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Classification of complete left-invariant affine structures on the oscillator group*

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Abstract. The goal of this paper is to provide a method, based on the theory of extensions of left-symmetric algebras, for classifying left-invariant affine structures on a given solvable Lie group of low dimension. To illustrate our method better, we shall apply it to classify all complete left-invariant affine structures on the oscillator group.

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1. Introduction

It is a well known result (see [1, 19]) that a simply connected Lie group G which admits a complete left-invariant affine structure, or equivalently G acts simply transitively by affine transformations on $\mathbb{R}^n$, must be solvable. It is also well known that not every solvable (even nilpotent) Lie group can admit an affine structure [3].

The goal of this paper is to provide a method for classifying all complete left-invariant affine structures on a given solvable Lie group of low dimension. Since the classification has been completely achieved up to dimension four in the nilpotent case (see [10, 14, 17]), we will illustrate our method by applying it to the remarkable solvable non-nilpotent 4-dimensional Lie group O_4 known as the *oscillator group*. Since complete left-invariant affine structures on a Lie group G are in one-to-one correspondence with complete (in the sense of [22]) left-symmetric structures on its Lie algebra $\mathcal{G}$ [14], we will carry out the classification in terms of complete left-symmetric structures on the oscillator algebra $\mathcal{O}_4$.

The paper is organized as follows. In Section 2, we will recall the notion of extensions of Lie algebras and its relationship to the notion of $\mathcal{G}$-kernels. In Section 3, we will give some necessary definitions and basic results on left-symmetric algebras and their extensions. In Section 4, given a complete left-symmetric algebra A_4 whose associated Lie algebra is $\mathcal{O}_4$, we will use the complexification of A_4 and some results in [5] and [15] to show first that A_4 is not simple. Precisely, we will show that A_4 has a proper two-sided ideal whose associated Lie algebra is isomorphic to the

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center $Z(\mathcal{O}_4) \cong \mathbb{R}$ or the commutator ideal $[\mathcal{O}_4, \mathcal{O}_4] \cong \mathcal{H}_3$ of $\mathcal{O}_4$. In the latter case, we will show that the so-called center of A_4 is nontrivial, and therefore we can get A_4 as a central extension (in some sense that we will define later) of a complete 3-dimensional left-symmetric algebra A_3 by the trivial left-symmetric algebra $\mathbb{R}$ (i.e., the vector space $\mathbb{R}$ with the trivial left-symmetric product). In Section 5, we will show that in both cases we have a short exact sequence (which turns out to be central) of left-symmetric algebras of the form $0 \to \mathbb{R} \xrightarrow{i} A_4 \xrightarrow{\pi} A_3 \to 0$, where here A_3 is a complete left-symmetric algebra whose Lie algebra is isomorphic to the Lie algebra $\mathcal{E}(2)$ of the group of Euclidean motions of the plane. We will then show that, up to left-symmetric isomorphism, there are only two non-isomorphic complete left-symmetric structures on $\mathcal{E}(2)$, and we will use these to carry out all complete left-symmetric structures on $\mathcal{O}_4$. We will see that one of these two left-symmetric structures on $\mathcal{E}(2)$ yields exactly one complete left-symmetric structure on $\mathcal{O}_4$. However, the second one yields a two-parameter family of complete left-symmetric algebras $A_4(s,t)$ whose associated Lie algebra is $\mathcal{O}_4$, and the conjugacy class of $A_4(s,t)$ is given as follows: $A_4(s',t')$ is isomorphic to $A_4(s,t)$ if and only if $(s',t') = (\alpha s, \pm t)$ for some $\alpha \in \mathbb{R}^*$. By using the Lie group exponential maps, we will deduce the classification of all complete left-invariant affine structures on the oscillator group O_4 in terms of simply transitive actions of subgroups of the affine group $Aff(\mathbb{R}^4) = GL(\mathbb{R}^4) \ltimes \mathbb{R}^4$ (see Theorem 3).

Throughout this paper, all vector spaces, Lie algebras, and left-symmetric algebras are supposed to be over the field $\mathbb{R}$, unless otherwise specified. We shall also suppose that all Lie groups are connected and simply connected.

2. Extensions of Lie algebras

The notion of extensions of a Lie algebra $\mathcal{G}$ by an abelian Lie algebra $\mathcal{A}$ is well known (see, for instance, books [8] and [13]). In light of [21], we will briefly describe here the notion of extension $\widetilde{\mathcal{G}}$ of a Lie algebra $\mathcal{G}$ by a Lie algebra $\mathcal{A}$ which is not necessarily abelian.

Suppose that a vector space extension $\widetilde{\mathcal{G}}$ of a Lie algebra $\mathcal{G}$ by another Lie algebra $\mathcal{A}$ is known, and we want to define a Lie structure on $\widetilde{\mathcal{G}}$ in terms of the Lie structures of $\mathcal{G}$ and $\mathcal{A}$. Let $\sigma : \mathcal{G} \to \widetilde{\mathcal{G}}$ be a section, that is, a linear map such that $\pi \circ \sigma = id$. Then the linear map $\Psi : (a,x) \mapsto i(a) + \sigma(x)$ from $\mathcal{A} \oplus \mathcal{G}$ onto $\widetilde{\mathcal{G}}$ is an isomorphism of vector spaces. For (a,x) and (b,y) in $\mathcal{A} \oplus \mathcal{G}$, a commutator on $\widetilde{\mathcal{G}}$ must satisfy

$$\begin{aligned}[i(a) + \sigma(x), i(b) + \sigma(y)] = i([a,b]) &+ [\sigma(x), i(b)] \\ &+ [i(a), \sigma(y)] + [\sigma(x), \sigma(y)]\end{aligned} \tag{1}$$

Now we define a linear map $\phi : \mathcal{G} \to End(\mathcal{A})$ by

$$\phi(x)a = [\sigma(x), i(a)] \tag{2}$$

On the other hand, since $\pi([\sigma(x), \sigma(y)]) = \pi(\sigma([x,y]))$, it follows that there exists an alternating bilinear map $\omega : \mathcal{G} \times \mathcal{G} \to \mathcal{A}$ such that $[\sigma(x), \sigma(y)] = \sigma[x,y] + \omega(x,y)$.

To sum up, by means of the isomorphism above, $\widetilde{\mathcal{G}} \cong \mathcal{A} \oplus \mathcal{G}$ and its elements may be denoted by (a, x) with $a \in \mathcal{A}$ and x is simply characterized by its coordinates in $\mathcal{G}$. The commutator defined by (1) is now given by

$$[(a,x),(b,y)] = ([a,b] + \phi(x)\,b - \phi(y)\,a + \omega(x,y), [x,y]), \tag{3}$$

for all $(a,x) \in \widetilde{\mathcal{G}} \cong \mathcal{A} \oplus \mathcal{G}$.

It is easy to see that this is actually a Lie bracket (i.e, it verifies the Jacobi identity) if and only if the following three conditions are satisfied

1. $\phi(x)[b,c] = [\phi(x)\,b, c] + [b, \phi(x)\,c]$,
2. $[\phi(x), \phi(y)] = \phi([x,y]) + ad_{\omega(x,y)}$,
3. $\omega([x,y],z) - \omega(x,[y,z]) + \omega(y,[x,z]) = \phi(x)\,\omega(y,z) + \phi(y)\,\omega(z,x) + \phi(z)\,\omega(x,y)$.

Remark 1. *We see that condition (1) above is equivalent to say that $\phi(x)$ is a derivation of $\mathcal{A}$. In other words, $\mathcal{G}$ is actually acting by derivations, that is, $\phi : \mathcal{G} \to Der(\mathcal{A})$. Condition (2) indicates clearly that if $\mathcal{A}$ is supposed to be abelian, then $\mathcal{A}$ becomes a $\mathcal{G}$-module in a natural way, because in this case the linear map $\phi : \mathcal{G} \to Der(\mathcal{A})$ given by $\phi(x)\,a = [\sigma(x), i(a)]$ is well defined. Condition (3) is equivalent to the fact that, if $\mathcal{A}$ is abelian, ω is a 2-cocycle (i.e., $\delta_\phi\omega = 0$, where δ_ϕ refers to the coboundary operator corresponding to the action ϕ).*

If now $\sigma' : \mathcal{G} \to \widetilde{\mathcal{G}}$ is another section, then $\sigma' - \sigma = \tau$ for some linear map $\tau : \mathcal{G} \to \mathcal{A}$, and it follows that the corresponding morphism and the 2-cocycle are $\phi' = \phi + ad \circ \tau$ and $\omega' = \omega + \delta_\phi\tau + \frac{1}{2}[\tau,\tau]$, respectively, where ad stands here and below (if there is no ambiguity) for the adjoint representation in $\mathcal{A}$, and where $[\tau,\tau]$ has the following meaning: Given two linear maps $\alpha, \beta : \mathcal{G} \to \mathcal{A}$, we define $[\alpha,\beta](x,y) = [\alpha(x), \beta(y)] - [\alpha(y), \beta(x)]$. In particular, we have $\frac{1}{2}[\tau,\tau](x,y) = [\tau(x), \tau(y)]$. Note here that the Lie algebra $\mathcal{A}$ is not necessarily abelian. Therefore, $\omega' - \omega$ is a 2 -coboundary if and only if $[\tau(x), \tau(y)] = 0$ for all $x, y \in \mathcal{G}$. Equivalently, $\omega' - \omega$ is a 2-coboundary if and only if τ has its range in the center $Z(\mathcal{A})$ of $\mathcal{A}$. In that case, we get $\omega' - \omega = \delta_\phi\tau \in B^2_\phi(\mathcal{G}, Z(\mathcal{A}))$, the group of 2-coboundaries for $\mathcal{G}$ with values in $Z(\mathcal{A})$.

