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Nonexistence of positive solutions for a system of nonlinear multi-point boundary value problems on time scales

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Abstract. We determine intervals for two eigenvalues for which there exists no positive solution of a system of nonlinear differential equations subject to multi-point boundary value problems on time scales.

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1. Introduction

The theory of dynamic equation on time scales (or measure chains) was initiated by Stefan Hilger in his PhD thesis in 1988 [13] (supervised by Bernd Aulbach) as a means of unifying the structure for the study of differential equations in the continuous case and the study of finite difference equations in the discrete case. In recent years, it has gained a considerable amount of interest and attracted the attention of many researchers. It is still a new area, and the research in this area is rapidly growing. The study of time scales [6] has led to several important applications, e.g., in the study of insect population models, heat transfer, neural networks, phytoremediation of metals, wound healing and epidemic models.

Multi-point boundary value problems (BVPs) for ordinary differential or difference equation arise in different areas of applied mathematics and physics such as the deflection of a curved beam having a constant or varying cross section, three layers beam, electromagnetic waves or gravity driven flow and so on. For example, the vibrations of a guy wire of a uniform cross-section and composed of N parts of different densities can be set up as a multi-point BVP [27] and many problems in the theory of elastic stability can also be handled as multi-point problems [31]. The study of multi-point BVPs for second order differential equations was introduced by II'in and Moiseev [22, 23]. The few papers that motivated this work are as follows Agarwal et al. [3], Anderson [4, 5], Benchohra et al. [7], Chyan [8], Huang [20], Kameswararao et al. [24] and Prasad et al. [30] on time scales.

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In this paper, we consider the system of nonlinear boundary value problems on time scales

$$
\begin{aligned}
u^{\Delta\Delta} + \lambda p(t) f(u(t), v(t)) = 0, \quad t \in [t_1, t_m], \\
v^{\Delta\Delta} + \mu q(t) g(u(t), v(t)) = 0, \quad t \in [t_1, t_m],
\end{aligned} \tag{1}
$$

satisfying the multi-point boundary conditions

$$
\begin{aligned}
u(t_1) = 0, \quad \alpha u(\sigma(t_m)) + \beta u^{\Delta}(\sigma(t_m)) = \sum_{k=2}^{m-1} u^{\Delta}(t_k), \\
v(t_1) = 0, \quad \alpha v(\sigma(t_m)) + \beta v^{\Delta}(\sigma(t_m)) = \sum_{l=2}^{m-1} v^{\Delta}(t_l),
\end{aligned} \tag{2}
$$

where $\mathbb{T}$ is the time scales with $t_1, \sigma^2(t_m) \in \mathbb{T},\ 0 \leq t_1 < t_2 < t_3 < .... < t_{m-1} < \sigma(t_m), \alpha > 0, \beta > m-2$ are real numbers and $m \geq 3$.
We assume the following conditions hold throughout the paper:

(A1) The functions $f, g : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous,

(A2) The functions $p, q : [t_1, \sigma(t_m)] \to \mathbb{R}^+$ are continuous and p, q do not vanish identically on any closed subinterval of $[t_1, \sigma(t_m)]$,

(A3) $\alpha > 0,\ \beta > m-2$ and $\alpha > \frac{\beta}{t_2 - t_1}$.

The rest of the paper is organized as follows. In Section 2, we construct the Green's function for the homogeneous problem corresponding to (1)-(2) and estimate bounds for the Green's function. In Section 3, we consider the conditions of the nonexistence of a positive solution. Finally, in Section 4, we give an example to illustrate our result.

2. Green's function and bounds

In this section, we construct the Green's function for the homogeneous problem corresponding to (1)-(2) and estimate bounds for the Green's function.

Let $G(t, s)$ be the Green's function of a homogeneous boundary value problem

$$
-y^{\Delta\Delta}(t) = 0,\ t \in [t_1, t_m] \tag{3}
$$

$$
y(t_1) = 0, \quad \alpha y(\sigma(t_m)) + \beta y^{\Delta}(\sigma(t_m)) = \sum_{k=2}^{m-1} y^{\Delta}(t_k),\ m \geq 3. \tag{4}
$$

