has $n - m$ of its zeros at ∞).

Frappier, Rahman and Ruscheweyh ([6], p. 97) showed that for the polynomial $f(z) := \{(1 - iz)^2 + z^{n-2}(z - i)^2\}/4$, which clearly belongs to $\mathcal{P}_n^\vee$, we have

$$\max_{|z|=1} |f(z)| = 1 = |f(i)| \quad \text{whereas} \quad |f'(-i)| = n - 1,$$

thus exhibiting a polynomial f in $\mathcal{P}_n^\vee$ for which

$$\max_{|z|=1} |f'(z)| \geq (n - 1) \max_{|z|=1} |f(z)|. \quad (5)$$

Later Frappier, Rahman and Ruscheweyh ([7, Theorem 2]) proved that for polynomials $f(z) := \sum_{\nu=0}^n a_\nu z^\nu$, whose constant term a_0 is equal to a_n (the coefficient of the leading term $a_n z^n$), we have

$$\max_{|z|=1} |f'(z)| \leq \left(n - \frac{1}{2} + \frac{1}{2(n+1)} \right) \max_{|z|=1} |f(z)|. \quad (6)$$

Since f belongs to $\mathcal{P}_n^\vee$ if and only if $a_k = a_{n-k}$ for each k ($k = 0 \dots n$), the above inequality certainly holds for polynomials in $\mathcal{P}_n^\vee$. Inequalities (5) and (6) show that by restricting ourselves to the subclass $\mathcal{P}_n^\vee$, we do not obtain a meaningful improvement on the Bernstein's inequality (1). This is quite surprising since the two classes $\mathcal{P}_n^\sim$ and $\mathcal{P}_n^\vee$ look similar; for $\mathcal{P}_n^\sim$ holds formula (4) by which $|f'(z)|$ at a point of the unit circle cannot be larger than $n/2$ times $M := \max_{|z|=1} |f(z)|$ if $f \in \mathcal{P}_n^\sim$ while it can be as large as $n - 1$ times M if f belongs to $\mathcal{P}_n^\vee$, as (5) says.

However, under some additional restrictions, either on the location of the zeros or on the coefficients of polynomials in $\mathcal{P}_n^\vee$, the bound in (6) can be improved. For example, Rahman and Tariq [16] (see also [11]) proved that for a polynomial $f(z) := \sum_{\nu=0}^n a_\nu z^\nu$ in $\mathcal{P}_n^\vee$, whose coefficients lie in a sector of opening $0 \leq \gamma < \pi$ with the vertex at the origin, we have

$$\max_{|z|=1} |f'(z)| \leq \frac{n}{2 \cos(\gamma/2)} |f(1)|. \quad (7)$$

In the case when n is an even integer, the equality holds in (7) for the polynomial $f(z) = z^n + 2e^{i\gamma} z^{n/2} + 1$.

On the other hand, if we assume that all the zeros of f are in the left half plane or in the right half plane [9], then

$$\max_{|z|=1} |f'(z)| \leq \frac{n}{\sqrt{2}} \max_{|z|=1} |f(z)|. \quad (8)$$

Very few sharp results are known about the class $\mathcal{P}_n^\vee$ although many papers have been written on the subject since 1976 (see for example, [9, 11, 16]). In fact, the sharp inequality analogous to (1) is still unknown even for $n = 3$.

The Bernstein's inequality has been generalized in many ways. For example, if f is a polynomial in $\mathcal{P}_n$, then by Zygmund [19] for any $p \geq 1$, we have

$$\int_{-\pi}^{\pi} |f'(e^{i\theta})|^p d\theta \leq n^p \int_{-\pi}^{\pi} |f(e^{i\theta})|^p d\theta, \quad f \in \mathcal{P}_n. \quad (9)$$

If we assume that f belongs to $\mathcal{P}_n^\sim$, the above inequality can be improved. In this case Dewan and Govil [5] proved the following result

$$\int_{-\pi}^{\pi} |f'(e^{i\theta})|^p d\theta \leq n^p C_p \int_{-\pi}^{\pi} |f(e^{i\theta})|^p d\theta, \quad f \in \mathcal{P}_n^\sim, \quad (10)$$

where

$$C_p = \frac{2\pi}{\int_{-\pi}^{\pi} |1 + e^{i\alpha}|^p d\alpha} = 2^{-p} \frac{\sqrt{\pi} \Gamma(p/2 + 1)}{\Gamma(p/2 + 1/2)}. \quad (11)$$

In this paper, we present a property of polynomials in $\mathcal{P}_n^\vee$ which have all their zeros in the left half plane. More precisely, we have the following

Theorem 1. *Let f be a polynomial in $\mathcal{P}_n^\vee$ having all its zeros in the left half plane. Suppose in addition that its zeros which lie in the second quadrant are of modulus at most 1. Then*

$$|f'(e^{-i\theta})| \leq |f'(e^{i\theta})|, \quad 0 \leq \theta \leq \pi. \quad (12)$$

As the first application of Theorem 1, we will prove the following L^p inequality for the subclass $\mathcal{P}_n^\vee$. We do not know if it is sharp.

Corollary 1. *Let f , which has all its zeros in the left half plane, belong to $\mathcal{P}_n^\vee$. Furthermore, the zeros in the second quadrant are in the unit disk $\{z : |z| \leq 1\}$. Then, for $p \geq 1$*

$$\int_{-\pi}^0 |f'(e^{i\theta})|^p d\theta \leq n^p C_p \int_{-\pi}^0 |f(e^{i\theta})|^p d\theta, \quad (13)$$

where C_p is as given in (11).

As the next application we state the following corollary.

Corollary 2. *Let f , which has all its zeros in the left half plane, belong to $\mathcal{P}_n^\vee$. Furthermore, the zeros in the second quadrant are in the unit disk $\{z : |z| \leq 1\}$. Suppose that $|f(e^{-i\theta})| \leq M$ for $0 \leq \theta \leq \pi$. Then*

$$|f'(e^{-i\theta})| \leq M \frac{n}{2}, \quad 0 \leq \theta \leq \pi. \quad (14)$$

The example $f(z) = (z^2 + 1)^{\frac{n}{2}}$ shows that the estimate is sharp when n is even. For odd n , the equality holds for $f(z) = (z + 1)^n$.

1.2. Transcendental entire functions of exponential type

For an entire function f and a real number $r > 0$, let $M(r) = M_f(r) := \max_{|z|=r} |f(z)|$. Unless f is a constant of modulus less than or equal to 1, its order, which is denoted by ρ , is defined to be $\limsup_{r \rightarrow \infty} (\log r)^{-1} \log \log M(r)$. Constants of modulus less than or equal to 1 are of order 0 by convention.

If f is of finite positive order ρ , then $T := \limsup_{r \rightarrow \infty} r^{-\rho} \log M(r)$ is called its type.

An entire function f is said to be of exponential type τ if for any $\varepsilon > 0$ there exists a constant $k(\varepsilon)$ such that $|f(z)| \leq k(\varepsilon)e^{(\tau+\varepsilon)|z|}$ for all $z \in \mathbb{C}$. Any entire function of order less than 1 is of exponential type τ , where τ can be taken to be any number greater than or equal to 0. Functions of order 1 type $T \leq \tau$ are also of exponential type τ .

If f is an entire function of exponential type, then its indicator function $h_f(\theta)$ is defined by $h_f(\theta) := \limsup_{r \rightarrow \infty} r^{-1} \log |f(re^{i\theta})|$. It describes the growth of f along the ray $\{z : \arg z = \theta\}$. $h_f(\theta)$ is either finite or $-\infty$ and is a continuous function of θ unless it is identically $-\infty$.

For a detailed discussion on entire functions of exponential type, we refer the reader to Boas [4].

Bernstein [2], (see also [3], p. 102) extended inequality (1) to arbitrary entire functions of exponential type bounded on the real line.

Theorem 2. *Let f be an entire function of exponential type $\tau > 0$ such that $|f(x)| \leq M$ on the real axis. Then*

$$\sup_{-\infty < x < \infty} |f'(x)| \leq M\tau. \quad (15)$$

The equality in (15) holds if and only if $f(z) \equiv ae^{i\tau z} + be^{-i\tau z}$, where $a, b \in \mathbb{C}$.

If $f \in \mathcal{P}_n^\vee$, then $g(z) := f(e^{iz})$ is an entire function of exponential type which satisfies the condition $g(z) \equiv e^{inz}g(-z)$. Moreover, its type is n . This suggests that the class of entire functions of exponential type that generalizes $\mathcal{P}_n^\vee$ consists of entire functions of exponential type f such that $f(z) \equiv e^{i\tau z}f(-z)$. Let us denote this class by $\mathcal{F}_\tau^\vee$ which has been studied by Govil [8], Rahman and Tariq [17, 18].