To overcome all these difficulties, we proceed as follows. Let $C^2(\mathcal{G},\mathcal{A})$ be the abelian group of all 2-cochains, i.e., alternating bilinear mappings $\mathcal{G} \times \mathcal{G} \to \mathcal{A}$. For a given $\phi : \mathcal{G} \to Der(\mathcal{A})$, let $T_\phi \in C^2(\mathcal{G},\mathcal{A})$ be defined by

$$T_\phi(x,y) = [\phi(x), \phi(y)] - \phi([x,y]), \quad \text{for all } x, y \in \mathcal{G}.$$

If there exists some $\omega \in C^2(\mathcal{G},\mathcal{A})$ such that $T_\phi = ad \circ \omega$ and $\delta_\phi\omega = 0$, then the pair (ϕ, ω) is called a *factor system* for $(\mathcal{G},\mathcal{A})$. Let $Z^2(\mathcal{G},\mathcal{A})$ be the set of all factor systems for $(\mathcal{G},\mathcal{A})$. It is well known that the equivalence classes of extensions of a Lie algebra $\mathcal{G}$ by a Lie algebra $\mathcal{A}$ are in one-to-one correspondence with the elements of the quotient space $Z^2(\mathcal{G},\mathcal{A})/C^1(\mathcal{G},\mathcal{A})$, where $C^1(\mathcal{G},\mathcal{A})$ is the space of linear maps from $\mathcal{G}$ into $\mathcal{A}$ (see, for instance, [21], Theorem II.7). Note that if we assume that $\mathcal{A}$ is abelian, then we meet the well known result (see, for instance, [7]) stating that for a given action $\phi : \mathcal{G} \to End(\mathcal{A})$, the equivalence classes of extensions of $\mathcal{G}$ by $\mathcal{A}$ are in one-to-one correspondence with the elements of the second cohomology group

$$H^2_\phi(\mathcal{G}, \mathcal{A}) = Z^2_\phi(\mathcal{G}, \mathcal{A}) / B^2_\phi(\mathcal{G}, \mathcal{A}).$$

In the present paper, we will be concerned with the special case where $\mathcal{A}$ is non-abelian and $\mathcal{G}$ is $\mathbb{R}$, and henceforth the cocycle ω is identically zero.

Remark 2. *It is worth noticing that the construction above is closely related to the notion of $\mathcal{G}$-kernels considered for Lie algebras firstly in [20].*

3. Left-symmetric algebras

The notion of a *left-symmetric algebra* arises naturally in various areas of mathematics and physics. It originally appeared in the works of Vinberg [23] and Koszul [16] concerning convex homogeneous cones and bounded homogeneous domains, respectively. It also appears, for instance, in connection with Yang-Baxter equation and integrable hydrodynamic systems (cf. [4, 12, 18]). A left-symmetric algebra $(A, .)$ is a finite-dimensional algebra A in which the products, for all $x, y, z \in A$, satisfy the identity

$$(x \cdot y) \cdot z - x \cdot (y \cdot z) = (y \cdot x) \cdot z - y \cdot (x \cdot z) \tag{4}$$

It is clear that an associative algebra is a left-symmetric algebra. Actually, if A is a left-symmetric algebra and $(x, y, z) = (x \cdot y) \cdot z - x \cdot (y \cdot z)$ is the associator of x, y, z, then we can see that (4) is equivalent to $(x, y, z) = (y, x, z)$. This means that the notion of a left-symmetric algebra is a natural generalization of the notion of an associative algebra. If A is a left-symmetric algebra, then the commutator

$$[x, y] = x \cdot y - y \cdot x \tag{5}$$

defines the structure of a Lie algebra on A, called the *associated Lie algebra*. Conversely, if $\mathcal{G}$ is a Lie algebra with a left-symmetric product $\cdot$ satisfying (5), then we say that the left-symmetric structure is *compatible* with the Lie structure on $\mathcal{G}$.

On the other hand, let G be a Lie group with a left-invariant flat affine connection ∇, and define a product $\cdot$ on the Lie algebra $\mathcal{G}$ of G by

$$x \cdot y = \nabla_x y, \text{ for all } x, y \in \mathcal{G}. \tag{6}$$

Then, conditions on the connection ∇ for being flat and torsion-free are now equivalent to conditions (4) and (5), respectively. Conversely, suppose that $\mathcal{G}$ is endowed with a left-symmetric product $\cdot$ which is compatible with the Lie bracket of $\mathcal{G}$. In this case, in order to obtain a left-invariant flat affine structure on G, we can define an operator ∇ on $\mathcal{G}$ according to identity (6) and then extend it by left-translations to the whole Lie group G. To sum up, the left-invariant flat affine structures on G are in one-to-one correspondence with the left-symmetric structures on $\mathcal{G}$ compatible with the Lie structure.

Let now A be a left-symmetric algebra, and let L_x and R_x be the left and right multiplications by the element x, that is, $L_x y = x \cdot y$ and $R_x y = y \cdot x$. We say that

A is *complete* if R_x is a nilpotent operator, for all $x \in A$. It turns out that, for a given simply connected Lie group G with Lie algebra $\mathcal{G}$, the complete left-invariant flat affine structures on G are in one-to-one correspondence with the complete left-symmetric structures on $\mathcal{G}$ compatible with the Lie structure (see, for example, [14]). It is also known that an n-dimensional simply connected Lie group admits a complete left-invariant flat affine structure if and only if it acts simply transitively on $\mathbb{R}^n$ by affine transformations (see [14]). A simply connected Lie group acting simply transitively on $\mathbb{R}^n$ by affine transformations must be solvable according to [1], but it is worth noticing that there exist solvable (even nilpotent) Lie groups which do not admit affine structures (see [3]).

We close this section by fixing some notations which we will use in what follows. For a left-symmetric algebra A, we can easily check that the subset

$$T(A) = \{x \in A : L_x = 0\} \tag{7}$$

is a two-sided ideal in A. Geometrically, if G is a Lie group which acts simply transitively on $\mathbb{R}^n$ by affine transformations, then $T(\mathcal{G})$ corresponds to the set of translational elements in G, where $\mathcal{G}$ is endowed with the complete left-symmetric product corresponding to the action of G on $\mathbb{R}^n$. It has been conjectured in [1] that every nilpotent Lie group G which acts simply transitively on $\mathbb{R}^n$ by affine transformations contains a translation which lies in the center of G, but this conjecture turned out to be false (see [9]).

3.1. Extensions of left-symmetric algebras

In this section, we will briefly discuss the problem of an extension of a left-symmetric algebras. To our knowledge, this notion has been considered for the first time in [14]. Suppose we are given a vector space A as an extension of a left-symmetric algebra K by another left-symmetric algebra E. We want to define a left-symmetric structure on A in terms of the left-symmetric structures given on K and E. In other words, we want to define a left-symmetric product on A for which E becomes a two-sided ideal in A such that $A/E \cong K$; or equivalently, $0 \to E \to A \to K \to 0$ becomes a short exact sequence of left-symmetric algebras.

Theorem 1 (See [14]). *There exists a left-symmetric structure on A extending a left-symmetric algebra K by a left-symmetric algebra E if and only if there exist two linear maps λ, $\rho : K \to End(E)$ and a bilinear map $g : K \times K \to E$ such that, for all $x, y, z \in K$ and $a, b \in E$, the following conditions are satisfied.*

(*i*) $\lambda_x (a \cdot b) = \lambda_x (a) \cdot b + a \cdot \lambda_x (b) - \rho_x (a) \cdot b$,

(*ii*) $\rho_x ([a, b]) = a \cdot \rho_x (b) - b \cdot \rho_x (a)$,

(*iii*) $[\lambda_x, \lambda_y] = \lambda_{[x,y]} + L_{g(x,y)-g(y,x)}$, *where $L_{g(x,y)-g(y,x)}$ denotes the left multiplication in E by $g(x, y) - g(y, x)$,*

(*iv*) $[\lambda_x, \rho_y] = \rho_{x\cdot y} - \rho_y \circ \rho_x + R_{g(x,y)}$, *where $R_{g(x,y)}$ denotes the right multiplication in E by $g(x, y)$,*

(v) $g(x, y\cdot z) - g(y, x\cdot z) + \lambda_x(g(y,z)) - \lambda_y(g(x,z)) - g([x,y],z) - \rho_z(g(x,y) - g(y,x)) = 0$.

