MATHEMATICAL COMMUNICATIONS
Math. Commun. **19**(2014), 453–468

Method of asymptotic partial domain decomposition for non-steady problems: heat equation on a thin structure*

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Received October 18, 2013; accepted March 8, 2014

Abstract. The non-steady heat equation is considered in thin structures. The asymptotic expansion of the solution is constructed.The error estimates for high order asymptotic approximations are proved. The method of asymptotic partial domain decomposition is justified for the non-steady heat equation.

AMS subject classifications: 35B25, 35B40, 35B27

Key words: heat equation, thin structures, asymptotics, partial domain decomposition

1. Introduction

The method of asymptotic partial decomposition for thin structures was proposed in [11], and then developed in [12]. Thin rod structures are connected finite unions of thin finite cylinders (in the $2D$ case respectively thin rectangles), where the ratio of the diameter and the height of cylinders is the small parameter ε. Each such structure may be schematically represented by its graph: letting the thickness of cylinders to zero we find out that cylinders degenerate to segments. Although the method is developed for the steady problems, there are only few examples of its application to non-steady equations (see [13]). In the present paper, the heat equation set on the thin structure with the Neumann boundary condition at the lateral boundary is considered. An asymptotic expansion of the solution to the problem is constructed. It has a regular part, expansion in powers of ε with coefficients depending on the time variable and the longitudinal space variable only, and the boundary layer correctors depending on the dilated space variables x/ε and the time and decaying exponentially with respect to space variables, so that their values at some small distance from the bases of the cylinders become of order of ε^J for any J. This property of asymptotic expansion allows us to "cut " the cylinders at the distance of order $\varepsilon|ln(\varepsilon)|$ from the bases of cylinders, to reduce dimension in the truncated middle parts of cylinders and to set at the truncated sections some special asymptotically justified interface conditions between the 1D and multi-dimensional parts

*The present work is partially supported by the Research Federative Structures MODMAD FED 4169 and FR CNRS 3490, by the French-German grant PROCOPE EGIDE 28481WB "Homogenization based optimization for elasticity on the network of beams", and by LABEX MILYON (ANR-10-LABX-0070) of University of Lyon, within the program "Investissements d'Avenir" (ANR-11-IDEX-0007) operated by the French National Research Agency (ANR).

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(see [11]). Note that in the non-steady case these conditions are the same as in the steady case [11].

Notice that earlier the dimension reduction of parabolic problems set in thin structures was considered in [3] (Chapter 8) and [8]. Another method which can be applied to the construction of an asymptotic expansion of the solution is the method of matching (see [4, 5, 6, 9]); however, here we use the method developed in [3] and [12].

2. Graphs

Let $O_1, O_2, \ldots, O_N$ be N different points in $\mathbb{R}^n, n = 2, 3$, and $e_1, e_2, \ldots, e_M$ M closed segments each connecting two of these points (i.e., each $e_j = \overline{O_{i_j} O_{k_j}}$, where $i_j, k_j \in \{1, \ldots, N\}, i_j \neq k_j$). All points O_i are supposed to be the ends of some segments e_j. The segments e_j are called edges of the graph. A point O_i is called a node if it is the common end of at least two edges and O_i is called a vertex if it is the end of only one edge. Any two edges e_j and e_i can intersect only at the common node. The set of vertices is supposed to be non-empty.

By $\mathcal{B} = \bigcup_{j=1}^{M} e_j$ denote the union of edges and assume that $\mathcal{B}$ is a connected set. The graph $\mathcal{G}$ is defined as the collection of nodes, vertices and edges.

The union of all edges having the same end point in O_l is called the bundle $\mathcal{B}^{(l)}$.

Let e be some edge, $e = \overline{O_i O_j}$. Consider two Cartesian coordinate systems in $\mathbb{R}^n$. The first one has the origin in O_i and the axis $O_i x_1^{(e)}$ has the direction of the ray $[O_i O_j)$; the second one has the origin in O_j and the opposite direction, i.e., $O_i \tilde{x}_1^{(e)}$ is directed over the ray $[O_j O_i)$.

Further, in various situations we will choose one or another coordinate system denoting the local variable in both cases as x^e and pointing out which end is taken as the origin of the coordinate system.

3. Rod structures

With every edge e_j we associate a bounded domain $\sigma^j \subset \mathbb{R}^{n-1}$ having Lipschitz boundary $\partial\sigma^j, j = 1, \ldots, M$. For every edge $e_j = e$ and associated $\sigma^j = \sigma^{(e)}$ by $B_\varepsilon^{(e)}$ we denote the cylinder

$$B_\varepsilon^{(e)} = \{x^{(e)} \in \mathbb{R}^n : x_1^{(e)} \in (0, |e|), \frac{x^{(e)'}}{\varepsilon} \in \sigma^{(e)}\},$$

where $x^{(e)'} = (x_2^{(e)}, \ldots, x_n^{(e)})$, $|e|$ is the length of the edge e and $\varepsilon > 0$ is a small parameter. Notice that the edges e_j and Cartesian coordinates of nodes and vertices O_j, as well as domains σ_j, do not depend on ε.