Lemma 1. *Let $d = \alpha(\sigma(t_m) - t_1) + \beta - m + 2 \neq 0$. Then the Green's function $G(t, s)$*

for the homogeneous boundary value problem (3)-(4) *is given by*

$$G(t,s)=\begin{cases} \underset{t\in[t_1,t_2]}{G_1(t,s)} = \begin{cases} G_{1_1}(t,s), & t_1\le\sigma(s)\le t\le t_2<...<\sigma(t_m),\\ G_{1_2}(t,s), & t_1\le t\le s\le t_2<t_3<...<\sigma(t_m),\\ G_{1_3}(t,s), & t_1<t\le t_2\le s\le t_3<...<\sigma(t_m),\\ . \\ . \\ . \\ G_{1_{m-1}}(t,s), & t_1<t\le ...\le s\le t_{m-1}<\sigma(t_m),\\ G_{1_m}(t,s), & t_1<t\le ...\le t_{m-1}\le s\le\sigma(t_m), \end{cases} \\ \\ \underset{t\in[t_2,t_3]}{G_2(t,s)} = \begin{cases} G_{2_1}(t,s), & t_1\le\sigma(s)\le t_2\le t<...<\sigma(t_m),\\ G_{2_2}(t,s), & t_1<t_2\le\sigma(s)\le t\le t_3<...<\sigma(t_m),\\ G_{2_3}(t,s), & t_1<t_2\le t\le s\le t_3<...<\sigma(t_m),\\ . \\ . \\ . \\ G_{2_{m-1}}(t,s), & t_1<t_2<t\le ...\le s\le t_{m-1}<\sigma(t_m),\\ G_{2_m}(t,s), & t_1<t_2<t\le ...\le t_{m-1}\le s\le\sigma(t_m), \end{cases} \\ . \\ . \\ . \\ \underset{t\in[t_{m-1},\sigma(t_m)]}{G_{m-1}(t,s)} = \begin{cases} G_{m-1_1}(t,s), & t_1\le\sigma(s)\le ...\le t<\sigma(t_m),\\ G_{m-1_2}(t,s), & t_1<t_2\le\sigma(s)\le ...\le t<\sigma(t_m),\\ G_{m-1_3}(t,s), & t_1<t_2<t_3\le ...\le t<\sigma(t_m),\\ . \\ . \\ . \\ G_{m-1_{m-1}}(t,s), & t_1<...\le\sigma(s)\le t\le\sigma(t_m),\\ G_{m-1_m}(t,s), & t_1<...<t_{m-1}\le t\le s\le\sigma(t_m), \end{cases} \end{cases} \tag{5}$$

where

$$\begin{aligned} G_{i_j}(t,s) &= \frac{1}{d}\Big[(\alpha(\sigma(t_m)-t)+\beta-m+j+1)(\sigma(s)-t_1) \\ &\quad +(j-1)(t-\sigma(s))\Big],\ 1\le j\le i,\\ G_{i_{j+1}}(t,s) &= \frac{1}{d}(t-t_1)\Big[\alpha(\sigma(t_m)-\sigma(s))+\beta-m+j+1\Big],\\ &\quad i\le j\le m-1,\quad for\ all\quad i=1,2,...,m-1. \end{aligned}$$

Proof. It is easy to see that, if $h(t)\in C([t_1,\sigma(t_m)],\mathbb{R}^+)$, then the following boundary value problem

$$-y^{\Delta\Delta}(t)=h(t),\quad t\in[t_1,\sigma(t_m)],$$
$$y(t_1)=0,\quad\text{and}\quad \alpha y(\sigma(t_m))+\beta y^{\Delta}(\sigma(t_m))=\sum_{k=2}^{m-1}y^{\Delta}(t_k),\ m\ge 3,$$

has a unique solution

$$y(t) = \frac{1}{d}(t-t_1)\left[\int_{t_1}^{\sigma(t_m)}(\alpha(\sigma(t_m)-\sigma(s))+\beta)h(s)\Delta s - \sum_{k=2}^{m-1}\int_{t_1}^{t_k} h(s)\Delta s\right]$$
$$-\int_{t_1}^{t}(t-\sigma(s))h(s)\Delta s.$$

Rearranging the terms, it can be written as

$$y(t) = \frac{1}{d}(t-t_1)\left[\int_{t_1}^{\sigma(t_m)}(\alpha(\sigma(t_m)-\sigma(s))+\beta)h(s)\Delta s\right.$$
$$\left.-\sum_{j=1}^{m-2}(m-j-1)\int_{t_j}^{t_{j+1}} h(s)\Delta s\right] + \int_{t_1}^{t}(\sigma(s)-t)h(s)\Delta s.$$