If the conditions of Theorem 1 are fulfilled, then the extended left-symmetric product on $A \cong K \times E$ is given by

$$(x,a)\cdot(y,b) = (x\cdot y, a\cdot b + \lambda_x(b) + \rho_y(a) + g(x,y)). \tag{8}$$

It is remarkable that if the left-symmetric product of E is trivial, then the conditions of Theorem 1 simplify to the following three conditions:

(i) $[\lambda_x, \lambda_y] = \lambda_{[x,y]}$, i.e., λ is a representation of Lie algebras,

(ii) $[\lambda_x, \rho_y] = \rho_{x\cdot y} - \rho_y \circ \rho_x$,

(iii) $g(x, y\cdot z) - g(y, x\cdot z) + \lambda_x(g(y,z)) - \lambda_y(g(x,z)) - g([x,y],z) - \rho_z(g(x,y) - g(y,x)) = 0$.

In this case, E becomes a K-bimodule and the extended product given in (8) simplifies, too. Recall that if K is a left-symmetric algebra and V is a vector space, then we say that V is a K-bimodule if there exist two linear maps $\lambda, \rho : K \to End(V)$ which satisfy conditions (i) and (ii) stated above.

Let K be a left-symmetric algebra, and let V be a K-bimodule. Let $L^p(K,V)$ be the space of all p-linear maps from K to V, and define two coboundary operators $\delta_1 : L^1(K,V) \to L^2(K,V)$ and $\delta_2 : L^2(K,V) \to L^3(K,V)$ as follows: For a linear map $h \in L^1(K,V)$ we set

$$\delta_1 h(x,y) = \rho_y(h(x)) + \lambda_x(h(y)) - h(x\cdot y), \tag{9}$$

and for a bilinear map $g \in L^2(K,V)$ we set

$$\begin{aligned}\delta_2 g(x,y,z) = {} & g(x, y\cdot z) - g(y, x\cdot z) + \lambda_x(g(y,z)) - \lambda_y(g(x,z)) \\ & - g([x,y],z) - \rho_z(g(x,y) - g(y,x)).\end{aligned} \tag{10}$$

It is straightforward to check that $\delta_2 \circ \delta_1 = 0$. Therefore, if we set $Z^2_{\lambda,\rho}(K,V) = \ker\delta_2$ and $B^2_{\lambda,\rho}(K,V) = \operatorname{Im}\delta_1$, we can define a notion of second cohomology for the actions λ and ρ by simply setting $H^2_{\lambda,\rho}(K,V) = Z^2_{\lambda,\rho}(K,V)/B^2_{\lambda,\rho}(K,V)$. As in the case of extensions of Lie algebras, we can prove that for given linear maps $\lambda, \rho : K \to End(V)$, the equivalence classes of extensions $0 \to V \to A \to K \to 0$ of K by V are in one-to-one correspondence with the elements of the second cohomology group $H^2_{\lambda,\rho}(K,V)$. We close this subsection with the following lemma on completeness of left-symmetric algebras (see [6, Proposition 3.4]).

Lemma 1. *Let $0 \to E \to A \to K \to 0$ be a short exact sequence of left-symmetric algebras. Then, A is complete if and only if E and K are so.*

3.2. Central extensions of left-symmetric algebras

The notion of central extensions known for Lie algebras may analogously be defined for left-symmetric algebras. Let A be a left-symmetric extension of a left-symmetric algebra K by another left-symmetric algebra E, and let $\mathcal{G}$ be the Lie algebra associated to A. Define *the center* of A to be $C(A) = T(A) \cap Z(\mathcal{G})$, that is,

$$C(A) = \{x \in A : x \cdot y = y \cdot x = 0, \text{ for all } y \in A\}, \tag{11}$$

where $Z(\mathcal{G})$ is the center of the Lie algebra $\mathcal{G}$ and $T(A)$ is the two-sided ideal of A defined by (7).

Definition 1. *The extension $0 \to E \overset{i}{\to} A \overset{\pi}{\to} K \to 0$ of left-symmetric algebras is said to be* central *(resp.* exact*) if $i(E) \subseteq C(A)$ (resp. $i(E) = C(A)$).*

Remark 3. *It is not difficult to show that if the extension $0 \to E \overset{i}{\to} A \overset{\pi}{\to} K \to 0$ is central, then both the left-symmetric product and the K-bimodule on E are trivial (i.e., $a \cdot b = 0$ for all $a, b \in E$, and $\lambda = \rho = 0$). It is also easy to show that if $[g]$ is the cohomology class associated to this extension, and if*

$$I_{[g]} = \{x \in K : x \cdot y = y \cdot x = 0 \text{ and } g(x,y) = g(y,x) = 0, \text{ for all } y \in K\},$$

then the extension is exact if and only if $I_{[g]} = 0$ (see [14]). We note here that $I_{[g]}$ is well defined because any other element in $[g]$ takes the form $g + \delta_1 h$, with $\delta_1 h(x,y) = -h(x \cdot y)$.

Let now K be a left-symmetric algebra, and E a trivial K-bimodule. Denote by $(A, [g])$ the central extension $0 \to E \to A \to K \to 0$ corresponding to the cohomology class $[g] \in H^2(K,E)$. Let $(A, [g])$ and $(A', [g'])$ be two central extensions of K by E, and let $\mu \in Aut(E) = GL(E)$ and $\eta \in Aut(K)$, where $Aut(E)$ and $Aut(K)$ are the groups of left-symmetric automorphisms of E and K, respectively. It is clear that if $h \in L^1(K,E)$, then the linear mapping $\psi : A \to A'$ defined by $\psi(x,a) = (\eta(x), \mu(a) + h(x))$ is an isomorphism provided $g'(\eta(x), \eta(y)) = \mu(g(x,y)) - \delta_1 h(x,y)$ for all $(x,y) \in K \times K$, i.e. $\eta^*[g'] = \mu_*[g]$. This allows us to define an action of the group $G = Aut(E) \times Aut(K)$ on $H^2(K,E)$ by setting

$$(\mu, \eta) . [g] = \mu_* \eta^* [g], \tag{12}$$

or equivalently, $(\mu, \eta) . g(x,y) = \mu(g(\eta(x), \eta(y)))$ for all $x, y \in K$.

Denoting the set of all exact central extensions of K by E by

$$H^2_{ex}(K,E) = \left\{[g] \in H^2(K,E) : I_{[g]} = 0\right\},$$

and the orbit of $[g]$ by $G_{[g]}$, it turns out that the following result is valid (see [14]).

Proposition 1. *Let $[g]$ and $[g']$ be two classes in $H^2_{ex}(K,E)$. Then, the central extensions $(A, [g])$ and $(A', [g'])$ are isomorphic if and only if $G_{[g]} = G_{[g']}$. In other words, the classification of the exact central extensions of K by E is, up to left-symmetric isomorphism, the orbit space of $H^2_{ex}(K,E)$ under the natural action of $G = Aut(E) \times Aut(K)$.*

3.3. Complexification of a real left-symmetric algebra

Let A be a real left-symmetric algebra of dimension n, and let $A^{\mathbb{C}}$ denote the real vector space $A \oplus A$. Let $J : A \oplus A \to A \oplus A$ be the linear map on $A \oplus A$ defined by $J(x, y) = (-y, x)$. For $\alpha + i\beta \in \mathbb{C}$ and $x, x', y, y' \in A$, we define

$$(\alpha + i\beta)(x, y) = (\alpha x - \beta y, \alpha y + \beta x), \tag{13}$$

$$(x, y) \cdot (x', y') = (xx' - yy', xy' + yx'). \tag{14}$$

We endow the set $A^{\mathbb{C}}$ with the componentwise addition, multiplication by complex numbers defined by (13), and the product defined by (14). It is then straightforward to verify that $A^{\mathbb{C}}$, when endowed with the product defined by (14), becomes a complex left-symmetric algebra called *the complexification* of A. The left-symmetric algebra A can be identified with the set of elements in $A^{\mathbb{C}}$ of the form $(x, 0)$, where $x \in A$. If $e_1, \ldots, e_n$ is a basis of A, then the elements $(e_1, 0), \ldots, (e_n, 0)$ form a basis of the complex vector space $A^{\mathbb{C}}$. It follows that $\dim_{\mathbb{C}}\left(A^{\mathbb{C}}\right) = \dim_{\mathbb{R}}(A)$.

Since $A^{\mathbb{C}}$ is a left-symmetric algebra, we know that the commutator $[(x, y), (x', y')] = (x, y) \cdot (x', y') - (x', y') \cdot (x, y)$ defines a Lie algebra $\mathcal{G}^{\mathbb{C}}$ on $A^{\mathbb{C}}$. Computing this commutator, we get the following lemma.

Lemma 2. *The complex Lie algebra $\mathcal{G}^{\mathbb{C}}$ associated to the complex left-symmetric algebra $A^{\mathbb{C}}$ is isomorphic to the complexification of the Lie algebra $\mathcal{G}$ associated to the left-symmetric algebra A.*

Therefore, if $e_1, \ldots, e_n$ is a basis of A, then the elements $(e_1, 0), \ldots, (e_n, 0)$ form a basis of $\mathcal{G}^{\mathbb{C}}$, and the structural constants of $\mathcal{G}^{\mathbb{C}}$ are real since they coincide with the structural constants of $\mathcal{G}$ in the basis $e_1, \ldots, e_n$.