Rahman and Tariq ([17, Theorem 2]) proved the following Theorem which is akin to (5), a result proved by Frappier, Rahman and Ruscheweyh [7] for polynomials.

Theorem 3. *For a given positive number ε , as small as we please, there exists an entire function $f_\varepsilon \in \mathcal{F}_\tau^\vee$ such that*

$$\sup_{-\infty < x < \infty} |f'_\varepsilon(x)| \geq (\tau - \varepsilon) \sup_{-\infty < x < \infty} |f_\varepsilon(x)|. \quad (16)$$

Like polynomials, improved inequalities for $\mathcal{F}_\tau^\vee$ can be obtained if we impose some additional restriction on it. For example, Rahman and Tariq ([17, Theorem 1]) proved the following theorem for functions in $\mathcal{F}_\tau^\vee$ which are uniformly almost periodic on the real line. It is clearly an extension of (7) for entire functions of exponential type.

Theorem 4. *Let $f \in \mathcal{F}_\tau^\vee$ be uniformly almost periodic on the real line. Furthermore, suppose that the coefficients $A_1, A_2, \dots$ of the Fourier series $\sum_{n=1}^{\infty} A_n e^{i\Lambda_n x}$ of f lie in a sector of opening $0 \leq \gamma < \pi$ with the vertex at the origin. Then*

$$\sup_{-\infty < x < \infty} |f'(x)| \leq \frac{\tau}{2 \cos(\gamma/2)} |f(0)|. \quad (17)$$

The result is best possible as the equality holds for $f(z) = e^{i\tau z} + 2e^{i\gamma} e^{i\tau z/2} + 1$.

Let $p > 0$ be a real number. We say that a function f belongs to L^p on the real line if, $\int_{-\infty}^{\infty} |f(x)|^p dx < \infty$. Inequalities (9) and (10) have been generalized for entire functions of exponential type as well. For example, as a generalization of (9) we have

Theorem 5. *Let f be an entire function of exponential type τ that belongs to L^p on the real line, where $p > 0$ is a real number. Then*

$$\int_{-\infty}^{\infty} |f'(x)|^p dx \leq \tau^p \int_{-\infty}^{\infty} |f(x)|^p dx. \quad (18)$$

For various refinement and detailed information we refer the reader to the paper of Rahman and Schemeisser [14].

For functions f in $\mathcal{F}_\tau^\vee$ that belong to L^2 on the real line, Rahman and Tariq ([18, Theorem 3]) proved that

$$\int_{-\infty}^{\infty} |f'(x)|^2 dx \leq \frac{\tau^2}{2} \int_{-\infty}^{\infty} |f(x)|^2 dx, \quad (19)$$

where the coefficient $\tau^2/2$ of $\int_{-\infty}^{\infty} |f(x)|^2 dx$ cannot be replaced by a smaller number.

In this paper, we present the following theorem for functions in $\mathcal{F}_\tau^\vee$ that have all their zeros in the first and the third quadrants. It is clearly an extension of Theorem 1 for entire functions of exponential type.

Theorem 6. *Let f , which has all its zeros in the first and the third quadrants, belong to $\mathcal{F}_\tau^\vee$. Then*

$$|f'(-x)| \leq |f'(x)|, \quad x > 0. \quad (20)$$

As applications of Theorem 6, we state the following inequality about functions in $\mathcal{F}_\tau^\vee$. We do not know if it is sharp.

Corollary 3. Let f , which has all its zeros in the first and the third quadrants, belong to $\mathcal{F}_\tau^\vee$. Further suppose that $f \in L^p$ on $(-\infty, 0)$. Then, for $p \geq 1$

$$\int_{-\infty}^0 |f'(x)|^p dx \leq \tau^p C_p \int_{-\infty}^0 |f(x)|^p dx, \quad (21)$$

where C_p is as given in (11).

Corollary 4. Let f , which has all its zeros in the first and the third quadrants, belong to $\mathcal{F}_\tau^\vee$. Further assume that $|f(x)| \leq M$ on $(-\infty, 0)$. Then

$$|f'(x)| \leq \frac{M\tau}{2}, \quad x \leq 0. \quad (22)$$

The estimate is sharp as the example $M(1 + e^{i\tau z})/2$ shows.

Corollary 5. Let f , which has all its zeros in the first and the third quadrants, belong to $\mathcal{F}_\tau^\vee$. Further assume that $|f(x)| \leq M$ on $(-\infty, 0)$. Then

$$|f'(x + iy)| \leq \frac{M\tau}{2} e^{-\tau y}, \quad x < 0, y < 0. \quad (23)$$

The estimate is sharp as the example $M(1 + e^{i\tau z})/2$ shows.

1.3. Mean value of entire functions of exponential type

Let $p > 0$ be a real number. For a function f , the mean of order p on the real line is defined by

$$M^p f(x) = \limsup_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |f(x)|^p dx. \quad (24)$$

We say that f has a bounded mean of order p , if $M^p f(x) < \infty$. It can be easily seen that a function bounded on the real axis will always have a bounded mean. However, there are functions which have a bounded mean but not bounded on the real line. Harvey [12] considered the problems of the mean value of entire functions of exponential type. Here is one of his results.

Theorem 7. If f is an entire function of exponential type τ , then

$$M^p f'(x) \leq \frac{(p+2)2^{p+2}}{\pi\tau p \delta^{p+1}} (e^{\tau\delta p} - 1) M^p f(x), \quad p > 0, \quad (25)$$

where δ is an arbitrary positive number.

However, when p is greater than one, the constant in the above theorem can be replaced by τ^p . More precisely, Harvey [12] proved that

Theorem 8. If f is an entire function of exponential type τ , then

$$M^p f'(x) \leq \tau^p M^p f(x), \quad p > 1. \quad (26)$$

As to the mean value of functions in $\mathcal{F}_\tau^\vee$, Rahman and Tariq [18] considered the case when $p = 2$ and obtained the following inequality.

Theorem 9. *If f , which is a uniformly almost periodic function on the real line, belongs to $\mathcal{F}_\tau^\vee$, then*

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \int_{-T}^0 |f'(x)|^2 dx \leq \frac{\tau^2}{2} \limsup_{T \rightarrow \infty} \frac{1}{T} \int_{-T}^0 |f(x)|^2 dx. \quad (27)$$

Inequality (27) is sharp as the example $f(z) = (1 + e^{i\tau z})/2$ shows.

Here, we will prove the following inequality about the mean value theorem for functions in $\mathcal{F}_\tau^\vee$. We do not know if it is sharp.

Theorem 10. *Let f , which has all its zeros in the first and the third quadrants, belong to $\mathcal{F}_\tau^\vee$. Assume further that f has a bounded mean of order p where $p \geq 1$. Then*

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \int_{-T}^0 |f'(x)|^p dx \leq \tau^p C_p \limsup_{T \rightarrow \infty} \frac{1}{T} \int_{-T}^0 |f(x)|^p dx, \quad (28)$$

where C_p is as given in (11).

The rest of the paper is organized as follows. In Section 2, we list all the lemmas needed in our proofs. Section 3 deals with the proofs of Theorem 1, Theorem 6 and Theorem 10 and their corollaries discussed above.

2. Lemmas

The first two Lemmas have been proved by Rahman and Tariq [18].

Lemma 1. *Let f belong to $\mathcal{F}_\tau^\vee$ such that $|f(x)|$ is bounded on the real line. Then, for any real γ and $s = -\gamma/\tau$, we have*

$$-i \{e^{i\gamma} f'(x) + e^{i\tau x} f'(-x)\} = \sum_{n=-\infty}^{\infty} c_n f\left(x - s + \frac{n\pi}{\tau}\right), \quad x \in \mathbb{R}, \quad (29)$$

where

$$c_n = \frac{1}{(s\tau - n\pi)^2} \{1 + (-1)^n\} \{1 - (-1)^n \cos \gamma\} \tau, \quad n = 0, \pm 1, \pm 2, \dots$$

and $\sum_{n=-\infty}^{\infty} |c_n| = \tau$.

Lemma 2. *Let f belong to $\mathcal{F}_\tau^\vee$ such that $|f(x)| \leq M$ on the real line. Then*

$$|f'(x)| + |f'(-x)| \leq M\tau, \quad x \in \mathbb{R}. \quad (30)$$

We will make use of the following interpolation formula due to Aziz and Mohammad [1].