Case (i) Let $t_j \le s \le \sigma(s) \le t_{j+1}$ for $j = 1, 2, ..., m-2$ and $\sigma(s) \le t$. Then we have

$$G(t,s) = \frac{1}{d}(t-t_1)\big[\alpha(\sigma(t_m)-\sigma(s))+\beta-(m-j-1)\big]+\sigma(s)-t$$
$$= \frac{1}{d}\big[(\alpha(\sigma(t_m)-t)+\beta-m+j+1)(\sigma(s)-t_1)+(j-1)(t-\sigma(s))\big].$$

Case (ii) Let $t_j \le s \le \sigma(s) \le t_{j+1}$ for $j = 1, 2, ..., m-2$ and $t \le s$. Then we have

$$G(t,s) = \frac{1}{d}(t-t_1)\big[\alpha(\sigma(t_m)-\sigma(s))+\beta-m+j+1\big].$$

Case (iii) Let $t_{m-1} \le s \le \sigma(s) \le \sigma(t_m)$ and $\sigma(s) \le t$. Then we have

$$G(t,s) = \frac{1}{d}(t-t_1)\big[\alpha(\sigma(t_m)-\sigma(s))+\beta\big]+\sigma(s)-t$$
$$= \frac{1}{d}\big[(\alpha(\sigma(t_m)-t)+\beta)(\sigma(s)-t_1)+(m-2)(t-\sigma(s))\big].$$

Case (iv) Let $t_{m-1} \le s \le \sigma(s) \le \sigma(t_m)$ and $t \le s$. Then we have

$$G(t,s) = \frac{1}{d}(t-t_1)\big[\alpha(\sigma(t_m)-\sigma(s))+\beta\big].$$

□

Lemma 2. *Assume that condition (A3) is satisfied. Then the Green's function $G(t,s)$ of (3)-(4) is positive for all $(t,s) \in (t_1, \sigma(t_m)) \times (t_1, t_m)$,*

Proof. By simple algebraic calculations, we can easily establish the positivity of the Green's function. □

Theorem 1. *Assume that condition (A3) is satisfied. Then the Green's function $G(t,s)$ in (5) satisfies the following inequality:*

$$g(t)G(\sigma(s),s) \le G(t,s) \le G(\sigma(s),s) \text{ for all } (t,s) \in [t_1,\sigma(t_m)] \times [t_1,t_m], \quad (6)$$

where

$$g(t) = \min\left\{\frac{\sigma(t_m) - t}{\sigma(t_m) - t_1}, \frac{t - t_1}{\sigma(t_m) - t_1}\right\}.$$

Proof. The Green's function $G(t,s)$ is given in (5). In each case, we prove the inequality as in (6).

Case (i) Let $s \in [t_1, t_m]$ and $\sigma(s) \leq t$. Then

$$\begin{aligned}\frac{G(t,s)}{G(\sigma(s),s)} &= \frac{(\alpha(\sigma(t_m) - t) + \beta - m + j + 1)(\sigma(s) - t_1) + (j-1)(t - \sigma(s))}{(\alpha(\sigma(t_m) - \sigma(s)) + \beta - m + j + 1)(\sigma(s) - t_1)} \\ &\leq \frac{(\alpha(\sigma(t_m) - t) + \beta - m + j + 1) + \alpha(t - \sigma(s))}{(\alpha(\sigma(t_m) - \sigma(s)) + \beta - m + j + 1)} = 1\end{aligned}$$

and also

$$\begin{aligned}\frac{G(t,s)}{G(\sigma(s),s)} &= \frac{(\alpha(\sigma(t_m) - t) + \beta - m + j + 1)(\sigma(s) - t_1) + (j-1)(t - \sigma(s))}{(\alpha(\sigma(t_m) - \sigma(s)) + \beta - m + j + 1)(\sigma(s) - t_1)} \\ &\geq \frac{\sigma(t_m) - t}{\sigma(t_m) - t_1}.\end{aligned}$$

Case (ii) Let $s \in [t_1, t_m]$ and $t \leq s$. Then

$$\frac{G(t,s)}{G(\sigma(s),s)} = \frac{t - t_1}{\sigma(s) - t_1} \leq 1$$

and also

$$\frac{G(t,s)}{G(\sigma(s),s)} = \frac{t - t_1}{\sigma(s) - t_1} \geq \frac{t - t_1}{\sigma(t_m) - t_1}.$$

From the above cases, we have

$$g(t)G(\sigma(s),s) \leq G(t,s) \leq G(\sigma(s),s) \text{ for all } (t,s) \in [t_1, \sigma(t_m)] \times [t_1, t_m],$$

where

$$g(t) = \min\left\{\frac{\sigma(t_m) - t}{\sigma(t_m) - t_1}, \frac{t - t_1}{\sigma(t_m) - t_1}\right\}.$$