Let $O_1, \ldots, O_{N_1}$ be nodes and $O_{N_1+1}, \ldots, O_N$ vertices. Let $\omega^1, \ldots, \omega^N$ be bounded independent of ε domains in $\mathbb{R}^n$ with Lipschitz boundaries $\partial\omega^j$; introduce the nodal domains $\omega_\varepsilon^j = \{x \in \mathbb{R}^n : \dfrac{x - O_j}{\varepsilon} \in \omega^j\}$.

Every vertex O_j is the end of one and only one edge e_k. By a rod structure we call the following domain

$$B_\varepsilon = \Big(\bigcup_{j=1}^{M} B_\varepsilon^{(e_j)}\Big) \bigcup \Big(\bigcup_{j=1}^{N} \omega_\varepsilon^j\Big).$$

Assume that it is a connected set and that the boundary ∂B_ε of B_ε is C^2-smooth. Without loss of generality, assume that

$$\Big(B_\varepsilon^{(e_j)} \backslash \Big(\bigcup_{i=1}^{N} \omega_\varepsilon^i\Big)\Big) \cap \Big(B_\varepsilon^{(e_k)} \backslash \Big(\bigcup_{i=1}^{N} \omega_\varepsilon^i\Big)\Big) = \emptyset$$

for $j \neq k$. Denote $\gamma_\varepsilon^i = \partial B_\varepsilon \cap \partial \omega_\varepsilon^i$, $i = N_1+1, ..., N$ (these values of i correspond to the vertices), and $\gamma_\varepsilon = \bigcup_{i=N_1+1}^{N} \gamma_\varepsilon^i$.

Let us introduce some Sobolev spaces:

$$\begin{aligned} H^{1,0}_{\gamma,0}(B_\varepsilon \times (0,T)) =& \{v \in L_2(B_\varepsilon \times (0,T)) | \|u\|_{L_2(B_\varepsilon \times (0,T))} \\ & + \|\nabla u\|_{L_2(B_\varepsilon \times (0,T))} < +\infty, v|_{\gamma_\varepsilon} = 0\}, \\ H^{1,0}_{\gamma,0}(B_\varepsilon \times (0,T)) =& \{v \in L_2(B_\varepsilon \times (0,T)) | \|u\|_{L_2(B_\varepsilon \times (0,T))} \\ & + \|\nabla u\|_{L_2(B_\varepsilon \times (0,T))} < +\infty, v|_{\gamma_\varepsilon} = 0\}. \end{aligned}$$

4. Formulation of the heat equation in a rod structure

Consider the initial boundary value problem for the non-steady heat equations in the tube structure B_ε

$$\begin{aligned} \frac{\partial u_\varepsilon}{\partial t} - \Delta u_\varepsilon &= f(x,t), \quad x \in B_\varepsilon, t \in (0,T), \\ \frac{\partial u_\varepsilon}{\partial n} &= 0, \quad x \in \partial B_\varepsilon \backslash \gamma_\varepsilon, t \in (0,T), \\ u_\varepsilon &= 0, \quad x \in \gamma_\varepsilon, t \in (0,T), \\ u_\varepsilon(x,0) &= 0, x \in B_\varepsilon. \end{aligned} \tag{1}$$

The right-hand side f is a function defined on $B_\varepsilon \times [0,T]$ such that $f(x,t) = f_j(x_1,t)$, if $x \in B_\varepsilon^{(e_j)}$, $\quad j = 1, ..., M$, where f_j are independent of ε $C^{J+4}-$smooth functions and they are constant with respect to x in some neighborhood of the nodes and vertices. The values of f in the domains ω_ε^i are equal to its value in the node or vertex O_i. We assume that $f_j(.,t) = 0$ for $t \leq \tau$, $\quad \tau > 0$.

The variational formulation of problem (1) is: find $u_\varepsilon \in H^1_{\gamma,0}(B_\varepsilon \times (0,T))$ such that for almost all $t \in (0,T)$,

$$\int_{B_\varepsilon} \Big(\frac{\partial u_\varepsilon}{\partial t} v + \nabla u_\varepsilon \cdot \nabla v\Big) dx = \int_{B_\varepsilon} f v dx, \quad v \in H^1_{\gamma,0}(B_\varepsilon), \tag{2}$$

$$u_\varepsilon|_{t=0} = 0. \tag{3}$$

This variational formulation implies:

$$\int_{B_\varepsilon\times(0,T)}\Big(\frac{\partial u_\varepsilon}{\partial t}v+\nabla u_\varepsilon\cdot\nabla v\Big)dxdt=\int_{B_\varepsilon\times(!0,T)}fvdxdt,\quad v\in H^{1,0}_{\gamma,0}(B_\varepsilon\times(0,T)),\tag{4}$$

$$u_\varepsilon|_{t=0}=0.\tag{5}$$

Identity (4) will be used in Section 7.