Lemma 3. Let f belong to $\mathcal{P}_n$ and let $\xi_1, \xi_2, \dots, \xi_n$ be the zeros of $z^n + a$, where $a \neq -1$ is an arbitrary complex number. Then for any complex number z we have

$$zf'(z) = \frac{n}{1+a}f(z) + \frac{1+a}{na} \sum_{\nu=1}^n c_\nu(a)f(z\xi_\nu), \quad (31)$$

where

$$\sum_{\nu=1}^n c_\nu(a) = \sum_{\nu=1}^n \frac{\xi_\nu}{(\xi_\nu - 1)^2} = -\frac{n^2 a}{(1+a)^2}.$$

The next inequality, that can be found in Malik [13] (also see, Govil and Rahman ([10], Inequality (3.2)) where this inequality is given for any order derivatives) is well-known and widely used in the study of polynomials.

Lemma 4. Let f belong to $\mathcal{P}_n$. Define $g(z) \equiv z^n \overline{f(1/\bar{z})}$, a polynomial in $\mathcal{P}_n$. Then

$$|f'(z)| + |g'(z)| \leq n \max_{|z|=1} |f(z)|, \quad |z| = 1. \quad (32)$$

Lemma 5. Let us denote $\omega_\nu = z_\nu + 1/z_\nu$ and $\omega_\mu = z_\mu + 1/z_\mu$, where z_ν, z_μ are complex numbers such that $\pi/2 \leq \arg z_\nu, \arg z_\mu \leq \pi$ and $|z_\nu| \leq 1, |z_\mu| \leq 1$. Define

$$\begin{aligned} F(x; \omega_\nu, \omega_\mu) &= -4x^2 \Im(\omega_\nu - \bar{\omega}_\mu) + 4x \Im(\omega_\nu \omega_\mu) - (|\omega_\mu|^2 \Im \omega_\nu + |\omega_\nu|^2 \Im \omega_\mu), \\ G(x; \omega_\nu) &= -2(x+1) \Im \omega_\nu. \end{aligned}$$

Then for $-1 \leq x \leq 1$,

$$\begin{aligned} F(x; \omega_\nu, \omega_\mu) &\geq 0, \\ G(x; \omega_\nu) &\geq 0. \end{aligned}$$

Proof. First, we note that ω_ν may be written as $\omega_\nu := x_\nu R_\nu + iy_\nu L_\nu$, where

$$R_\nu = \left(1 + \frac{1}{x_\nu^2 + y_\nu^2}\right) \geq 0, \quad L_\nu = \left(1 - \frac{1}{x_\nu^2 + y_\nu^2}\right) \leq 0,$$

$x_\nu = \Re z_\nu$ and $y_\nu = \Im z_\nu$. Since $\arg z_\nu$ lies in $[\pi/2, \pi]$, it implies that $\Re \omega_\nu \leq 0$ and $\Im \omega_\nu \leq 0$. Similarly, $\omega_\mu := x_\mu R_\mu + iy_\mu L_\mu$, where $R_\mu \geq 0, L_\mu \leq 0, \Re \omega_\mu \leq 0$ and $\Im \omega_\mu \leq 0$.

$F(x; \omega_\nu, \omega_\mu)$ is a quadratic function of the form $ax^2 + bx + c$, where $a = -4\Im(\omega_\nu - \bar{\omega}_\mu) \geq 0, b = 4\Im(\omega_\nu \omega_\mu) \geq 0$, and $c = -(|\omega_\mu|^2 \Im \omega_\nu + |\omega_\nu|^2 \Im \omega_\mu) \geq 0$. Its vertex is $(-b/2a, F(-b/2a; \omega_\nu, \omega_\mu))$, where $-b/2a = \Im(\omega_\nu \omega_\mu)/2\Im(\omega_\nu - \bar{\omega}_\mu)$ and

$$\begin{aligned} F(-b/2a; \omega_\nu, \omega_\mu) &= \frac{(\Im(\omega_\nu \omega_\mu))^2}{\Im(\omega_\nu - \bar{\omega}_\mu)} - (|\omega_\mu|^2 \Im \omega_\nu + |\omega_\nu|^2 \Im \omega_\mu) \\ &= \frac{(\Im(\omega_\nu \omega_\mu))^2 - (|\omega_\mu|^2 \Im \omega_\nu + |\omega_\nu|^2 \Im \omega_\mu) \Im(\omega_\nu - \bar{\omega}_\mu)}{\Im(\omega_\nu - \bar{\omega}_\mu)} \end{aligned}$$

The numerator $(\Im(\omega_\nu \omega_\mu))^2 - (|\omega_\mu|^2 \Im \omega_\nu + |\omega_\nu|^2 \Im \omega_\mu) \Im(\omega_\nu - \bar{\omega}_\mu)$ of the above expression is equal to

$$\begin{aligned} & (x_\mu R_\mu y_\nu L_\nu + x_\nu R_\nu y_\mu L_\mu)^2 - \{y_\nu L_\nu (x_\mu^2 R_\mu^2 + y_\mu^2 L_\mu^2) + y_\mu L_\mu (x_\nu^2 R_\nu^2 + y_\nu^2 L_\nu^2)\} (y_\nu L_\nu + y_\mu L_\mu) \\ &= -[y_\mu L_\mu y_\nu L_\nu \{(x_\mu R_\mu - x_\nu R_\nu)^2 + (y_\mu^2 L_\mu^2 + y_\nu^2 L_\nu^2)\} + 2y_\nu^2 L_\nu^2 y_\mu^2 L_\mu^2] \\ &\leq 0. \end{aligned} \quad (33)$$

Since $\Im(\omega_\nu - \bar{\omega}_\mu) \leq 0$, we have $a \geq 0$ and $F(-b/2a; \omega_\nu, \omega_\mu) \geq 0$. Also, $F(x; \omega_\nu, \omega_\mu)$ will attain the minimum value at the vertex. Thus, for any real number x we have $F(x; \omega_\nu, \omega_\mu) \geq F(-b/2a; \omega_\nu, \omega_\mu) \geq 0$ and hence in particular for $-1 \leq x \leq 1$.

As far as the function G is concerned, we just have to note that $\Im \omega_\nu \leq 0$, which shows that $G(x; \omega_\nu) = -2(x+1)\Im \omega_\nu \geq 0$ for $-1 \leq x \leq 1$. $\square$

3. Proofs of Theorem 1, Theorem 6 and Theorem 10

3.1. Proof of Theorem 1 and its corollaries

Case 1. f has all its zeros on the unit circle

Let us assume that $z_\nu := e^{i\theta_\nu}$, where $\pi/2 \leq \theta_\nu \leq \pi$ ($\nu = 1, 2, \dots, l$) are l zeros of f . Since f belongs to $\mathcal{P}_n^\vee$, for every ν , $1/z_\nu = e^{-i\theta_\nu}$ is also a zero of f . Assume further that f has a zero of multiplicity m at -1 , where $m \geq 0$. Thus f may be written as

$$f(z) = (z+1)^m \prod_{\nu=1}^l (z - e^{i\theta_\nu})(z - e^{-i\theta_\nu}),$$

where $n = 2l + m$. Let θ be a number in $[0, \pi]$ such that $\theta \neq \theta_\nu$, ($\nu = 1, 2, \dots, l$). Then

$$\frac{f'(e^{i\theta})}{f(e^{i\theta})} = \Re \frac{f'(e^{i\theta})}{f(e^{i\theta})} + i \Im \frac{f'(e^{i\theta})}{f(e^{i\theta})},$$

where

$$\Re \frac{f'(e^{i\theta})}{f(e^{i\theta})} = \frac{m(\cos \theta + 1)}{|1 + e^{i\theta}|^2} + \sum_{\nu=1}^l \frac{\cos \theta - \cos \theta_\nu}{|e^{i\theta} - e^{i\theta_\nu}|^2} + \frac{\cos \theta - \cos \theta_\nu}{|e^{i\theta} - e^{-i\theta_\nu}|^2}$$

and

$$\Im \frac{f'(e^{i\theta})}{f(e^{i\theta})} = -\frac{m \sin \theta}{|1 + e^{i\theta}|^2} + \sum_{\nu=1}^l \frac{\sin \theta_\nu - \sin \theta}{|e^{i\theta} - e^{i\theta_\nu}|^2} - \frac{\sin \theta_\nu + \sin \theta}{|e^{i\theta} - e^{-i\theta_\nu}|^2}.$$

Note that $\Re(f'(e^{i\theta})/f(e^{i\theta}))$ is an even function of θ and $\Im(f'(e^{i\theta})/f(e^{i\theta}))$ is an odd function of θ . So $|f'(e^{i\theta})|/|f(e^{i\theta})|$ is equal to

$$\begin{aligned} & \sqrt{\left(\Re \frac{f'(e^{i\theta})}{f(e^{i\theta})}\right)^2 + \left(\Im \frac{f'(e^{i\theta})}{f(e^{i\theta})}\right)^2} = \sqrt{\left(\Re \frac{f'(e^{-i\theta})}{f(e^{-i\theta})}\right)^2 + \left(-\Im \frac{f'(e^{-i\theta})}{f(e^{-i\theta})}\right)^2} \\ &= \left| \frac{f'(e^{-i\theta})}{f(e^{-i\theta})} \right|. \end{aligned} \quad (34)$$

For $f \in \mathcal{P}_n^\vee$ and $\theta \in [0, \pi]$, we have $|f(e^{-i\theta})| = |f(e^{i\theta})|$. So we conclude, from (34)

$$|f'(e^{-i\theta})| \leq |f'(e^{i\theta})|, \quad 0 \leq \theta \leq \pi, f(e^{i\theta}) \neq 0.$$

By continuity, the same must hold for those θ for which $f(e^{i\theta}) = 0$.