4. Left-symmetric structures on the oscillator algebra

Recall that the Heisenberg group H_3 is the 3-dimensional Lie group diffeomorphic to $\mathbb{R} \times \mathbb{C}$ with the group law

$$(v_1, z_1) \cdot (v_2, z_2) = (v_1 + v_2 + \frac{1}{2}\operatorname{Im}(\overline{z_1} z_2), z_1 + z_2),$$

for all $v_1, v_2 \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{C}$. Let $\lambda > 0$, and let $G = \mathbb{R} \ltimes H_3$ be equipped with the group law

$$(t_1, v_1, z_1) \cdot (t_2, v_2, z_2) = (t_1 + t_2, v_1 + v_2 + \frac{1}{2}\operatorname{Im}(\overline{z_1} z_2 e^{i\lambda t_1}), z_1 + z_2 e^{i\lambda t_1}),$$

for all $t_1, t_2 \in \mathbb{R}$ and $(v_1, z_1), (v_2, z_2) \in H_3$. This is a 4-dimensional Lie group with Lie algebra $\mathcal{G}$ having a basis $\{e_1, e_2, e_3, e_4\}$ such that

$$[e_1, e_2] = e_3,\ [e_4, e_1] = \lambda e_2,\ [e_4, e_2] = -\lambda e_1,$$

and all the other brackets are zero. It follows that the derived series is given by

$$\mathcal{D}^1\mathcal{G} = [\mathcal{G}, \mathcal{G}] = span\{e_1, e_2, e_3\},\ \mathcal{D}^2\mathcal{G} = span\{e_3\},\ \mathcal{D}^3\mathcal{G} = \{0\},$$

and therefore $\mathcal{G}$ is a (non-nilpotent) 3-step solvable Lie algebra. When $\lambda = 1$, G is known as the *oscillator group*. We will denote it by O_4, and we shall denote its Lie algebra by $\mathcal{O}_4$ and call it the *oscillator algebra*.

From now on, A_4 will be a complete real left-symmetric algebra whose associated Lie algebra is $\mathcal{O}_4$. We begin by proving the following proposition which will be crucial to the classification of complete left-symmetric structures on $\mathcal{O}_4$.

Proposition 2. *A_4 is not simple (i.e., A_4 contains a proper two-sided ideal).*

Proof. Assume to the contrary that A_4 is simple, and let $A_4^{\mathbb{C}}$ be its complexification. By [15], Lemma 2.10, it follows that $A_4^{\mathbb{C}}$ is either simple or a direct sum of two simple ideals having the same dimension. If $A_4^{\mathbb{C}}$ is simple, then we can apply Proposition 5.1 in [5] to deduce that, being simple and complete, $A_4^{\mathbb{C}}$ is necessarily isomorphic to the complex left-symmetric algebra B_4 having a basis $\{e_1, e_2, e_3, e_4\}$ such that

$$\begin{aligned} &e_1 \cdot e_2 = e_2 \cdot e_1 = e_4,\ e_2 \cdot e_3 = 2e_1, \\ &e_3 \cdot e_2 = e_4 \cdot e_1 = e_1,\ e_4 \cdot e_2 = -e_2,\ e_4 \cdot e_3 = 2e_3, \end{aligned}$$

and all other products are zero. It follows that the Lie algebra $\mathcal{G}_4$ associated to B_4 admits a basis $\{e_1, e_2, e_3, e_4\}$ such that

$$[e_2, e_3] = [e_4, e_1] = e_1,\ \ [e_2, e_4] = e_2,\ \ [e_3, e_4] = -2e_3.$$

This leads to a contradiction since, according to Lemma 2, $\mathcal{G}_4$ should be isomorphic to the complexification of the Lie algebra $\mathcal{O}_4$, but this is obviously not the case. This contradiction shows that $A_4^{\mathbb{C}}$ cannot be simple.

If $A_4^{\mathbb{C}}$ is a direct sum of two simple ideals having the same dimension, say $A_4^{\mathbb{C}} = A_1 \oplus A_2$, it follows that $\dim A_1 = \dim A_2 = \frac{1}{2} \dim A_4^{\mathbb{C}} = 2$. In this case, by Corollary 4.1 in [5], A_1 and A_2 are both isomorphic to the unique two-dimensional complex simple left-symmetric algebra having a basis

$$B_2 = \langle e_1, e_2 : e_1 \cdot e_1 = 2e_1,\ e_1 \cdot e_2 = e_2,\ e_2 \cdot e_2 = e_1 \rangle .$$

This is a contradiction, since A_1 and A_2 are complete but B_2 is not. This contradiction shows that $A_4^{\mathbb{C}}$ cannot be direct sum of two simple ideals. We deduce that A_4 is not simple, and this completes the proof of the proposition. □

Before we return to the algebra A_4, we need to give the following lemmas.

Lemma 3. *Let A be a left-symmetric algebra with Lie algebra $\mathcal{G}$, and R a two-sided ideal in A. Then, the Lie algebra $\mathcal{R}$ associated to R is an ideal in $\mathcal{G}$.*

Proof. Let $x \in \mathcal{R}$ and $y \in \mathcal{G}$. Since R is a two-sided ideal, then $x \cdot y$ and $y \cdot x$ belong to R. It follows that $[x, y] = x \cdot y - y \cdot x \in R$, and therefore $\mathcal{R}$ is an ideal in $\mathcal{G}$. □

Lemma 4. *The oscillator algebra $\mathcal{O}_4$ contains only two proper ideals which are $Z(\mathcal{O}_4) \cong \mathbb{R}$ and $[\mathcal{O}_4, \mathcal{O}_4] \cong \mathcal{H}_3$.*

Proof. It is clear that $\mathcal{Z}(\mathcal{O}_4) \cong \mathbb{R}$ and $[\mathcal{O}_4, \mathcal{O}_4] \cong \mathcal{H}_3$ are proper ideals in $\mathcal{O}_4$. If $\mathcal{I}$ is a proper ideal in $\mathcal{O}_4$, then $\mathcal{I}$ should be unimodular. If $\dim(\mathcal{I}) = 1$, then $\mathcal{I}$ is isomorphic to $\mathcal{Z}(\mathcal{O}_4) \cong \mathbb{R}$. If $\dim(\mathcal{I}) = 2$, then being unimodular, $\mathcal{I}$ is isomorphic to $\mathbb{R}^2$. In particular, $\mathcal{I}$ contains $\mathcal{Z}(\mathcal{O}_4)$ and thus $\mathcal{O}_4/\mathcal{I}$ is abelian, a contradiction since $\mathcal{O}_4$ is not nilpotent. Hence, $\mathcal{O}_4$ contains no two-dimensional ideals. If $\dim(\mathcal{I}) = 3$, then being unimodular and solvable, $\mathcal{I}$ is isomorphic to either $\mathcal{H}_3$, the Lie algebra $\mathcal{E}(2)$ of the group of the rigid motions of the plane, or the Lie algebra $\mathcal{E}(1,1)$ of the group of the rigid motions of the Minkowski plane. However, it is straightforward to show that $\mathcal{O}_4$ cannot be obtained as an extension of $\mathcal{E}(2)$ or $\mathcal{E}(1,1)$. We have therefore proved the lemma. □

By the above proposition, A_4 is not simple and hence it has a proper two-sided ideal I, so we get a short exact sequence of complete left-symmetric algebras

$$0 \to I \xrightarrow{i} A_4 \xrightarrow{\pi} J \to 0. \tag{15}$$

In fact, according to Lemma 1, the completeness of I and J comes from that of A_4. If $\mathcal{I}$ is the Lie subalgebra associated to I then, by Lemma 3, $\mathcal{I}$ is an ideal in $\mathcal{O}_4$. From Lemma 4, it follows that there are two cases to consider according to whether $\mathcal{I}$ is isomorphic to $\mathcal{H}_3$ or $\mathbb{R}$. Next, we will focus on the case where $\mathcal{I}$ is isomorphic to $\mathcal{H}_3 \cong [\mathcal{O}_4, \mathcal{O}_4]$. In this case, the short exact sequence (15) becomes

$$0 \to I_3 \xrightarrow{i} A_4 \xrightarrow{\pi} I_0 \to 0, \tag{16}$$

where I_3 is a complete 3-dimensional left-symmetric algebra whose Lie algebra is $\mathcal{H}_3$, and $I_0 = \{e_0 : e_0 \cdot e_0 = 0\}$ the trivial one-dimensional real left-symmetric algebra. It is easy to prove the following proposition (cf. [10, Theorem 3.5]).

Proposition 3. *Up to left-symmetric isomorphism, the complete left-symmetric structures on the Heisenberg algebra $\mathcal{H}_3$ are classified as follows: There is a basis $\{e_1, e_2, e_3\}$ of $\mathcal{H}_3$ relative to which the left-symmetric product is given by one of the following classes:*

(*i*) $e_1 \cdot e_1 = pe_3$, $e_2 \cdot e_2 = qe_3$, $e_1 \cdot e_2 = \frac{1}{2}e_3$, $e_2 \cdot e_1 = -\frac{1}{2}e_3$, *where* $p, q \in \mathbb{R}$.