□

Lemma 3. *Assume that condition* $(A3)$ *is satisfied and* $s \in [t_1, t_m]$. *Then the Green's function* $G(t,s)$ *in* (5) *satisfies*

$$\min_{t \in [t_{m-1}, \sigma(t_m)]} G(t,s) \geq \gamma G(\sigma(s), s),$$

where

$$\gamma = \frac{\beta - m + 2}{\alpha(\sigma(t_m) - t_1) + \beta - m + 2} < 1.$$

Proof. The Green's function $G(t,s)$ is given in (5). Then, by Theorem 1, we obtain

$$\min_{t\in[t_{m-1},\sigma(t_m)]} G(t,s) \geq \gamma G(\sigma(s),s),$$

where

$$\gamma = \frac{\beta - m + 2}{\alpha(\sigma(t_m) - t_1) + \beta - m + 2} < 1.$$

□

3. Main results

In this section, we give some sufficient conditions for the nonexistence of a positive solution to the BVP (1)-(2).

We introduce the following extreme limits

$$\begin{aligned}
f_0^s &= \lim_{u+v\to 0^+} \sup \frac{f(u,v)}{u+v}, & g_0^s &= \lim_{u+v\to 0^+} \sup \frac{g(u,v)}{u+v},\\
f_0^i &= \lim_{u+v\to 0^+} \inf \frac{f(u,v)}{u+v}, & g_0^i &= \lim_{u+v\to 0^+} \inf \frac{g(u,v)}{u+v},\\
f_\infty^s &= \lim_{u+v\to \infty} \sup \frac{f(u,v)}{u+v}, & g_\infty^s &= \lim_{u+v\to \infty} \sup \frac{g(u,v)}{u+v},\\
f_\infty^i &= \lim_{u+v\to \infty} \inf \frac{f(u,v)}{u+v}, & g_\infty^i &= \lim_{u+v\to \infty} \inf \frac{g(u,v)}{u+v}.
\end{aligned}$$

By using the Green's function $G(t,s)$ from Section 2, our problem (1)-(2) can be written equivalently as the following nonlinear system of integral equations

$$\begin{cases} u(t) = \lambda \int_{t_1}^{\sigma(t_m)} G(t,s)p(s)f(u(s),v(s))\Delta s, \ t_1 \leq t \leq \sigma(t_m), \\ \\ v(t) = \mu \int_{t_1}^{\sigma(t_m)} G(t,s)q(s)g(u(s),v(s))\Delta s, \ t_1 \leq t \leq \sigma(t_m). \end{cases}$$

We consider the Banach space $X = C[t_1, \sigma(t_m)]$ with supremum norm $\|\cdot\|$, and the Banach space $Y = X \times X$ with the norm $\| (u,v) \| = \| u \| + \| v \|$. We define the cone $\kappa \subset Y$ by

$$\kappa = \Big\{ (u,v) \in Y : u(t) \geq 0, \ v(t) \geq 0, \ \ \forall\, t \in [t_1, \sigma(t_m)] \ \text{ and } \min_{t\in\ [t_{m-1},\sigma(t_m)]} (u(t)+v(t)) \geq \gamma \| (u,v) \| \Big\}.$$

For $\lambda,\ \mu > 0$, we define the operators $Q_\lambda, Q_\mu : Y \to X$ as

$$Q_\lambda(u,v)(t) = \lambda \int_{t_1}^{\sigma(t_m)} G(t,s)p(s)f(u(s),v(s))\Delta s, \quad t_1 \leq t \leq \sigma(t_m),$$

$$Q_\mu(u,v)(t) = \mu \int_{t_1}^{\sigma(t_m)} G(t,s)q(s)g(u(s),v(s))\Delta s, \quad t_1 \leq t \leq \sigma(t_m),$$

and an operator $Q : Y \to Y$ as

$$Q(u,v) = \Big(Q_\lambda(u,v), Q_\mu(u,v)\Big), \quad (u,v) \in Y.$$

It is clear that the existence of a positive solution to the system (1)-(2) is equivalent to the existence of fixed points of the operator Q.

Lemma 4. *$Q : \kappa \to \kappa$ is completely continuous.*

Proof. By using standard arguments, we can easily show that, under assumptions $(A1) - (A3)$, the operator Q is completely continuous and we need only to prove $Q(\kappa) \subset \kappa$.