Case 2. f is a second degree polynomial

Let z_ν be the zero of f such that $\pi/2 \leq \arg z_\nu \leq \pi$ and $|z_\nu| \leq 1$. The polynomial f and its derivative f' may be written as $f(z) = (z - z_\nu)(z - 1/z_\nu)$ and $f'(z) = 2z - \omega_\nu$, respectively, where $\omega_\nu := z_\nu + 1/z_\nu = x_\nu R_\nu + i y_\nu L_\nu$, $-1 \leq x_\nu = \Re z_\nu \leq 0$,

$$R_\nu = \left(1 + \frac{1}{x_\nu^2 + y_\nu^2}\right) > 0, 0 \leq y_\nu = \Im z_\nu \leq 1 \text{ and } L_\nu = \left(1 - \frac{1}{x_\nu^2 + y_\nu^2}\right) < 0.$$

The conditions on $R_\nu, L_\nu, x_\nu, y_\nu$ ensure that ω_ν lies in the third quadrant. Thus, we have

$$|f'(e^{-i\theta})| = |2e^{-i\theta} - \omega_\nu| \leq |2e^{i\theta} - \omega_\nu| = |f'(e^{i\theta})|, \quad 0 \leq \theta \leq \pi.$$

This proves the theorem when f is a polynomial of degree 2. We also note that for $0 \leq \theta \leq \pi$, $|f(e^{-i\theta})| = |f(e^{i\theta})|$. Thus, for any f in $\mathcal{P}_2^\vee$ we have

$$\left| \frac{f'(e^{-i\theta})}{f(e^{-i\theta})} \right| \leq \left| \frac{f'(e^{i\theta})}{f(e^{i\theta})} \right|, \quad 0 \leq \theta \leq \pi. \quad (35)$$

Case 3. Not all the zeros of f are on the unit circle

Let z_ν ($\nu = 1, 2, \dots, l$) be the zeros of f such that $\pi/2 \leq \arg z_\nu \leq \pi$ and $|z_\nu| \leq 1$. Also suppose that f has a zero of multiplicity m at -1 where $m \geq 0$. Then f can be represented as

$$f(z) = (z + 1)^m \prod_{\nu=1}^l g_\nu(z),$$

where $g_\nu(z) = (z - z_\nu)(z - 1/z_\nu)$ is a second degree polynomial in $\mathcal{P}_2^\vee$ for each ν . For any z on the unit circle such that $f(z) \neq 0$ we have

$$\frac{f'(z)}{f(z)} = \frac{m}{z + 1} + \sum_{\nu=1}^l \frac{g'_\nu(z)}{g_\nu(z)}.$$

A straightforward calculation gives us

$$\begin{aligned} \left| \frac{f'(z)}{f(z)} \right|^2 &= \left| \frac{m}{z + 1} \right|^2 \\ &+ \sum_{\nu=1}^l \left(\left| \frac{g'_\nu(z)}{g_\nu(z)} \right|^2 + 2\Re \left(\frac{m}{(z + 1)} \frac{g'_\nu(z)}{g_\nu(z)} \right) + 2 \sum_{\mu=\nu+1}^l \Re \left(\frac{g'_\nu(z)}{g_\nu(z)} \frac{\overline{g'_\mu(z)}}{\overline{g_\mu(z)}} \right) \right). \end{aligned} \quad (36)$$

There are four parts in the above equation. We will compare the value of each part at $e^{-i\theta}$ and $e^{i\theta}$, respectively.

The first part $|m/(z+1)|^2$ gives us

$$\left| \frac{m}{e^{-i\theta} + 1} \right|^2 = \left| \frac{m}{e^{i\theta} + 1} \right|^2, \quad 0 \leq \theta \leq \pi. \quad (37)$$

Since $g_\nu(z)$ belongs to $\mathcal{P}_2^\vee$ for each ν , from (35) the second part $|g'_\nu(z)/g_\nu(z)|^2$ gives us

$$\left| \frac{g'_\nu(e^{-i\theta})}{g_\nu(e^{-i\theta})} \right|^2 \leq \left| \frac{g'_\nu(e^{i\theta})}{g_\nu(e^{i\theta})} \right|^2, \quad 0 \leq \theta \leq \pi. \quad (38)$$

Let $z = x + iy$ be a point on the unit circle. From Case 2 again, it is easy to verify that

$$\begin{aligned} \frac{m}{(z+1)} \frac{g'_\nu(z)}{g_\nu(z)} &= \frac{m}{(z+1)} \frac{2z - \omega_\nu}{z^2 - \omega_\nu z + 1} \\ &= \frac{m}{|z+1|^2} \frac{2z - \omega_\nu}{|z^2 - \omega_\nu z + 1|^2} (z+1)(\bar{z}^2 - \bar{\omega}_\nu \bar{z} + 1) \\ &= \frac{m}{|z+1|^2} \frac{(Q_1(x; \omega_\nu) + y S_1(x; \omega_\nu)) + i(Q_2(x; \omega_\nu) + y S_2(x; \omega_\nu))}{|z^2 - \omega_\nu z + 1|^2}, \end{aligned} \quad (39)$$

where

$$\begin{aligned} Q_1(x; \omega_\nu) &= (x+1)(4x - 2(x+1)\Re\omega_\nu + |\omega_\nu|^2); \\ S_1(x; \omega_\nu) &= -2(x+1)\Im\omega_\nu; \\ Q_2(x; \omega_\nu) &= 2(1-x^2)\Im\omega_\nu; \\ S_2(x; \omega_\nu) &= (4x + 2(x-1)\Re\omega_\nu - |\omega_\nu|^2). \end{aligned} \quad (40)$$

Thus from Lemma 5, (39) and the fact that

$$|e^{-2i\theta} - e^{-i\theta}\omega_\nu + 1|^2 = |e^{-2i\theta}||e^{2i\theta} - e^{i\theta}\omega_\nu + 1|^2 = |e^{2i\theta} - e^{i\theta}\omega_\nu + 1|^2,$$

the third part $\Re\left(mg'_\nu(z)/\overline{(z+1)}g_\nu(z)\right)$ gives us

$$\begin{aligned} \Re\left(\frac{m}{(e^{-i\theta} + 1)} \frac{g'_\nu(e^{-i\theta})}{g_\nu(e^{-i\theta})}\right) &= m \frac{Q_1(\cos(-\theta); \omega_\nu) + \sin(-\theta) S_1(\cos(-\theta); \omega_\nu)}{|e^{-i\theta} + 1|^2 |e^{-2i\theta} - e^{-i\theta}\omega_\nu + 1|^2} \\ &= m \frac{Q_1(\cos\theta; \omega_\nu) - \sin\theta S_1(\cos\theta; \omega_\nu)}{|e^{i\theta} + 1|^2 |e^{2i\theta} - e^{i\theta}\omega_\nu + 1|^2} \\ &\leq m \frac{Q_1(\cos\theta; \omega_\nu) + \sin\theta S_1(\cos\theta; \omega_\nu)}{|e^{i\theta} + 1|^2 |e^{2i\theta} - e^{i\theta}\omega_\nu + 1|^2} \\ &= \Re\left(\frac{m}{(e^{i\theta} + 1)} \frac{g'_\nu(e^{i\theta})}{g_\nu(e^{i\theta})}\right), \quad 0 \leq \theta \leq \pi. \end{aligned} \quad (41)$$