(*ii*) $e_1 \cdot e_2 = me_3$, $e_2 \cdot e_1 = (m-1)e_3$, $e_2 \cdot e_2 = e_1$, *where* $m \in \mathbb{R}$.

Remark 4. *It is noticeable that the left-symmetric products on $\mathcal{H}_3$ belonging to class (i) in Proposition 3 are obtained by central extensions (in the sense of fixed in Subsection 3.1) of $\mathbb{R}^2$ endowed with some complete left-symmetric structure by I_0. However, the left-symmetric products on A_3 belonging to class (ii) are obtained by central extensions of the non-abelian two-dimensional Lie algebra $\mathcal{G}_2$ endowed with its unique complete left-symmetric structure by I_0.*

Now we return to the short exact sequence (16). First, let $\sigma : I_0 \to A_4$ be a section, and set $\sigma(e_0) = x_0 \in A_4$. Define two linear maps λ, $\rho \in End(I_3)$ by putting $\lambda(y) = x_0 \cdot y$ and $\rho(y) = y \cdot x_0$, and put $e = x_0 \cdot x_0$ (clearly $e \in I_3$). Let $g : I_0 \times I_0 \to I_3$ be the bilinear map defined by $g(e_0, e_0) = e$. It is obvious, using the notation of Subsection 3.1, to verify that $\delta_2 g = 0$, i.e. $g \in Z^2_{\lambda,\rho}(I_0, I_3)$. The extended left-symmetric product on $I_3 \oplus I_0$ given by (8) turns out to take the simplified form $(x, ae_0) \cdot (y, be_0) = (x \cdot y + a\lambda(y) + b\rho(x) + abe, 0)$, for all $x, y \in I_3$ and $a, b \in \mathbb{R}$. The conditions in Theorem 1 can be simplified to the following conditions:

$$\lambda(x \cdot y) = \lambda(x) \cdot y + x \cdot \lambda(y) - \rho(x) \cdot y \tag{17}$$

$$\rho([x, y]) = x \cdot \rho(y) - y \cdot \rho(x) \tag{18}$$

$$[\lambda, \rho] + \rho^2 = R_e \tag{19}$$

Let $\phi : \mathbb{R} \to End(\mathcal{H}_3)$ be the linear map defined by formula (2). As we mentioned in Remark 1, $\mathbb{R}$ acts on $\mathcal{H}_3$ by derivations, that is, $\phi : \mathbb{R} \to Der(\mathcal{H}_3)$. In particular, we deduce in view of (3) that $\lambda = D + \rho$ for some derivation D of $\mathcal{H}_3$. The derivations of $\mathcal{H}_3$ are given by the following lemma, whose proof is straightforward and is therefore omitted.

Lemma 5. *In a basis $\{e_1, e_2, e_3\}$ of $\mathcal{H}_3$ satisfying $[e_1, e_2] = e_3$, a derivation D of $\mathcal{H}_3$ takes the form*

$$D = \begin{pmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & a_1 + b_2 \end{pmatrix}.$$

On the other hand, observe that $(x, ae_0) \in T(A_4)$ if and only if $(x, ae_0) \cdot (y, be_0) = (0, 0)$ for all $(y, be_0) \in I_3 \oplus I_0$, or equivalently, $x \cdot y + a\lambda(y) + b\rho(x) + abe = 0$ for all $(y, be_0) \in I_3 \oplus I_0$. Since y and b are arbitrary, we conclude that this is also equivalent to say that $(L_x)_{|A_3} = -a\lambda$ and $\rho(x) = -ae$. In particular, an element $x \in I_3$ belongs to $T(A_4)$ if and only if $(L_x)_{|I_3} = 0$ and $\rho(x) = 0$, or equivalently,

$$I_3 \cap T(A_4) = T(I_3) \cap \ker \rho. \tag{20}$$

The following lemma will be crucial for the next section.

Lemma 6. *The center $C(A_4) = T(A_4) \cap Z(\mathcal{O}_4)$ is non-trivial.*

Proof. In view of Proposition 3, we have to consider two cases.

Case 1. Assume that there is a basis $\{e_1, e_2, e_3\}$ of $\mathcal{H}_3$ relative to which the left-symmetric product of I_3 is given by : $e_1 \cdot e_1 = pe_3$, $e_2 \cdot e_2 = qe_3$, $e_1 \cdot e_2 = \frac{1}{2}e_3$, $e_2 \cdot e_1 = -\frac{1}{2}e_3$, where $p, q \in \mathbb{R}$. Substituting $x = e_1$ and $y = e_2$ into (18), we find that the operator ρ takes the form

$$\rho = \begin{pmatrix} \alpha_1 & \beta_1 & 0 \\ \alpha_2 & \beta_2 & 0 \\ \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix},$$

with $\gamma_3 = p\beta_1 - q\alpha_2 + \frac{1}{2}(\alpha_1 + \beta_2)$. Since $\lambda = D + \rho$ for some $D \in \mathcal{H}_3$, we use Lemma 5 to deduce that

$$\lambda = \begin{pmatrix} \alpha_1 + a_1 & \beta_1 + b_1 & 0 \\ \alpha_2 + a_2 & \beta_2 + b_2 & 0 \\ \alpha_3 + a_3 & \beta_3 + b_3 & \gamma_3 + a_1 + b_2 \end{pmatrix}.$$

Since $(L_{e_3})_{|_{I_3}} = 0$ and $e \in I_3$, then (19), when applied to e_3, gives

$$\gamma_3^2 e_3 = e_3 \cdot e = 0,$$

from which we get $\gamma_3 = 0$, i.e., $\rho(e_3) = 0$. It follows from (20) that $e_3 \in T(A_4)$. Since $Z(\mathcal{O}_4) = \mathbb{R}e_3$, we deduce that $C(A_4) = T(A_4) \cap Z(\mathcal{O}_4) \neq 0$, as required.

Case 2. Assume now that there is a basis $\{e_1, e_2, e_3\}$ of $\mathcal{H}_3$ relative to which the left-symmetric product of I_3 is given by : $e_1 \cdot e_2 = me_3$, $e_2 \cdot e_1 = (m-1)e_3$, $e_2 \cdot e_2 = e_1$, where m is a real number.

Substituting successively $x = e_1,\ y = e_2$ and $x = e_2,\ y = e_3$ into equation (18), we find that the operator ρ takes the form

$$\rho = \begin{pmatrix} \alpha_1 & \beta_1 & -\alpha_2 \\ \alpha_2 & \beta_2 & 0 \\ \alpha_3 & \beta_3 & m\beta_2 - (m-1)\alpha_1 \end{pmatrix}, \tag{21}$$

with $(m-1)\alpha_2 = 0$.

We claim that $\alpha_2 = 0$. To prove this, let us assume to the contrary that $\alpha_2 \neq 0$. It follows that $m = 1$, and therefore

$$\begin{aligned} \rho(e_3) &= -\alpha_2 e_1 + \beta_2 e_3 \\ \rho^2(e_3) &= -\alpha_2(\alpha_1 + \beta_2) e_1 - \alpha_2^2 e_2 + \left(\beta_2^2 - \alpha_2\alpha_3\right) e_3 \end{aligned}$$

Since $\alpha_2 \neq 0$, we deduce that e_3, $\rho(e_3)$, $\rho^2(e_3)$ form a basis of I_3. Since ρ is nilpotent (by completeness of the left-symmetric structure), it follows that $\rho^3(e_3) = 0$. In other words, ρ has the form

$$\rho = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

with respect to the basis $e_1' = -\rho(e_3)$, $e_2' = \rho^2(e_3)$, $e_3' = -e_3$.

Using the fact that $\alpha_1 + 2\beta_2 = 0$ which follows from the identity $\rho^3(e_3) = 0$, we see that $e_1' \cdot e_2' = \alpha_2^3 e_3'$, $e_2' \cdot e_2' = \alpha_2^3 e_1'$, and all other products are zero.

For simplicity, assume without loss of generality that $\alpha_2 = 1$. Since $\lambda = D + \rho$ for some $D \in \mathcal{H}_3$, Lemma 5 tells us that, with respect to the basis e_1', e_2', e_3', the operator λ takes the form

$$\lambda = \begin{pmatrix} a_1 & b_1 & 1 \\ a_2 - 1 & b_2 & 0 \\ a_3 & b_3 & a_1 + b_2 \end{pmatrix}.$$

Applying formula (19) to e_3' and recalling that $e_3' \cdot e = 0$ since $\mathbf{e} \in I_3$, we deduce that $a_2 = 1$ and $b_2 = a_3 = 0$. Then, substituting $x = y = e_2'$ into equation (17), we get $a_1 = b_1 = 0$. Thus, the form of λ reduces to

$$\lambda = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & b_3 & 0 \end{pmatrix}.$$

Now, by setting $\mathbf{e} = ae_1 + be_2 + ce_3$ and applying (19) to e_1, we get that $b_3 = -b$. By using (8), we deduce that the nonzero left-symmetric products are

$$\begin{aligned} &e_1' \cdot e_2' = e_3', \quad e_2' \cdot e_2' = e_1', \\ &e_1' \cdot e_4' = -e_2', \quad e_4' \cdot e_2' = -be_3' \\ &e_3' \cdot e_4' = e_4' \cdot e_3' = e_1', \quad e_4' \cdot e_4' = \mathbf{e}. \end{aligned}$$

This implies, in particular, that $\dim [\mathcal{O}_4, \mathcal{O}_4] = \dim [A_4, A_4] = 2$, a contradiction. It follows that $\alpha_2 = 0$, as desired.