In fact, for any $(t,s) \in\ [t_{m-1}, \sigma(t_m)] \times [t_1, t_m]$, from Lemma 3 we have

$$\begin{aligned}
\min_{t\in[t_{m-1},\sigma(t_m)]} &\Big[Q_\lambda(u,v)(t) + Q_\mu(u,v)(t)\Big] \\
&= \min_{t\in[t_{m-1},\sigma(t_m)]} \Big[\lambda \int_{t_1}^{\sigma(t_m)} G(t,s)p(s)f(u(s),v(s))\Delta s \\
&\quad + \mu \int_{t_1}^{\sigma(t_m)} G(t,s)q(s)g(u(s),v(s))\Delta s\Big] \\
&\geq \lambda\gamma \int_{t_1}^{\sigma(t_m)} G(\sigma(s),s)p(s)f(u(s),v(s))\Delta s \\
&\quad + \mu\gamma \int_{t_1}^{\sigma(t_m)} G(\sigma(s),s)q(s)g(u(s),v(s))\Delta s \\
&= \gamma \parallel Q_\lambda(u,v) \parallel + \gamma \parallel Q_\mu(u,v) \parallel = \gamma \parallel Q(u,v) \parallel;
\end{aligned}$$

hence,

$$\min_{t\in[t_{m-1},\sigma(t_m)]} \Big[Q_\lambda(u,v)(t) + Q_\mu(u,v)(t)\Big] \geq \gamma \parallel Q(u,v) \parallel .$$

Therefore, $Q(\kappa) \subset \kappa$. Standard arguments involving the Arzela-Ascoli theorem show that Q is a completely continuous operator. $\square$

Theorem 2. *Assume that $(A1) - (A3)$ hold. If $f_0^s, f_\infty^s, g_0^s, g_\infty^s < \infty$, then there exist positive constants λ_0, μ_0 such that for every $\lambda \in (0, \lambda_0)$ and $\mu \in (0, \mu_0)$ the boundary value problem (1)-(2) has no positive solution.*

Proof. Since $f_0^s, f_\infty^s < \infty$, we deduce that there exist $M_1^{'}, M_1^{''}, r_1, r_1^{'} > 0, r_1 < r_1^{'}$ such that

$$\begin{aligned}
f(u,v) &\leq M_1^{'}(u+v),\ \forall\ \ u,v \geq 0, \quad u+v \in [0, r_1], \\
f(u,v) &\leq M_1^{''}(u+v), \forall\ \ u,v \geq 0, \quad u+v \in [r_1^{'}, \infty).
\end{aligned}$$

We consider

$$M_1 = \max\Big\{M_1^{'}, M_1^{''}, \max_{r_1 \leq u+v \leq r_1^{'}} \frac{f(u,v)}{u+v}\Big\} > 0.$$

Then we obtain

$$f(u,v) \leq M_1(u+v), \quad \forall\ \ u,v \geq 0.$$

Since $g_0^s, g_\infty^s < \infty$, we deduce that there exist $M_2^{'}, M_2^{''}, r_2, r_2^{'} > 0, r_2 < r_2^{'}$ such that

$$g(u,v) \leq M_2^{'}(u+v),\ \forall\ u,v \geq 0, \quad u+v \in [0, r_2],$$
$$g(u,v) \leq M_2^{''}(u+v), \forall\ u,v \geq 0, \quad u+v \in [r_2^{'}, \infty).$$

We consider

$$M_2 = \max\left\{M_2^{'}, M_2^{''}, \max_{r_2 \leq u+v \leq r_2^{'}} \frac{g(u,v)}{u+v}\right\} > 0.$$

Then we obtain

$$g(u,v) \leq M_2(u+v), \quad \forall\ u,v \geq 0.$$

We define $\lambda_0 = \frac{1}{2M_1 B}$ and $\mu_0 = \frac{1}{2M_2 D}$, where

$$B = \int_{t_1}^{\sigma(t_m)} G(\sigma(s), s) p(s) \Delta s \ \text{ and } \ D = \int_{t_1}^{\sigma(t_m)} G(\sigma(s), s) q(s) \Delta s.$$

We shall show that for every $\lambda \in (0, \lambda_0)$ and $\mu \in (0, \mu_0)$, the problem (1)-(2) has no positive solution.