Let us turn to the fourth part. As in the third part, let $z = x + iy$ be a point on

the unit circle. Then for any μ and ν , it can be verified that

$$\begin{aligned} \frac{g'_\nu(z) \overline{g'_\mu(z)}}{g_\nu(z) \overline{g_\mu(z)}} &= \frac{2z - \omega_\nu}{z^2 - z\omega_\nu + 1} \frac{\overline{2z - \omega_\mu}}{\overline{z^2 - z\omega_\mu + 1}} = \frac{4 - 2z\bar{\omega}_\mu - 2\bar{z}\omega_\nu + \omega_\nu\bar{\omega}_\mu}{4x^2 - 2x(\bar{\omega}_\mu + \omega_\nu) + \omega_\nu\bar{\omega}_\mu} \\ &= \frac{(Q_3(x; \omega_\nu, \omega_\mu) + y S_3(x; \omega_\nu, \omega_\mu)) + i(Q_4(x; \omega_\nu, \omega_\mu) + y S_4(x; \omega_\nu, \omega_\mu))}{|4x^2 - 2x(\omega_\nu + \bar{\omega}_\mu) + \omega_\nu\bar{\omega}_\mu|^2}, \end{aligned} \quad (42)$$

where

$$\begin{aligned} Q_3(x; \omega_\nu, \omega_\mu) &= 16x^2 - 16x\Re(\bar{\omega}_\nu + \omega_\mu) + 8xy^2\Re(\bar{\omega}_\mu + \omega_\nu) + 8\Re(\bar{\omega}_\nu\omega_\mu) \\ &\quad - 4y^2\Re(\bar{\omega}_\mu\omega_\nu) + 4x^2|\bar{\omega}_\mu + \omega_\mu|^2 + |\omega_\mu\omega_\nu|^2 - 4x\Re(\bar{\omega}_\mu|\omega_\nu|^2 + \omega_\nu|\omega_\mu|^2); \\ S_3(x; \omega_\nu, \omega_\mu) &= -8x^2\Im(\omega_\nu - \bar{\omega}_\mu) + 8x\Im(\omega_\nu\omega_\mu) + 2\Im\bar{\omega}_\nu|\omega_\mu|^2 - 2\Im\omega_\mu|\omega_\nu|^2; \\ Q_4(x; \omega_\nu, \omega_\mu) &= 8xy^2\Im(\bar{\omega}_\mu + \omega_\nu) - 4y^2\Im(\bar{\omega}_\mu\omega_\nu); \\ S_4(x; \omega_\nu, \omega_\mu) &= 8x^2\Re(\omega_\nu - \bar{\omega}_\mu) - 4x(|\omega_\nu|^2 - |\omega_\mu|^2) + 2|\omega_\nu|^2\Re\bar{\omega}_\mu - 2|\omega_\mu|^2\Re\bar{\omega}_\nu. \end{aligned} \quad (43)$$

Thus from Lemma 5 and (42), the fourth part $\sum_{\mu=\nu+1}^l \Re(g'_\nu(z)\overline{g'_\mu(z)}/g_\nu(z)\overline{g_\mu(z)})$ gives us

$$\begin{aligned} &\sum_{\mu=\nu+1}^l \Re\left(\frac{g'_\nu(e^{-i\theta})}{g_\nu(e^{-i\theta})} \frac{\overline{g'_\mu(e^{-i\theta})}}{\overline{g_\mu(e^{-i\theta})}}\right) \\ &= \sum_{\mu=\nu+1}^l \left(\frac{Q_3(\cos(-\theta); \omega_\nu) + \sin(-\theta) S_3(\cos(-\theta); \omega_\nu)}{|4\cos(-\theta)^2 - 2(\omega_\nu + \bar{\omega}_\mu)\cos(-\theta) + \omega_\nu\bar{\omega}_\mu|^2}\right) \\ &= \sum_{\mu=\nu+1}^l \left(\frac{Q_3(\cos\theta; \omega_\nu) - \sin\theta S_3(\cos\theta; \omega_\nu)}{|4\cos^2\theta - 2(\omega_\nu + \bar{\omega}_\mu)\cos\theta + \omega_\nu\bar{\omega}_\mu|^2}\right) \\ &\leq \sum_{\mu=\nu+1}^l \left(\frac{Q_3(\cos\theta; \omega_\nu) + \sin\theta S_3(\cos\theta; \omega_\nu)}{|4\cos^2\theta - 2(\omega_\nu + \bar{\omega}_\mu)\cos\theta + \omega_\nu\bar{\omega}_\mu|^2}\right) \\ &= \sum_{\mu=\nu+1}^l \Re\left(\frac{g'_\nu(e^{i\theta})}{g_\nu(e^{i\theta})} \frac{\overline{g'_\mu(e^{i\theta})}}{\overline{g_\mu(e^{i\theta})}}\right), \quad 0 \leq \theta \leq \pi. \end{aligned} \quad (44)$$

Using (37), (38), (41) and (44) in (36), we conclude that

$$\left|\frac{f'(e^{-i\theta})}{f(e^{-i\theta})}\right|^2 \leq \left|\frac{f'(e^{i\theta})}{f(e^{i\theta})}\right|^2, \quad 0 \leq \theta \leq \pi. \quad (45)$$

Since for any θ , $|f(e^{-i\theta})| = |f(e^{i\theta})|$, we get from (45)

$$|f'(e^{-i\theta})| \leq |f'(e^{i\theta})|, \quad 0 \leq \theta \leq \pi, f(e^{i\theta}) \neq 0.$$

By continuity, the same must hold for those θ for which $f(e^{i\theta}) = 0$. This completes the proof of Theorem 1.

Proof of Corollary 1. For polynomials f in $\mathcal{P}_n^\vee$, we have

$$z^{n-1} f'(\frac{1}{z}) + z f'(z) = n f(z).$$

From the interpolation formula (31) of Aziz and Mohammad given in Lemma 3, with $a = e^{i\alpha}$, where $\alpha \in \mathbb{R}$ and $z = e^{i\theta}$ is a complex number on the unit circle, we get

$$e^{i(\theta+\alpha)} f'(e^{i\theta}) - e^{i(n-1)\theta} f'(e^{-i\theta}) = \frac{(1 + e^{i\alpha})^2}{n e^{i\alpha}} \sum_{\nu=1}^n c_\nu(a) f(e^{i\theta} \xi_\nu),$$

which can be written as

$$e^{i(\theta+\alpha)} f'(e^{i\theta}) - e^{i(n-1)\theta} f'(e^{-i\theta}) = n \sum_{\nu=1}^n d_\nu(e^{i\alpha}) f(e^{i\theta} \xi_\nu),$$

where

$$\sum_{\nu=0}^n |d_\nu(e^{i\alpha})| = \sum_{\nu=0}^n \left| \frac{c_\nu(e^{i\alpha})}{n^2 e^{i\alpha} / (1 + e^{i\alpha})^2} \right| = 1.$$

For $p \geq 1$, we have

$$\left| e^{i(\theta+\alpha)} f'(e^{i\theta}) - e^{i(n-1)\theta} f'(e^{-i\theta}) \right|^p \leq n^p \sum_{\nu=1}^n d_\nu(e^{i\alpha}) |f(e^{i\theta} \xi_\nu)|^p.$$

Integrating both sides with respect to θ from $-\pi$ to π , we get

$$\int_{-\pi}^{\pi} \left| e^{i(\theta+\alpha)} f'(e^{i\theta}) - e^{i(n-1)\theta} f'(e^{-i\theta}) \right|^p d\theta \leq n^p \int_{-\pi}^{\pi} |f(e^{i\theta})|^p d\theta.$$

Since the above inequality is true for every α in $[0, 2\pi]$, integrating both sides with respect to α and changing the order of integration, we get

$$\int_{-\pi}^{\pi} \int_0^{2\pi} \left| e^{i(\theta+\alpha)} f'(e^{i\theta}) - e^{i(n-1)\theta} f'(e^{-i\theta}) \right|^p d\alpha d\theta \leq 2\pi n^p \int_{-\pi}^{\pi} |f(e^{i\theta})|^p d\theta. \quad (46)$$

The left-hand side of the inequality (46) is

$$\begin{aligned} & \int_{-\pi}^{\pi} \int_0^{2\pi} \left| e^{i(\theta+\alpha)} f'(e^{i\theta}) - e^{i(n-1)\theta} f'(e^{-i\theta}) \right|^p d\alpha d\theta \\ &= \int_{-\pi}^0 \int_0^{2\pi} |f'(e^{i\theta})|^p \left| 1 - e^{i(n-2)\theta-i\alpha} \frac{f'(e^{-i\theta})}{f'(e^{i\theta})} \right|^p d\alpha d\theta \\ & \quad + \int_0^{\pi} \int_0^{2\pi} |f'(e^{-i\theta})|^p \left| 1 - e^{i(2-n)\theta+i\alpha} \frac{f'(e^{i\theta})}{f'(e^{-i\theta})} \right|^p d\alpha d\theta \\ &\geq 2 \int_{-\pi}^0 |f'(e^{i\theta})|^p d\theta \int_0^{2\pi} |1 + e^{i\alpha}|^p d\alpha. \end{aligned} \quad (47)$$