We now return to (21). Since $\alpha_2 = 0$, we have

$$\rho = \begin{pmatrix} \alpha_1 & \beta_1 & 0 \\ 0 & \beta_2 & 0 \\ \alpha_3 & \beta_3 & m\beta_2 - (m-1)\,\alpha_1 \end{pmatrix},$$

and since $\lambda = D + \rho$ for some $D \in \mathcal{H}_3$ then, in view of Lemma 5, the operator λ takes the form

$$\lambda = \begin{pmatrix} \alpha_1 + a_1 & \beta_1 + b_1 & 0 \\ a_2 & \beta_2 + b_2 & 0 \\ \alpha_3 + a_3 & \beta_3 + b_3 & a_1 + b_2 + m\beta_2 - (m-1)\,\alpha_1 \end{pmatrix}.$$

Once again, by applying (19) to e_3 and recalling that $e_3 \cdot e = 0$ since $\mathbf{e} \in I_3$, we deduce that $(m\beta_2 - (m-1)\,\alpha_1)^2 = 0$, thereby showing that $\rho\,(e_3) = 0$. Now, in view of (20) we get $e_3 \in T\,(A_4)$, and since $Z\,(\mathcal{O}_4) = \mathbb{R}e_3$ we deduce that $C\,(A_4) = T\,(A_4) \cap Z\,(\mathcal{O}_4) \neq 0$, as desired. This completes the proof of the lemma. □

5. Classification

We know from Section 4 that A_4 has a proper two-sided ideal I which is isomorphic to either the trivial one-dimensional real left-symmetric algebra $I_0 = \{e_0 : e_0 \cdot e_0 = 0\}$ or a 3 -dimensional left-symmetric algebra I_3 (as described in Proposition 3) whose associated Lie algebra is the Heisenberg algebra $\mathcal{H}_3$. In the case where $I \cong I_3$, we know by Lemma 6 that $C\,(A_4) \neq \{0\}$. Since in our situation $\dim Z\,(\mathcal{O}_4) = 1$, it follows that $C\,(A_4) \cong I_0$, so that we have a central short exact sequence of left-symmetric algebras of the form

$$0 \to I_0 \to A_4 \to I_3 \to 0. \tag{22}$$

In general, one has that the center of a left-symmetric algebra is a part of the center of the associated Lie algebra, and therefore the following lemma is proved.

Lemma 7. *The Lie algebra associated to I_3 is isomorphic to the Lie algebra $\mathcal{E}\,(2)$ of the group of Euclidean motions of the plane.*

Recall that $\mathcal{E}\,(2)$ is solvable non-nilpotent and has a basis $\{e_1, e_2, e_3\}$ which satisfies $[e_1, e_2] = e_3$ and $[e_1, e_3] = -e_2$.

In the case where $I \cong I_0$, we know by Lemma 3 that the associated Lie algebra is $\mathcal{I} \cong \mathbb{R}$. Since, by Lemma 4, $\mathcal{O}_4$ has only two proper ideals which are $Z\,(\mathcal{O}_4) \cong \mathbb{R}$ and

$[\mathcal{O}_4, \mathcal{O}_4] \cong \mathcal{H}_3$, it follows that $\mathcal{I} \cong \mathbb{R}$ coincides with the center $Z(\mathcal{O}_4)$. We deduce from this that, if $\mathcal{J}$ denotes the Lie algebra of the left-symmetric algebra J in the short exact sequence (15), then $\mathcal{J}$ is isomorphic to $\mathcal{E}(2)$. Therefore, we have a short sequence of left-symmetric algebras which looks like (22), except that it would not necessarily be central. But, as we will see a little later, this is necessarily a central extension (i.e., $I \cong C(A_4) \cong I_0$).

To summarize, each complete left-symmetric structure on $\mathcal{O}_4$ may be obtained by an extension of a complete 3-dimensional left-symmetric algebra A_3 whose associated Lie algebra is $\mathcal{E}(2)$ by I_0. Next, we shall determine all the complete left-symmetric structures on $\mathcal{E}(2)$. These are described by the following lemma that we state without proof (see [10], Theorem 4.1).

Lemma 8. *Up to left-symmetric isomorphism, any complete left-symmetric structure on $\mathcal{E}(2)$ is isomorphic to the following one which is given in a basis $\{e_1, e_2, e_3\}$ of $\mathcal{E}(2)$ by the relations $e_1 \cdot e_2 = e_3$, $e_1 \cdot e_3 = -e_2$, $e_2 \cdot e_2 = e_3 \cdot e_3 = \varepsilon e_1$.*

There are exactly two non-isomorphic conjugacy classes according to whether $\varepsilon = 0$ or $\varepsilon \neq 0$.

From now on, A_3 will denote the vector space $\mathcal{E}(2)$ endowed with one of the complete left-symmetric structures described in Lemma 8. The extended Lie bracket on $\mathcal{E}(2) \oplus \mathbb{R}$ is given by

$$[(x,a),(y,b)] = ([x,y], \omega(x,y)), \tag{23}$$

with $\omega \in Z^2(\mathcal{E}(2), \mathbb{R})$. The extended left-symmetric product on $A_3 \oplus I_0$ is given by

$$(x, ae_0) \cdot (y, be_0) = (x \cdot y, b\lambda_x(e_0) + a\rho_y(e_0) + g(x,y)), \tag{24}$$

with λ, $\rho : A_3 \to End(I_0)$ and $g \in Z^2_{\lambda,\rho}(A_3, I_0)$.

As we have noticed in Section 3, I_0 is an A_3-bimodule, or equivalently, the conditions in Theorem 1 simplify to the following conditions:

(i) $\lambda_{[x,y]} = 0$,

(ii) $\rho_{x \cdot y} = \rho_y \circ \rho_x$,

(iii) $g(x, y \cdot z) - g(y, x \cdot z) + \lambda_x(g(y,z)) - \lambda_y(g(x,z)) - g([x,y],z) - \rho_z(g(x,y) - g(y,x)) = 0$.

By using (23) and (24), we deduce from $[(x,a),(y,b)] = (x, ae_0) \cdot (y, be_0) - (y, be_0) \cdot (x, ae_0)$ that

$$\omega(x,y) = g(x,y) - g(y,x) \text{ and } \lambda = \rho. \tag{25}$$

By applying identity (ii) above to $e_i \cdot e_i$, $1 \leq i \leq 3$, we deduce that $\rho = 0$, and a fortiori $\lambda = 0$. In other words, the extension A_4 is always central (i.e., $I \cong C(A_4)$ even in the case where $\mathcal{I} \cong \mathbb{R}$). In fact, we have

Claim 1. *The extension $0 \to I_0 \to A_4 \to A_3 \to 0$ is exact.*

Proof. We recall from Subsection 3.1 that the extension given by the short sequence (22) is exact, i.e., $i(I_0) = C(A_4)$, if and only if $I_{[g]} = 0$, where

$$I_{[g]} = \{x \in A_3 : x \cdot y = y \cdot x = 0 \text{ and } g(x,y) = g(y,x) = 0, \text{ for all } y \in A_3\}.$$

To show that $I_{[g]} = 0$, let x be an arbitrary element in $I_{[g]}$, and put $x = ae_1 + be_2 + ce_3 \in I_{[g]}$. Now, by computing all the products $x \cdot e_i = e_i \cdot x = 0$, $1 \leq i \leq 3$, we easily deduce that $x = 0$. □

Our aim is to classify complete left-symmetric structures on $\mathcal{O}_4$, up to left-symmetric isomorphisms. By Proposition 1, the classification of exact central extensions of A_3 by I_0 is nothing but the orbit space of $H^2_{ex}(A_3, I_0)$ under the natural action of $G = Aut(I_0) \times Aut(A_3)$. Accordingly, we must compute $H^2_{ex}(A_3, I_0)$. Since I_0 is a trivial A_3-bimodule, we see first from (9) and (10) that the coboundary operator δ simplifies as follows:

$$\delta_1 h(x,y) = -h(x \cdot y), \quad \delta_2 g(x,y,z) = g(x, y \cdot z) - g(y, x \cdot z) - g([x,y], z),$$

where $h \in L^1(A_3, I_0)$ and $g \in L^2(A_3, I_0)$.

In view of Lemma 8, there are two cases to be considered.

Case 1. $A_3 = \langle e_1, e_2, e_3 : e_1 \cdot e_2 = e_3,\ e_1 \cdot e_3 = -e_2 \rangle$.