Let $\lambda \in (0, \lambda_0)$ and $\mu \in (0, \mu_0)$. We suppose that (1)-(2) has a positive solution $(u(t), v(t)), t \in [t_1, \sigma(t_m)]$. Then we have

$$\begin{aligned}
u(t) = Q_\lambda(u,v)(t) &= \lambda \int_{t_1}^{\sigma(t_m)} G(t,s) p(s) f(u(s), v(s)) \Delta s \\
&\leq \lambda \int_{t_1}^{\sigma(t_m)} G(\sigma(s), s) p(s) f(u(s), v(s)) \Delta s \\
&\leq \lambda \int_{t_1}^{\sigma(t_m)} G(\sigma(s), s) p(s) M_1(u(s) + v(s)) \Delta s \\
&\leq \lambda M_1 \int_{t_1}^{\sigma(t_m)} G(\sigma(s), s) p(s) (\| u \| + \| v \|) \Delta s \\
&= \lambda M_1 B \| (u,v) \|, \quad \forall\ t \in [t_1, \sigma(t_m)].
\end{aligned}$$

Therefore, we conclude

$$\| u \| \leq \lambda M_1 B \| (u,v) \| < \lambda_0 M_1 B \| (u,v) \| = \frac{1}{2} \| (u,v) \| .$$

In a similar manner,

$$\begin{aligned}
v(t) = Q_\mu(u,v)(t) &= \mu \int_{t_1}^{\sigma(t_m)} G(t,s) q(s) g(u(s), v(s)) \Delta s \\
&\leq \mu \int_{t_1}^{\sigma(t_m)} G(\sigma(s), s) q(s) g(u(s), v(s)) \Delta s \\
&\leq \mu \int_{t_1}^{\sigma(t_m)} G(\sigma(s), s) q(s) M_2(u(s) + v(s)) \Delta s \\
&\leq \mu M_2 \int_{t_1}^{\sigma(t_m)} G(\sigma(s), s) q(s) (\| u \| + \| v \|) \Delta s \\
&= \mu M_2 D \| (u,v) \|, \quad \forall\ t \in [t_1, \sigma(t_m)].
\end{aligned}$$

Therefore, we conclude

$$\| v \| \leq \mu M_2 D \| (u,v) \| < \mu_0 M_2 D \| (u,v) \| = \frac{1}{2} \| (u,v) \| .$$

Hence, $\| (u,v) \| = \| u \| + \| v \| < \frac{1}{2} \| (u,v) \| + \frac{1}{2} \| (u,v) \| = \| (u,v) \|$, which is a contradiction. So, the boundary value problem (1)-(2) has no positive solution. $\square$

Theorem 3. *Assume that $(A1) - (A3)$ hold.*

(i) If $f_0^i, f_\infty^i > 0$, then there exists a positive constant $\tilde{\lambda_0}$ such that for every $\lambda > \tilde{\lambda_0}$ and $\mu > 0$, the boundary value problem (1)-(2) has no positive solution.

(ii) If $g_0^i, g_\infty^i > 0$, then there exists a positive constant $\tilde{\mu_0}$ such that for every $\mu > \tilde{\mu_0}$ and $\lambda > 0$, the boundary value problem (1)-(2) has no positive solution.

(iii) If $f_0^i, f_\infty^i, g_0^i, g_\infty^i > 0$, then there exist positive constants $\tilde{\tilde{\lambda_0}}$ and $\tilde{\tilde{\mu_0}}$ such that for every $\lambda > \tilde{\tilde{\lambda_0}}$ and $\mu > \tilde{\tilde{\mu_0}}$, the boundary value problem (1)-(2) has no positive solution.

Proof. (i) Since $f_0^i, f_\infty^i > 0$, we deduce that there exist $m_1^{'}, m_1^{''}, r_3, r_3^{'} > 0, r_3 < r_3^{'}$ such that

$$f(u,v) \geq m_1^{'}(u+v), \ \forall \ u,v \geq 0, \quad u+v \in [0,r_3],$$
$$f(u,v) \geq m_1^{''}(u+v), \forall \ u,v \geq 0, \quad u+v \in [r_3^{'},\infty).$$

We introduce

$$m_1 = \min\left\{m_1^{'}, m_1^{''}, \min_{r_3 \leq u+v \leq r_3^{'}} \frac{f(u,v)}{u+v}\right\} > 0.$$

Then, we obtain

$$f(u,v) \geq m_1(u+v), \ \forall \ u,v \geq 0.$$

We define $\tilde{\lambda_0} = \frac{1}{\gamma^2 m_1 A} > 0$, where $A = \int_{t_{m-1}}^{\sigma(t_m)} G(\sigma(s),s)p(s)\Delta s$. We shall show that for every $\lambda > \tilde{\lambda_0}$ and $\mu > 0$ the problem (1)-(2) has no positive solution.