Inequality (47) follows from the fact that

$$\begin{aligned} |f'(e^{-i\theta})/f'(e^{i\theta})| &\geq 1 \text{ for } -\pi \leq \theta \leq 0, \\ |f'(e^{i\theta})/f'(e^{-i\theta})| &\geq 1 \text{ for } 0 \leq \theta \leq \pi, \end{aligned}$$

and

$$\int_0^{2\pi} |1 + re^{i\gamma}|^p d\gamma \geq \int_0^{2\pi} |1 + e^{i\gamma}|^p d\gamma \text{ for every } |r| \geq 1 \text{ and } p \geq 1.$$

Also, for $f \in \mathcal{P}_n^\vee$, $|f(e^{-i\theta})| = |f(e^{i\theta})|$ for $0 \leq \theta \leq \pi$. From (46) and (47) we conclude that

$$\int_{-\pi}^0 |f'(e^{i\theta})|^p d\theta \leq n^p C_p \int_{-\pi}^0 |f(e^{i\theta})|^p d\theta,$$

where C_p is as given in (11). $\square$

Proof of Corollary 2. Let f be a polynomial in $\mathcal{P}_n^\vee$ such that $|f(e^{-i\theta})| \leq M$ for $0 \leq \theta \leq \pi$. Since $|f(e^{-i\theta})| = |f(e^{i\theta})|$ for every f in $\mathcal{P}_n^\vee$, it implies that $|f(e^{i\theta})| \leq M$ for $-\pi \leq \theta \leq \pi$. We also observe that $g(z) \equiv z^n f(1/\bar{z}) = \overline{f(\bar{z})}$. Then from inequality (32) in Lemma 4, for $z = e^{i\theta}$

$$|f'(e^{i\theta})| + |g'(e^{i\theta})| = |f'(e^{-i\theta})| + |f'(e^{i\theta})| \leq nM, \quad -\pi \leq \theta \leq \pi. \quad (48)$$

From Theorem 1, $|f'(e^{-i\theta})| \leq |f'(e^{i\theta})|$ for $0 \leq \theta \leq \pi$. So, from (48) we get

$$2|f'(e^{-i\theta})| \leq |f'(e^{-i\theta})| + |f'(e^{i\theta})| \leq nM, \quad 0 \leq \theta \leq \pi. \quad (49)$$

The result follows from (49). It is easy to verify that the equality holds for $f(z) = (z^2 + 1)^{\frac{n}{2}}$, when n is even and $f(z) = (z + 1)^n$, when n is odd. $\square$

3.2. Proof of Theorem 6 and its corollaries

Let $\{z_\nu\}, \nu = 1, 2, \dots$ be the zeros of f other than 0 in $\{z \in \mathbb{C} : \Re z \geq 0, \Im z \geq 0\}$. The number of such zeros can be finite or infinite. Besides, to each zero z_ν there corresponds a zero $-z_\nu$. A zero of f at the origin, if there is any, must be of even multiplicity, say $2k$. For these reasons, the Hadamard factorization of f takes the form

$$f(z) = cz^{2k} e^{i\tau z/2} \prod_{\nu} \left(1 - \frac{z^2}{z_\nu^2}\right),$$

where c is a constant and k is a non-negative integer. Now, let us write

$$x_\nu = \Re z_\nu \text{ and } y_\nu = \Im z_\nu$$

so that $x_\nu \geq 0$ and $y_\nu \geq 0$.

Case 1. f has only real zeros

In this case, for any real x different from 0 that is not a zero of f , we have

$$\frac{f'(x)}{f(x)} = \frac{2k}{x} + \sum_{\nu} \left(\frac{1}{x_\nu + x} - \frac{1}{x_\nu - x} \right) + i\frac{\tau}{2}.$$

The real part of $f'(x)/f(x)$ is clearly an odd function of x and so

$$\frac{f'(-x)}{f(-x)} = - \left(\frac{2k}{x} + \sum_{\nu} \left(\frac{1}{x_{\nu} + x} - \frac{1}{x_{\nu} - x} \right) \right) + i \frac{\tau}{2}.$$

From the definition of the class $\mathcal{F}_{\tau}^{\vee}$ it is clear that $|f(-x)| = |f(x)|$ for any real x . Hence $|f'(-x)| = |f'(x)|$. Since it holds for any x such that $f(x) \neq 0$, by continuity it also holds for those values for x for which $f(x) = 0$.

Case 2. The zeros of f are not all real

In this case, for any real x different from 0 that is not a zero of f , we have

$$\frac{f'(x)}{f(x)} = A_f(x) + i \left(\frac{\tau}{2} + B_f(x) \right),$$

where

$$A_f(x) := \frac{2k}{x} + \sum_{\nu} \left(\frac{x_{\nu} + x}{(x_{\nu} + x)^2 + y_{\nu}^2} - \frac{x_{\nu} - x}{(x_{\nu} - x)^2 + y_{\nu}^2} \right)$$

and

$$B_f(x) := 4x \sum_{\nu} \left(\frac{x_{\nu} y_{\nu}}{((x_{\nu} + x)^2 + y_{\nu}^2)((x_{\nu} - x)^2 + y_{\nu}^2)} \right).$$

Consequently, for any real $x \neq 0$ such that $f(x) \neq 0$ we have

$$\left| \frac{f'(x)}{f(x)} \right| = \sqrt{(A_f(x))^2 + \left(B_f(x) + \frac{\tau}{2} \right)^2}.$$

Now note that $B_f(x)$ is an odd function that is positive for $x > 0$. Hence

$$\left| B_f(-x) + \frac{\tau}{2} \right| < \left| B_f(x) + \frac{\tau}{2} \right|, \quad x > 0, f(x) \neq 0.$$

Since $|f(-x)| = |f(x)|$, we find that $|f'(-x)| \leq |f'(x)|$ for any positive x if $f(x) \neq 0$. However, by continuity, the same must also hold for those values of x for which $f(x) = 0$. The proof of Theorem 6 is thus complete.

Proof of Corollary 3. Let $p \geq 1$ be any real number. From the interpolation formula (29) given in Lemma 1, we get

$$\left| \frac{e^{i\gamma} f'(x) + e^{i\tau x} f'(-x)}{\tau} \right|^p \leq \sum_{n=-\infty}^{\infty} \frac{c_n}{\tau} \left| f \left(x - s + \frac{n\pi}{\tau} \right) \right|^p.$$

If we integrate both sides of the above inequality with respect to x on the real line, we have

$$\int_{-\infty}^{\infty} |e^{i\gamma} f'(x) + e^{i\tau x} f'(-x)|^p dx \leq \tau^p \int_{-\infty}^{\infty} |f(x)|^p dx.$$

The above integral is true for any $0 \leq \gamma \leq 2\pi$, therefore by integrating both sides with respect to γ on the interval $[0, 2\pi]$ we get

$$\int_0^{2\pi} \int_{-\infty}^{\infty} |e^{i\gamma} f'(x) + e^{i\tau x} f'(-x)|^p dx d\gamma \leq 2\pi \tau^p \int_{-\infty}^{\infty} |f(x)|^p dx. \quad (50)$$

The integral on the left-hand side of (50) may be written as

$$\int_0^{2\pi} \int_{-\infty}^0 |e^{i\gamma} f'(x) + e^{i\tau x} f'(-x)|^p dx d\gamma + \int_0^{2\pi} \int_0^\infty |e^{i\gamma} f'(x) + e^{i\tau x} f'(-x)|^p dx d\gamma. \quad (51)$$

The first integral $\int_0^{2\pi} \int_{-\infty}^0 |e^{i\gamma} f'(x) + e^{i\tau x} f'(-x)|^p dx d\gamma$ in (51), after the change of order of integration can be written as

$$\begin{aligned} & \int_{-\infty}^0 \int_0^{2\pi} |e^{i\gamma} f'(x) + e^{i\tau x} f'(-x)|^p dx d\gamma \\ &= \int_{-\infty}^0 |f'(x)|^p dx \int_0^{2\pi} \left| 1 + e^{i\tau x - i\gamma} \frac{f'(-x)}{f'(x)} \right|^p d\gamma \\ &\geq \int_{-\infty}^0 |f'(x)|^p dx \int_0^{2\pi} |1 + e^{i\gamma}|^p d\gamma. \end{aligned} \quad (52)$$

Inequality (52) follows because for $x \leq 0$, $|f'(-x)/f'(x)| \geq 1$ from Theorem 6 and $\int_0^{2\pi} |1 + re^{i\gamma}|^p d\gamma \geq \int_0^{2\pi} |1 + e^{i\gamma}|^p d\gamma$ for every $|r| \geq 1$ and $p \geq 1$.