In this case, using the first formula above for δ_1, we get

$$\delta_1 h = \begin{pmatrix} 0 & h_{12} & h_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $h_{12} = -h(e_3)$ and $h_{13} = h(e_2)$. Similarly, using the second formula above for δ_2, we verify easily that if g is a cocycle (i.e. $\delta_2 g = 0$) and $g_{ij} = g(e_i, e_j)$, then

$$g = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ 0 & g_{22} & g_{23} \\ 0 & -g_{23} & g_{22} \end{pmatrix},$$

that is, $g_{21} = g_{31} = 0$, $g_{32} = -g_{23}$, and $g_{33} = g_{22}$. We deduce that, in the basis above, the class $[g] \in H^2(A_3, \mathbb{R})$ of a cocycle g takes the simplified form

$$g = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & \gamma \\ 0 & -\gamma & \beta \end{pmatrix}.$$

We can now determine the extended left-symmetric structure on A_4. By setting $\widetilde{e}_i = (e_i, 0)$, $1 \leq i \leq 3$, and $\widetilde{e}_4 = (0,1)$, and using formula (24) which (since $\lambda = \rho = 0$) reduces to

$$(x, ae_0) \cdot (y, be_0) = (x \cdot y, g(x,y)), \tag{26}$$

we obtain

$$\begin{aligned} &\widetilde{e}_1 \cdot \widetilde{e}_1 = \alpha \widetilde{e}_4,\ \widetilde{e}_2 \cdot \widetilde{e}_2 = \widetilde{e}_3 \cdot \widetilde{e}_3 = \beta \widetilde{e}_4 \\ &\widetilde{e}_1 \cdot \widetilde{e}_2 = \widetilde{e}_3, \quad \widetilde{e}_1 \cdot \widetilde{e}_3 = -\widetilde{e}_2, \\ &\widetilde{e}_2 \cdot \widetilde{e}_3 = \gamma \widetilde{e}_4, \quad \widetilde{e}_3 \cdot \widetilde{e}_2 = -\gamma \widetilde{e}_4, \end{aligned} \tag{27}$$

and all the other products are zero. We observe here that we should have $\gamma \neq 0$, given that the underlying Lie algebra is $\mathcal{O}_4$. We denote by $A_4(\alpha,\beta,\gamma)$ the Lie algebra $\mathcal{O}_4$ endowed with the above complete left-symmetric product.

Let now $A_4(\alpha,\beta,\gamma)$ and $A_4(\alpha',\beta',\gamma')$ be two arbitrary left-symmetric structures on $\mathcal{O}_4$ given as above, and let $[g]$ and $[g']$ be the corresponding classes in $H^2_{ex}(A_3, I_0)$. By Proposition 1, we know that $A_4(\alpha,\beta,\gamma)$ is isomorphic to $A_4(\alpha',\beta',\gamma')$ if and only if the exists $(\mu,\eta) \in Aut(I_0) \times Aut(A_3)$ such that for all $x, y \in A_3$, we have

$$g'(x,y) = \mu(g(\eta(x),\eta(y))). \tag{28}$$

We shall first determine $Aut(I_0) \times Aut(A_3)$. We have $Aut(I_0) \cong \mathbb{R}^*$, and it is easy too to determine $Aut(A_3)$. Indeed, recall that the unique left-symmetric structure of A_3 is given by $e_1 \cdot e_2 = e_3,\ \ e_1 \cdot e_3 = -e_2$, and let $\eta \in Aut(A_3)$ be given in the basis $\{e_1, e_2, e_3\}$ by

$$\eta = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}.$$

From the identity $\eta(e_3) = \eta(e_1 \cdot e_2) = \eta(e_1) \cdot \eta(e_2)$, we get $c_1 = 0$, $c_2 = -a_1 b_3$, and $c_3 = a_1 b_2$. From the identity $-\eta(e_2) = \eta(e_1 \cdot e_3) = \eta(e_1) \cdot \eta(e_3)$ we get $b_1 = 0$, $b_2 = a_1 c_3$, and $b_3 = -a_1 c_2$. Since $\det \eta \neq 0$, we deduce that $a_1 = \pm 1$. It follows, by setting $\varepsilon = \pm 1$, that $b_3 = -\varepsilon c_2$ and $c_3 = \varepsilon b_2$. From the identity $\eta(e_1) \cdot \eta(e_1) = \eta(e_1 \cdot e_1) = 0$, we obtain $a_2 = a_3 = 0$. Therefore, η takes the form

$$\eta = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & b_2 & c_2 \\ 0 & -\varepsilon c_2 & \varepsilon b_2 \end{pmatrix}, \quad b_2^2 + c_2^2 \neq 0.$$

We now apply formula (28). For this we recall first that in the basis above the classes $[g]$ and $[g']$ corresponding to $A_4(\alpha,\beta,\gamma)$ and $A_4(\alpha',\beta',\gamma')$, respectively, have the forms

$$g = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & \gamma \\ 0 & -\gamma & \beta \end{pmatrix} \quad \text{and} \quad g' = \begin{pmatrix} \alpha' & 0 & 0 \\ 0 & \beta' & \gamma' \\ 0 & -\gamma' & \beta' \end{pmatrix},$$

respectively. From $g'(e_1, e_1) = \mu g(\eta(e_1), \eta(e_1))$, we get

$$\alpha' = \mu\alpha, \tag{29}$$

and from $g'(e_2, e_2) = \mu g(\eta(e_2), \eta(e_2))$, we get

$$\beta' = \mu\left(b_2^2 + c_2^2\right)\beta. \tag{30}$$

Similarly, from $g'(e_2, e_3) = \mu g(\eta(e_2), \eta(e_3))$ we get

$$\gamma' = \mu\varepsilon\left(b_2^2 + c_2^2\right)\gamma. \tag{31}$$

Recall here that $\mu \neq 0$, $\gamma \neq 0$, and $b_2^2 + c_2^2 \neq 0$.

Claim 2. *Each $A_4(\alpha,\beta,\gamma)$ is isomorphic to some $A_4(\alpha',\beta',1)$. Precisely, $A_4(\alpha,\beta,\gamma)$ is isomorphic to $A_4\left(\varepsilon\frac{\alpha}{\gamma}, \varepsilon\frac{\beta}{\gamma}, 1\right)$.*

Proof. By (29), (30), and (31), $A_4(\alpha,\beta,\gamma)$ is isomorphic to $A_4(\alpha',\beta',1)$ if and only if there exists $\mu\in\mathbb{R}^*$ and $b,c\in\mathbb{R}$, with $b^2+c^2\neq 0$, such that

$$\begin{aligned}\alpha' &= \mu\alpha, \beta' \\ &= \mu\left(b^2+c^2\right)\beta, 1 \\ &= \mu\varepsilon\left(b^2+c^2\right)\gamma.\end{aligned}$$

Now, by taking $b^2+c^2=1$ (for instance, $b=\cos\theta_0$ and $c=\sin\theta_0$ for some θ_0), the third equation yields $\mu=\frac{\varepsilon}{\gamma}$. Substituting the value of μ in the two first equations, we deduce that $\alpha'=\varepsilon\frac{\alpha}{\gamma}$ and $\beta'=\varepsilon\frac{\beta}{\gamma}$. Consequently, each $A_4(\alpha,\beta,\gamma)$ is isomorphic to $A_4\left(\varepsilon\frac{\alpha}{\gamma},\varepsilon\frac{\beta}{\gamma},1\right)$. □

Case 2. $A_3=\langle e_1,e_2,e_3 : e_1\cdot e_2=e_3,\ e_1\cdot e_3=-e_2,\ e_2\cdot e_2=e_3\cdot e_3=e_1\rangle$.
Similarly to the first case, we get

$$\delta_1 h=\begin{pmatrix}0 & h_{12} & h_{13}\\ 0 & h_{22} & 0\\ 0 & 0 & h_{22}\end{pmatrix}\quad\text{and}\quad g=\begin{pmatrix}0 & g_{12} & g_{13}\\ 0 & g_{22} & g_{23}\\ 0 & -g_{23} & g_{22}\end{pmatrix},$$

where $h_{12}=-h(e_3)$, $h_{13}=h(e_2)$, $h_{22}=-h(e_1)$, and $g_{ij}=g(e_i,e_j)$. It follows that in this case the class $[g]\in H^2(A_3,\mathbb{R})$ of a cocycle g takes the reduced form

$$g=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & \gamma\\ 0 & -\gamma & 0\end{pmatrix},\quad \gamma\neq 0.$$

By setting $\widetilde{e}_i=(e_i,0)$, $1\leq i\leq 3$, and $\widetilde{e}_4=(0,1)$, and using formula (26) we find that the nonzero relations are

$$\begin{aligned}&\widetilde{e}_1\cdot\widetilde{e}_2=\widetilde{e}_3,\ \widetilde{e}_1\cdot\widetilde{e}_3=-\widetilde{e}_2,\ \widetilde{e}_2\cdot\widetilde{e}_2=\widetilde{e}_3\cdot\widetilde{e}_3=\widetilde{e}_1\\ &\widetilde{e}_2\cdot\widetilde{e}_3=\gamma\widetilde{e}_4,\quad \widetilde{e}_3\cdot\widetilde{e}_2=-\gamma\widetilde{e}_4,\quad \gamma\neq 0.\end{aligned}\tag{32}$$

We can now state the main result of this paper.