Let $\lambda > \tilde{\lambda_0}$ and $\mu > 0$. We suppose that (1)-(2) has a positive solution $(u(t), v(t)), t \in [t_1, \sigma(t_m)]$. Then we obtain

$$\begin{aligned}
u(t) = Q_\lambda(u,v)(t) &= \lambda \int_{t_1}^{\sigma(t_m)} G(t,s)p(s)f(u(s),v(s))\Delta s \\
&\geq \lambda\gamma \int_{t_{m-1}}^{\sigma(t_m)} G(\sigma(s),s)p(s)f(u(s),v(s))\Delta s \\
&\geq \lambda\gamma \int_{t_{m-1}}^{\sigma(t_m)} G(\sigma(s),s)p(s)m_1(u(s)+v(s))\Delta s \\
&\geq \lambda\gamma^2 m_1 \int_{t_{m-1}}^{\sigma(t_m)} G(\sigma(s),s)p(s) \| (u,v) \| \Delta s \\
&= \lambda\gamma^2 m_1 A \| (u,v) \| .
\end{aligned}$$

Therefore, we deduce

$$\| u \| \geq u(t) \geq \lambda\gamma^2 m_1 A \| (u,v) \| > \tilde{\lambda_0}\gamma^2 m_1 A \| (u,v) \| = \| (u,v) \| .$$

and so $\| (u,v) \| = \| u \| + \| v \| \geq \| u \| > \| (u,v) \|$, which is a contradiction. Therefore, the boundary value problem (1)-(2) has no positive solution.

(ii) Since $g_0^i, g_\infty^i > 0$, we deduce that there exist $m_2^{'}, m_2^{''}, r_4, r_4^{'} > 0, r_4 < r_4^{'}$ such that

$$g(u,v) \geq m_2^{'}(u+v), \ \forall\ u,v \geq 0, \quad u+v \in [0, r_4],$$
$$g(u,v) \geq m_2^{''}(u+v), \ \forall\ u,v \geq 0, \quad u+v \in [r_4^{'}, \infty).$$

We introduce

$$m_2 = \min\Big\{m_2^{'}, m_2^{''}, \min_{r_4 \leq u+v \leq r_4^{'}} \frac{g(u,v)}{u+v}\Big\} > 0.$$

Then we obtain

$$g(u,v) \geq m_2(u+v), \quad \forall \ \ u,v \geq 0.$$

We define $\tilde{\mu_0} = \frac{1}{\gamma^2 m_2 C} > 0$, where $C = \int_{t_{m-1}}^{\sigma(t_m)} G(\sigma(s),s)q(s)\Delta s$. We shall show that for every $\mu > \tilde{\mu_0}$ and $\lambda > 0$ the problem (1)-(2) has no positive solution.

Let $\mu > \tilde{\mu_0}$ and $\lambda > 0$. We suppose that (1)-(2) has a positive solution $(u(t), v(t)), t \in [t_1, \sigma(t_m)]$. Then we obtain

$$\begin{aligned}
v(t) = Q_\mu(u,v)(t) &= \mu \int_{t_1}^{\sigma(t_m)} G(t,s)q(s)g(u(s),v(s))\Delta s \\
&\geq \mu\gamma \int_{t_{m-1}}^{\sigma(t_m)} G(\sigma(s),s)q(s)g(u(s),v(s))\Delta s \\
&\geq \mu\gamma \int_{t_{m-1}}^{\sigma(t_m)} G(\sigma(s),s)q(s)m_2(u(s)+v(s))\Delta s \\
&\geq \mu\gamma^2 m_2 \int_{t_{m-1}}^{\sigma(t_m)} G(\sigma(s),s)q(s) \| (u,v) \| \Delta s \\
&= \lambda\gamma^2 m_2 C \| (u,v) \| .
\end{aligned}$$

Therefore, we deduce

$$\| v \| \geq v(t) \geq \mu\gamma^2 m_2 C \| (u,v) \| > \tilde{\mu_0}\gamma^2 m_2 C \| (u,v) \| = \| (u,v) \| .$$

and so $\| (u,v) \| = \| u \| + \| v \| \geq \| v \| > \| (u,v) \|$, which is a contradiction. Therefore, the boundary value problem (1)-(2) has no positive solution.