Similar reasoning applied to the second integral $\int_0^{2\pi} \int_0^\infty |e^{i\gamma} f'(x) + e^{i\tau x} f'(-x)|^p dx d\gamma$ in (51) gives

$$\int_0^{2\pi} \int_0^\infty |e^{i\gamma} f'(x) + e^{i\tau x} f'(-x)|^p dx d\gamma \geq \int_0^{2\pi} |f'(-x)|^p dx \int_0^{2\pi} |1 + e^{i\gamma}|^p d\gamma, \quad (53)$$

as once again from Theorem 6 we have $|f'(x)/f'(-x)| \geq 1$ when $x \geq 0$.

Thus from (50), (52) and (53) we get

$$\int_0^{2\pi} |1 + e^{i\gamma}|^p d\gamma \left(\int_{-\infty}^0 |f'(x)|^p dx + \int_0^\infty |f'(-x)|^p dx \right) \leq 2\pi\tau^p \int_{-\infty}^\infty |f(x)|^p dx. \quad (54)$$

Note that

$$\int_{-\infty}^0 |f'(x)|^p dx + \int_0^\infty |f'(-x)|^p dx = 2 \int_{-\infty}^0 |f'(x)|^p dx. \quad (55)$$

Also, for $f \in \mathcal{F}_\tau^\vee$, we have $|f(x)| = |f(-x)|$, and so

$$\int_{-\infty}^\infty |f(x)|^p dx = 2 \int_{-\infty}^0 |f(x)|^p dx. \quad (56)$$

From (54), (55), and (56) we get

$$\int_{-\infty}^0 |f'(x)|^p dx \leq \tau^p C_p \int_{-\infty}^0 |f(x)|^p dx,$$

where C_p is as given in (11). □

Proof of Corollary 4. Let $f \in \mathcal{F}_\tau^\vee$ such that $|f(x)| \leq M$ for $x \leq 0$. Since $f \in \mathcal{F}_\tau^\vee$, we have $|f(x)| = |f(-x)|$ for $x \in \mathbb{R}$ and hence $|f(x)| \leq M$ for $-\infty < x < \infty$. So from inequality (30) in Lemma 2 we have

$$|f'(x)| + |f'(-x)| \leq M\tau, \quad x \in \mathbb{R}. \quad (57)$$

Also, from Theorem 6, $|f'(-x)| \geq |f'(x)|$ for $x \leq 0$, and (57) then gives us

$$|f'(x)| \leq \frac{M\tau}{2}, \quad x \leq 0.$$

It is easy to verify that the equality holds in (22) for $f(x) = M(1 + e^{i\tau z})/2$. $\square$

Proof of Corollary 5. Let f satisfy the conditions given in Corollary 5. Then according to Corollary 4, for $x \leq 0$, $|f'(x)| \leq M\tau/2$. From Rahman and Tariq ([18, Lemma 3]), $h_f(\pi/2) \leq 0$. Thus we have $h_{f'}(\pi/2) \leq h_f(\pi/2) \leq 0$ as well. Consider the function $g(z) = e^{i\tau z} \overline{f(\bar{z})}$. Then $g(z)$ is an entire function of exponential type τ and $g(z) = f(-z)$. From Corollary 4, $|g'(x)| \leq M\tau/2$ for $x \geq 0$. Also, $h_{g'}(\pi/2) = h_{f'}(-\pi/2) = \tau$. Then according to Theorem 6.2.3 ([4], page 82), for $x \geq 0, y \geq 0$,

$$|g'(x + iy)| \leq \frac{M\tau}{2} e^{\tau y}.$$

Since $g(z) = f(-z)$, we have for $x \leq 0, y \leq 0$,

$$|f'(x + iy)| \leq \frac{M\tau}{2} e^{-\tau y}.$$

It is easy to see that the equality holds for the function $M(1 + e^{i\tau z})/2$. $\square$

3.3. Proof of Theorem 10

Let f , whose zeros lie in the first and the third quadrants, belong to $\mathcal{F}_\tau^\vee$. Let $\varepsilon > 0$ be an arbitrary real number. Define the function g_ε as follows

$$g_\varepsilon(z) = e^{i\frac{\varepsilon}{2}z} \frac{\sin \frac{\varepsilon}{2}z}{\frac{\varepsilon}{2}z} f(z). \quad (58)$$

It is obvious that $g_\varepsilon(z)$ is an entire function of exponential type $\tau + \varepsilon$. Also,

$$e^{i(\tau+\varepsilon)z} g_\varepsilon(-z) = e^{i\frac{\varepsilon}{2}z} \frac{\sin \frac{\varepsilon}{2}z}{\frac{\varepsilon}{2}z} e^{i\tau z} f(-z) = e^{i\frac{\varepsilon}{2}z} \frac{\sin \frac{\varepsilon}{2}z}{\frac{\varepsilon}{2}z} f(z) = g_\varepsilon(z).$$

Thus, $g_\varepsilon(z)$ belongs to $\mathcal{F}_{\tau+\varepsilon}^\vee$.

Note that the zeros of $g_\varepsilon(z)$ are the zeros of $\sin \frac{\varepsilon}{2}z$ or the zeros of $f(z)$. Since the zeros of $\sin z$ are all real, the zeros of $g_\varepsilon(z)$ also lie in the first and third quadrants. Hence, according to Theorem 6,

$$|g'_\varepsilon(-x)| \leq |g'_\varepsilon(x)|, \quad x \geq 0. \quad (59)$$

Next, we will show that g_ε is bounded on the real line. The assumption that $M^p(f) < \infty$ gives us ([12, Theorem 1]), $f(x) = O(|x|^{\frac{1}{p}})$ as $|x| \rightarrow \infty$. It means there exist a positive real number $x_0 \in \mathbb{R}$ and a real number $N_1 \in \mathbb{R}$ such that $|f(x)| \leq N_1 |x|^{\frac{1}{p}}$ for $|x| \geq x_0$. Thus for $|x| \geq x_0$,

$$|g_\varepsilon(x)| = \left| e^{i\frac{\varepsilon}{2}x} \frac{\sin \frac{\varepsilon}{2}x}{\frac{\varepsilon}{2}x} f(x) \right| \leq N_1 \left| \frac{\sin \frac{\varepsilon}{2}x}{\frac{\varepsilon}{2}x} \right| |x|^{\frac{1}{p}} \leq N_1 \frac{2}{\varepsilon |x|^{1-\frac{1}{p}}} \leq N_1 \frac{2}{\varepsilon |x_0|^{1-\frac{1}{p}}}.$$

On the interval $[-x_0, x_0]$, g_ε is continuous and hence bounded. So there exists a real number N_2 such that $|g_\varepsilon(x)| \leq N_2$ for $x \in [-x_0, x_0]$. Let $K = \max(2N_1/\varepsilon |x_0|^{1-\frac{1}{p}}, N_2)$. Then $|g_\varepsilon(x)| \leq K$ for $x \in \mathbb{R}$. Thus g_ε is bounded on the real line and belongs to $\mathcal{F}_{\tau+\varepsilon}^\vee$. Hence Lemma 1 (with τ replaced by $\tau + \varepsilon$), when applied to the function $g_\varepsilon(z)$, gives us for $x \in \mathbb{R}$

$$-i \left\{ e^{i\gamma} g'_\varepsilon(x) + e^{i(\tau+\varepsilon)x} g'_\varepsilon(-x) \right\} = \sum_{n=-\infty}^{\infty} c_n g_\varepsilon \left(x - s + \frac{n\pi}{\tau + \varepsilon} \right),$$

where

$$c_n = \frac{1}{(s(\tau + \varepsilon) - n\pi)^2} \{1 + (-1)^n\} \{1 - (-1)^n \cos \gamma\} (\tau + \varepsilon), \quad n = 0, \pm 1, \pm 2, \dots,$$

γ is any real number, $s = -\gamma/(\tau + \varepsilon)$, and $\sum_{n=-\infty}^{\infty} |c_n| = \tau + \varepsilon$.