Theorem 1. *There exist a unique solution to problem* (2), (3).

Proof. The proof of the theorem is based on the Galerkin method and follows the same ideas as in [7, Chapter 4, Section 3]. Let $\phi_1, ..., \phi_m, ...$ be an orthogonal with respect to the inner product $\int_{B_\varepsilon}\Big(\nabla u\cdot\nabla v\Big)dx$ base of $H^1_{\gamma,0}(B_\varepsilon)$. Consider the span H_N of N first functions of the base and consider the projection of problem (2), (3) on this subspace. Its solution u^N is saught in the form of a sum $\sum_{l=1}^N c_l(t)\phi_l$ with $c_l\in H^1(0,T)$, so that for the unknown functions c_l satisfy the system of ordinary differential equations with homogeneous initial conditions. Multiplying its equations by c_l and adding them up, we get an estimate for u^N in the V^2 norm ($\|u\|_{V^2}=\sup_{t\in[0,T]}\|u(.,t)\|_{L_2(B_\varepsilon)}+\|\nabla u\|_{L_2(B_\varepsilon\times(0,T))}$). Multiplying then the equations by $\frac{dc_l}{dt}$ and adding them up, we get an estimate for u^N in the $H^1(B_\varepsilon\times(0,T))$ norm. Then we apply the standard argument of the weak compactness of a ball in the Hilbert space and find that a weak limit of some subsequence is a solution of (2), (3). The Poincaré-Friedrichs inequality holds with a constant independent of ε (see [12, Chapter 4, Appendices]).

The uniqueness follows from identity (4) written for $v=u_\varepsilon$. □

The estimates for u^N still hold for the weak limit u_ε, so that

Theorem 2. *The estimate holds*

$$\|u_\varepsilon\|_{H^1(B_\varepsilon\times(0,T))}\le C_{PF}\|f\|_{L_2(B_\varepsilon\times(0,T))},\tag{6}$$

where the constant C_{PF} is independent of ε.

Remark 1. *This estimate* (6) *holds in the case if the right-hand side is any function of $L_2(B_\varepsilon\times(0,T))$ free of the above regularity restrictions.Indeed, these restrictions were not used in the proof of Theorems 1 and 2.*

5. Construction of an asymptotic expansion

Let us seek the $J-$th approximation of an asymptotic expansion of the solution to problem (1) in the form of a sum of functions v_j defined on the graph $\mathcal{G}$, multiplied by the cut-off functions vanishing in the neighborhood of the nodes and vertices, and the boundary layer correctors V_i^{BL} depending on $\frac{x-O_i}{\varepsilon}$ and exponentially tending to zero as $|\frac{x-O_i}{\varepsilon}|\to\infty$. Namely, consider it in the form:

$$u_\varepsilon^{(J)} = \sum_{j=1}^{M} \zeta(\frac{x_1^{(e_j)}}{3r\varepsilon})\zeta(\frac{|e_j| - x_1^{(e_j)}}{3r\varepsilon}) v_j(x_1^{(e_j)}, t)\chi_j(x_1^{(e_j)})$$
$$+ \sum_{i=1}^{N} V_i^{BL}(\frac{x - O_i}{\varepsilon}, t)(1 - \zeta(\frac{x - O_i}{e_{min}})), \tag{7}$$

where r is the maximal diameter of domains ω_j, ζ is a smooth cut-off function independent of ε with $\zeta(\tau) = 0$ for $\tau \leq 1/3$, $\zeta(\tau) = 1$ for $\tau \geq 2/3$, $0 \leq \zeta(\tau) \leq 1$; e_{min} is the minimal length of the edges; $\chi_j(x_1^{(e_j)}) = 1$ iff $x_1^{(e_j)} \in (0, |e_j|)$, and it is equal to zero otherwise; functions v_j satisfy the heat equation on the graph $\mathcal{G}$ with some Kirchhoff-type junction conditions in the nodes O_i, $i = 1, ..., N_1$ and the Dirichlet condition in the vertices O_i, $i = N_1, ..., N$; V_i^{BL}, $i = 1, ..., N$, are the boundary layer correctors. Let us specify now v_j and V_i^{BL}:

$$v_j(x_1^{(e_j)}, t) = \sum_{l=0}^{J} \varepsilon^l v_{jl}(x_1^{(e_j)}, t), \tag{8}$$