Theorem 2. *Let A_4 be a complete non-simple real left-symmetric algebra whose associated Lie algebra is $\mathcal{O}(4)$. Then A_4 is isomorphic to one of the following left-symmetric algebras:*

(i) $A_4(s,t)$*: There exist real numbers s,t, and a basis $\{e_1,e_2,e_3,e_4\}$ of $\mathcal{O}(4)$ relative to which the nonzero left-symmetric relations are*

$$\begin{aligned}&e_1\cdot e_1=se_4,\quad e_2\cdot e_2=e_3\cdot e_3=te_4\\ &e_1\cdot e_2=e_3,\quad e_1\cdot e_3=-e_2,\\ &e_2\cdot e_3=\frac{1}{2}e_4,\quad e_3\cdot e_2=-\frac{1}{2}e_4.\end{aligned}$$

The conjugacy class of $A_4(s,t)$ is given as follows: $A_4(s',t')$ is isomorphic to $A_4(s,t)$ if and only if $(s',t')=(\alpha s,\pm t)$ for some $\alpha\in\mathbb{R}^*$.

(*ii*) B_4*: There is a basis* $\{e_1, e_2, e_3, e_4\}$ *of* $\mathcal{O}(4)$ *relative to which the nonzero left-symmetric relations are*

$$\begin{aligned}
&e_1 \cdot e_2 = e_3, \quad e_1 \cdot e_3 = -e_2, \ e_2 \cdot e_2 = e_3 \cdot e_3 = e_1 \\
&e_2 \cdot e_3 = \frac{1}{2}e_4, \quad e_3 \cdot e_2 = -\frac{1}{2}e_4.
\end{aligned}$$

Proof. According to the discussion above, there are two cases to be considered.

Case 1. $A_3 = \langle e_1, e_2, e_3 : e_1 \cdot e_2 = e_3, \ e_1 \cdot e_3 = -e_2 \rangle$.

In this case, Claim 2 asserts that A_4 is isomorphic to some $A_4(\alpha, \beta, 1)$; and according to equations (27), we know that there is a basis $\{\widetilde{e}_1, \widetilde{e}_2, \widetilde{e}_3, \widetilde{e}_4\}$ of $\mathcal{O}_4$ relative to which the nonzero relations for $A_4(\alpha, \beta, 1)$ are:

$$\begin{aligned}
&\widetilde{e}_1 \cdot \widetilde{e}_1 = \alpha \widetilde{e}_4, \ \widetilde{e}_2 \cdot \widetilde{e}_2 = \widetilde{e}_3 \cdot \widetilde{e}_3 = \beta \widetilde{e}_4 \\
&\widetilde{e}_1 \cdot \widetilde{e}_2 = \widetilde{e}_3, \quad \widetilde{e}_1 \cdot \widetilde{e}_3 = -\widetilde{e}_2, \\
&\widetilde{e}_2 \cdot \widetilde{e}_3 = \widetilde{e}_4, \quad \widetilde{e}_3 \cdot \widetilde{e}_2 = -\widetilde{e}_4.
\end{aligned}$$

Now, it is clear that by setting $s = \frac{\alpha}{2}$, $t = \frac{\beta}{2}$, $e_i = \widetilde{e}_i$ for $1 \leq i \leq 3$, and $e_4 = 2\widetilde{e}_4$, we get the desired two-parameter family $A_4(s, t)$. On the other hand, we see from (29), (30), and (31) that $A_4(s', t')$ is isomorphic to $A_4(s, t)$ if and only if there exists $\alpha \in \mathbb{R}^*$ and $b, c \in \mathbb{R}$, with $b^2 + c^2 \neq 0$, such that

$$\begin{aligned}
s' &= \alpha s, \\
t' &= \alpha \left(b^2 + c^2\right) t, \\
1 &= \alpha \varepsilon \left(b^2 + c^2\right).
\end{aligned}$$

From the third equation, we get $b^2 + c^2 = \frac{\varepsilon}{\alpha}$; and by substituting the latter into the second equation, we get $t' = \varepsilon t$. In other words, we have shown that $A_4(s', t')$ and $A_4(s, t)$ are isomorphic if and only if there exists $\alpha \in \mathbb{R}^*$ such that $s' = \alpha s$ and $t' = \pm t$.

Case 2. $A_3 = \langle e_1, e_2, e_3 : e_1 \cdot e_2 = e_3, \ e_1 \cdot e_3 = -e_2, \ e_2 \cdot e_2 = e_3 \cdot e_3 = e_1 \rangle$.

In this case, by (32), there is a basis $\{\widetilde{e}_1, \widetilde{e}_2, \widetilde{e}_3, \widetilde{e}_4\}$ of $\mathcal{O}_4$ relative to which the nonzero relations in A_4 are:

$$\begin{aligned}
&\widetilde{e}_1 \cdot \widetilde{e}_2 = \widetilde{e}_3, \ \widetilde{e}_1 \cdot \widetilde{e}_3 = -\widetilde{e}_2, \ \widetilde{e}_2 \cdot \widetilde{e}_2 = \widetilde{e}_3 \cdot \widetilde{e}_3 = \widetilde{e}_1 \\
&\widetilde{e}_2 \cdot \widetilde{e}_3 = \gamma \widetilde{e}_4, \quad \widetilde{e}_3 \cdot \widetilde{e}_2 = -\gamma \widetilde{e}_4, \gamma \neq 0.
\end{aligned}$$

By setting $e_i = \widetilde{e}_i$ for $1 \leq i \leq 3$, and $e_4 = 2\gamma \widetilde{e}_4$, we see that A_4 is isomorphic to B_4. This finishes the proof of the main theorem. □

Remark 5. *Recall that a left-symmetric algebra* A *is called* Novikov *if it satisfies the condition* $(x \cdot y) \cdot z = (x \cdot z) \cdot y$, *for all* $x, y, z \in A$.

Novikov left-symmetric algebras were introduced in [2] (see also [24] for some important results concerning this). We note here that $A_4(s, 0)$ *is Novikov and that* B_4 *is not.*

We can explicitly compute the exponential map $\exp : \mathcal{O}_4 \to O_4$ of the oscillator group in the parametrization given in Section 4. Details of the argument are left to the reader (see [11]). It is given by

$$\exp(v,z,t) = \begin{cases} \left(v + \frac{|z|^2}{4t}\left(1 - \frac{\sin 2t}{2t}\right), z\frac{\sin t}{t}, t\right), & t \neq 0 \\ (v,z,0), & t = 0 \end{cases}$$

On the other hand, we note that the mapping $X \mapsto (L_X, X)$ is a Lie algebra representation of $\mathcal{O}_4$ in $\mathfrak{aff}(\mathbb{R}^4) = End(\mathbb{R}^4) \oplus \mathbb{R}^4$. By using the exponential map of the affine group $Aff(\mathbb{R}^4) = GL(\mathbb{R}^4) \ltimes \mathbb{R}^4$, Theorem 2 can now be stated, in terms of simply transitive actions of subgroups of $Aff(\mathbb{R}^4)$, as follows. To state it, define the continuous functions

$$f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}, \quad g(x) = \begin{cases} \frac{1-\cos x}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases},$$
$$h(x) = \begin{cases} \frac{x-\sin x}{x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}, \quad k(x) = \begin{cases} \frac{1-\cos x}{x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases},$$

and set

$$\Phi_t(x) = \left(\frac{y}{2} + tz\right) g(x) - \left(\frac{z}{2} - ty\right) f(x),$$
$$\Psi_t(x) = \left(\frac{y}{2} + tz\right) f(x) + \left(\frac{z}{2} - ty\right) g(x).$$

Theorem 3. *Suppose that the oscillator group O_4 acts simply transitively by affine transformations on $\mathbb{R}^4$. Then, as a subgroup of $Aff(\mathbb{R}^4) = GL(\mathbb{R}^4) \ltimes \mathbb{R}^4$, O_4 is conjugate to one of the following subgroups:*

(*i*)

$$G_4 = \left\{ \begin{array}{l} \begin{bmatrix} 1 & yf(x) + zg(x) & zf(x) - yg(x) & 0 \\ 0 & \cos x & -\sin x & 0 \\ 0 & \sin x & \cos x & 0 \\ 0 & \Phi_0(x) & \Psi_0(x) & 1 \end{bmatrix} \times \begin{bmatrix} x + (y^2+z^2)k(x) \\ yf(x) - zg(x) \\ zf(x) + yg(x) \\ w + \frac{(y^2+z^2)}{2}h(x) \end{bmatrix} \\ : x,y,z,w \in \mathbb{R} \end{array} \right\},$$

(*ii*)

$$G_4(s,t) = \left\{ \begin{array}{l} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos x & -\sin x & 0 \\ 0 & \sin x & \cos x & 0 \\ sx & \Phi_t(x) & \Psi_t(x) & 1 \end{bmatrix} \times \begin{bmatrix} x \\ yf(x) - zg(x) \\ zf(x) + yg(x) \\ w + \frac{s}{2}x^2 + (y^2+z^2)\left(\frac{h(x)}{2} + tk(x)\right) \end{bmatrix} \\ : x,y,z,w \in \mathbb{R} \end{array} \right\},$$

where $s,t \in \mathbb{R}$. The only pairs of conjugate subgroups in $Aff(\mathbb{R}^4)$ are $G_4(s,t)$ and $G_4(\alpha s, \pm t)$ where $\alpha \in \mathbb{R}^*$.

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