From the above interpolation formula we have

$$\frac{-i \left\{ e^{i\gamma} g'_\varepsilon(x) + e^{i(\tau+\varepsilon)x} g'_\varepsilon(-x) \right\}}{\tau + \varepsilon} = \sum_{n=-\infty}^{\infty} d_n g_\varepsilon \left(x - s + \frac{n\pi}{\tau + \varepsilon} \right), \quad (60)$$

where $d_n = c_n/(\tau + \varepsilon)$ and $\sum_{n=-\infty}^{\infty} |d_n| = 1$. Thus right-hand side of (60) is a convex combination of $\{g_\varepsilon(x - s + n\pi/(\tau + \varepsilon))\}_{n=-\infty}^{\infty}$. So for $p \geq 1$ we get

$$\left| \frac{-i \left\{ e^{i\gamma} g'_\varepsilon(x) + e^{i(\tau+\varepsilon)x} g'_\varepsilon(-x) \right\}}{\tau + \varepsilon} \right|^p \leq \sum_{n=-\infty}^{\infty} |d_n| \left| g_\varepsilon \left(x - s + \frac{n\pi}{\tau + \varepsilon} \right) \right|^p,$$

which gives us

$$\left| e^{i\gamma} g'_\varepsilon(x) + e^{i(\tau+\varepsilon)x} g'_\varepsilon(-x) \right|^p \leq (\tau + \varepsilon)^p \sum_{n=-\infty}^{\infty} |d_n| \left| g_\varepsilon \left(x - s + \frac{n\pi}{\tau + \varepsilon} \right) \right|^p. \quad (61)$$

Let $T > 0$ be an arbitrary real number. Then, integrating both sides of (61) with respect to x we get

$$\begin{aligned} & \frac{1}{2T} \int_{-T}^T \left| e^{i\gamma} g'_\varepsilon(x) + e^{i(\tau+\varepsilon)x} g'_\varepsilon(-x) \right|^p dx \\ & \leq (\tau + \varepsilon)^p \frac{1}{2T} \int_{-T}^T \sum_{n=-\infty}^{\infty} |d_n| \left| g_\varepsilon \left(x - s + \frac{n\pi}{\tau + \varepsilon} \right) \right|^p dx \\ & = (\tau + \varepsilon)^p \sum_{n=-\infty}^{\infty} |d_n| \frac{1}{2T} \int_{-T}^T \left| g_\varepsilon \left(x - s + \frac{n\pi}{\tau + \varepsilon} \right) \right|^p dx. \end{aligned}$$

We can change the order of integration in the last inequality because the series on right-hand side of (61) is absolutely convergent and hence uniformly convergent. Applying Lemma 4 followed by Lemma 1 given in [12] we get

$$\begin{aligned} \limsup_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \left| e^{i\gamma} g'_\varepsilon(x) + e^{i(\tau+\varepsilon)x} g'_\varepsilon(-x) \right|^p dx \\ \leq (\tau + \varepsilon)^p \sum_{n=-\infty}^{\infty} |d_n| \limsup_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \left| g_\varepsilon \left(x - s + \frac{n\pi}{\tau + \varepsilon} \right) \right|^p dx \\ = (\tau + \varepsilon)^p \sum_{n=-\infty}^{\infty} |d_n| \limsup_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |g_\varepsilon(x)|^p dx = (\tau + \varepsilon)^p M^p g_\varepsilon(x). \end{aligned}$$

Thus $M^p \{e^{i\gamma} g'_\varepsilon(x) + e^{i(\tau+\varepsilon)x} g'_\varepsilon(-x)\}$, the mean value of $\{e^{i\gamma} g'_\varepsilon(x) + e^{i(\tau+\varepsilon)x} g'_\varepsilon(-x)\}$, exists for each real number γ and $\varepsilon > 0$. From the definition of limit superior, for every $\delta > 0$ there exists a positive $T_0 \in \mathbb{R}$ such that

$$\begin{aligned} \frac{1}{2T} \int_{-T}^T \left| e^{i\gamma} g'_\varepsilon(x) + e^{i(\tau+\varepsilon)x} g'_\varepsilon(-x) \right|^p dx < M^p \{e^{i\gamma} g'_\varepsilon(x) + e^{i(\tau+\varepsilon)x} g'_\varepsilon(-x)\} + \delta \\ \leq (\tau + \varepsilon)^p M^p g_\varepsilon(x) + \delta \end{aligned} \quad (62)$$

for all $T \geq T_0 > 0$, $\gamma \in \mathbb{R}$, and $\varepsilon > 0$.

Since (62) is true for each γ , integrating both sides with respect to γ from 0 to 2π and changing the order of integration which is justified by Fubini's Theorem as the function $|e^{i\gamma} g'_\varepsilon(x) + e^{i(\tau+\varepsilon)x} g'_\varepsilon(-x)|^p$ is continuous, we get

$$\frac{1}{2T} \int_{-T}^T \int_0^{2\pi} \left| e^{i\gamma} g'_\varepsilon(x) + e^{i(\tau+\varepsilon)x} g'_\varepsilon(-x) \right|^p d\gamma dx < 2\pi \{(\tau + \varepsilon)^p M^p g_\varepsilon(x) + \delta\}. \quad (63)$$

By considering the iterated integral on the left-hand side of (63), we get

$$\begin{aligned} \int_{-T}^T \int_0^{2\pi} \left| e^{i\gamma} g'_\varepsilon(x) + e^{i(\tau+\varepsilon)x} g'_\varepsilon(-x) \right|^p d\gamma dx \\ = \int_{-T}^0 \int_0^{2\pi} |g'_\varepsilon(x)|^p \left| 1 + e^{-i\gamma+i(\tau+\varepsilon)x} \frac{g'_\varepsilon(-x)}{g'_\varepsilon(x)} \right|^p d\gamma dx \\ + \int_0^T \int_0^{2\pi} |g'_\varepsilon(-x)|^p \left| 1 + e^{i\gamma-i(\tau+\varepsilon)x} \frac{g'_\varepsilon(x)}{g'_\varepsilon(-x)} \right|^p d\gamma dx \\ \geq \int_0^{2\pi} |1 + e^{i\gamma}|^p d\gamma \left(\int_{-T}^0 |g'_\varepsilon(x)|^p dx + \int_0^T |g'_\varepsilon(-x)|^p dx \right) \\ = 2 \int_0^{2\pi} |1 + e^{i\gamma}|^p d\gamma \left(\int_{-T}^0 |g'_\varepsilon(x)|^p dx \right). \end{aligned}$$

Then multiplying both sides by $1/2T$, from (63) we get

$$\begin{aligned} \frac{2}{2T} \int_0^{2\pi} |1 + e^{i\gamma}|^p d\gamma \left(\int_{-T}^0 |g'_\varepsilon(x)|^p dx \right) \\ \leq \frac{1}{2T} \int_{-T}^T \int_0^{2\pi} \left| e^{i\gamma} g'_\varepsilon(x) + e^{i(\tau+\varepsilon)x} g'_\varepsilon(-x) \right|^p d\gamma dx \\ \leq 2\pi \{(\tau + \varepsilon)^p M^p g_\varepsilon(x) + \delta\}. \end{aligned} \quad (64)$$

Inequality (64) is true for all $T \geq T_0$, so taking limit superior when $T \rightarrow \infty$, we get

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^{2\pi} |1 + e^{i\gamma}|^p d\gamma \left(\int_{-T}^0 |g'_\varepsilon(x)|^p dx \right) \leq 2\pi \{(\tau + \varepsilon)^p M^p g_\varepsilon(x) + \delta\}. \quad (65)$$

Since, δ is an arbitrary positive real number, letting $\delta \rightarrow 0$ we get

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^{2\pi} |1 + e^{i\gamma}|^p d\gamma \left(\int_{-T}^0 |g'_\varepsilon(x)|^p dx \right) \leq 2\pi(\tau + \varepsilon)^p \{M^p g_\varepsilon(x)\}. \quad (66)$$

Note that from (59) for every $x \in \mathbb{R}$ such that $x \geq 0$, $|g_\varepsilon(-x)| \leq |g_\varepsilon(x)|$, we have

$$M^p g_\varepsilon(x) = \limsup_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |g'_\varepsilon(x)|^p dx \leq 2 \limsup_{T \rightarrow \infty} \frac{1}{T} \int_{-T}^0 |g'_\varepsilon(x)|^p dx. \quad (67)$$

Then from (66) and (67), we get

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \int_{-T}^0 |g'_\varepsilon(x)|^p dx \leq (\tau + \varepsilon)^p C_p \limsup_{T \rightarrow \infty} \frac{1}{T} \int_{-T}^0 |g_\varepsilon(x)|^p dx, \quad (68)$$

where C_p is as given in (11).

For any $x \in \mathbb{R}$,

$$\lim_{\varepsilon \rightarrow 0} g_\varepsilon(x) = \lim_{\varepsilon \rightarrow 0} e^{i\frac{\varepsilon}{2}x} \frac{\sin \frac{\varepsilon}{2}x}{\frac{\varepsilon}{2}x} f(x) = f(x), \quad (69)$$

and

$$\lim_{\varepsilon \rightarrow 0} g'_\varepsilon(x) = f'(x). \quad (70)$$

Inequality (68) is true for every $\varepsilon > 0$, therefore by letting $\varepsilon \rightarrow 0$, and using (69) and (70), we get (28).

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