(iii) Because $f_0^i, f_\infty^i, g_0^i, g_\infty^i > 0$, we deduce as above that there exist $m_1, m_2 > 0$ such that

$$f(u,v) \geq m_1(u+v), \quad g(u,v) \geq m_2(u+v), \forall\ u,v \geq 0.$$

We define

$$\tilde{\tilde{\lambda}}_0 = \frac{1}{2\gamma^2 m_1 A}\Big(= \frac{\tilde{\lambda_0}}{2}\Big) \ \text{ and } \ \tilde{\tilde{\mu}}_0 = \frac{1}{2\gamma^2 m_2 C}\Big(= \frac{\tilde{\mu_0}}{2}\Big).$$

Then for every $\lambda > \tilde{\tilde{\lambda}}_0$ and $\mu > \tilde{\tilde{\mu}}_0$, the problem (1)-(2) has no positive solution.

Indeed, let $\lambda > \tilde{\tilde{\lambda}}_0$ and $\mu > \tilde{\tilde{\mu}}_0$. We suppose that (1)-(2) has a positive solution $(u(t), v(t)), t \in [t_1, \sigma(t_m)]$. Then in a similar manner as above, we deduce

$$\| u \| \geq \lambda\gamma^2 m_1 A \| (u,v) \|, \quad \| v \| \geq \mu\gamma^2 m_2 C \| (u,v) \|,$$

and so

$$\begin{aligned} \| (u,v) \| &= \| u \| + \| v \| \\ &\geq \lambda\gamma^2 m_1 A \| (u,v) \| + \mu\gamma^2 m_2 C \| (u,v) \| \\ &> \tilde{\tilde{\lambda}}_0 \gamma^2 m_1 A \| (u,v) \| + \tilde{\tilde{\mu}}_0 \gamma^2 m_2 C \| (u,v) \| \\ &= \frac{1}{2} \| (u,v) \| + \frac{1}{2} \| (u,v) \| = \| (u,v) \| \end{aligned}$$

which is a contradiction. Therefore, the boundary value problem (1)-(2) has no positive solution. □

3.1. Example

In this section, we give an example to illustrate our result. Let

$$\mathbb{T} = \left\{ \left(\frac{1}{2}\right)^n : n \in \mathbb{N}_0 \right\} \cup [1,2],$$

where $\mathbb{N}_0$ denotes the set of all non-negative integers. For the sake of simplicity, we take $m = 4,\ t_1 = \frac{1}{2},\ t_2 = 1,\ t_3 = \frac{3}{2},\ t_4 = 2,\ \alpha = 8,\ \beta = 3,\ p(t) = q(t)$ and $f(t) = g(t)$.

Consider the system of dynamic equation on time scales,

$$\begin{aligned} u^{\Delta\Delta} + \lambda \frac{1}{10} t \frac{k(u+v) + e^{2(u+v)}}{c + e^{(u+v)} + e^{2(u+v)}} &= 0, \quad t \in \left[\frac{1}{2}, 2\right], \\ v^{\Delta\Delta} + \mu \frac{1}{10} t \frac{k(u+v) + e^{2(u+v)}}{c + e^{(u+v)} + e^{2(u+v)}} &= 0, \quad t \in \left[\frac{1}{2}, 2\right] \end{aligned} \tag{7}$$

with the boundary conditions

$$\begin{aligned} u\left(\frac{1}{2}\right) &= 0, \quad 8u(\sigma(2)) + 3u^{\Delta}(\sigma(2)) = u^{\Delta}(1) + u^{\Delta}\left(\frac{3}{2}\right), \\ v\left(\frac{1}{2}\right) &= 0, \quad 8v(\sigma(2)) + 3v^{\Delta}(\sigma(2)) = v^{\Delta}(1) + v^{\Delta}\left(\frac{3}{2}\right). \end{aligned} \tag{8}$$

Here $p(t) = q(t) = \frac{1}{10}t,\ k = 100,\ c = 500,$

$$f(u,v) = \frac{k(u+v) + e^{2(u+v)}}{c + e^{(u+v)} + e^{2(u+v)}}, \quad g(u,v) = \frac{k(u+v) + e^{2(u+v)}}{c + e^{(u+v)} + e^{2(u+v)}}.$$

By simple calculation, we found that $\gamma = \frac{1}{7}, f_0 = g_0 = \frac{k}{c+2} = \frac{100}{502},\quad f_\infty = g_\infty = k = 100,,\quad M_1 = 140,\quad M_2 = 262,\quad B = 0.0421,\ D = 0.0224,\ \frac{1}{2M_1 B} = 0.0848,$ and $\frac{1}{2M_2 D} = 0.0851$. By Theorem 2, we deduce that problem (7)-(8) has no positive solutions for $0 < \lambda < 0.0848$ and $0 < \mu < 0.0851$.

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