$$V_i^{BL}(\xi, t) = \sum_{l=0}^{J} \varepsilon^l V_{il}^{BL}(\xi, t). \tag{9}$$

Substituting the first term of the expansion into the equation, we get the residual which has to be compensated by the boundary layer correctors. The result of the substitution has the form:

$$\sum_{j=1}^{M} \Big(\frac{\partial}{\partial t} - \Big(\frac{\partial}{\partial x_1^{(e_j)}}\Big)^2\Big)\{\zeta(\frac{x_1^{(e_j)}}{3r\varepsilon})\zeta(\frac{|e_j| - x_1^{(e_j)}}{3r\varepsilon}) v_j(x_1^{(e_j)}, t)\chi_j(x_1^{(e_j)})\}$$
$$= \sum_{j=1}^{M} \Big(\Big(\frac{\partial v_j((x_1^{(e_j)}, t)}{\partial t} - \frac{\partial^2 v_j(x_1^{(e_j)}, t)}{\partial x_1^{(e_j)\ 2}}\Big)\zeta(\frac{x_1^{(e_j)}}{3r\varepsilon})\zeta(\frac{|e_j| - x_1^{(e_j)}}{3r\varepsilon})$$
$$- \frac{2}{\varepsilon}\frac{\partial v_j(x_1^{(e_j)}, t)}{\partial x_1^{(e_j)}}\frac{\partial}{\partial \xi_1^{(e_j)}}\Big(\zeta(\frac{\xi_1^{(e_j)}}{3r})\zeta(\frac{|e_j| - \xi_1^{(e_j)}}{3r})\Big)$$
$$- \frac{1}{\varepsilon^2} v_j(x_1^{(e_j)}, t)\frac{\partial^2}{\partial \xi_1^{(e_j)2}}\Big(\zeta(\frac{\xi_1^{(e_j)}}{3r})\zeta(\frac{|e_j| - \xi_1^{(e_j)}}{3r})\Big)\Big)|_{\xi^{(e_j)} = x^{(e_j)}/\varepsilon}\chi_j(x_1^{(e_j)}). \tag{10}$$

Let us note that v_j are defined such that $\frac{\partial v_j}{\partial t} - \frac{\partial^2 v_j}{\partial x_1^{(e_j)2}} = f_j$, so that the first term of the sum is equal to

$$\sum_{j=1}^{M} f_j(x_1^{(e_j)}, t))\zeta(\frac{x_1^{(e_j)}}{3r\varepsilon})\zeta(\frac{|e_j| - x_1^{(e_j)}}{3r\varepsilon})\chi_j(x_1^{(e_j)}).$$

Note that $f_j(x_1^{(e_j)}, t))$ is a time dependent constant in every connected part of $supp\{\zeta(\frac{x_1^{(e_j)}}{3r\varepsilon})\zeta(\frac{|e_j| - x_1^{(e_j)}}{3r\varepsilon}) - 1\}$. These components are some neighborhoods of the extremities of the edge e_j. In particular, in the $e_{min}/2$-neighborhood of nodes and vertices O_i, we have:

$$f_j(x_1^{(e_j)},t)\{\zeta(\frac{x_1^{(e_j)}}{3r\varepsilon})\zeta(\frac{|e_j|-x_1^{(e_j)}}{3r\varepsilon})-1\}=f_j(0,t)\{\zeta(\frac{x_1^{(e_j)}}{3r\varepsilon})\zeta(\frac{|e_j|-x_1^{(e_j)}}{3r\varepsilon})-1\}$$

and

$$\sum_{j=1}^{M}f_j(x_1^{(e_j)},t)\zeta(\frac{x_1^{(e_j)}}{3r\varepsilon})\zeta(\frac{|e_j|-x_1^{(e_j)}}{3r\varepsilon})\chi_j(x_1^{(e_j)})$$
$$=f(x,t)+\sum_{i=1}^{N}f(O_i,t)\{\sum_{j:O_i\in e_j}\zeta(\frac{x_1^{(e_j)}}{3r\varepsilon})\chi_j(x_1^{(e_j)})-1\}\chi(\frac{|x-O_i|}{e_{min}}),$$

where $\chi(t)=1$ for $|t|<\frac{1}{2}$, $\chi(t)=0$ for $|t|\geq\frac{1}{2}$.

Let us expand now the functions v_{jl} and $\frac{\partial v_{jl}}{\partial x_1^{(e_j)}}$ according to Taylor's formula

$$v_{jl}(x_1^{(e_j)},t))=v_{jl}(0,t))+\sum_{m=1}^{J-l}\varepsilon^m\frac{1}{m!}\frac{\partial^m v_{jl}}{\partial x_1^{(e_j)m}}(0,t)\xi_1^{(e_j)m}$$
$$+\varepsilon^{J-l+1}\frac{1}{(J-l+1)!}\frac{\partial^{J-l+1}v_{jl}}{\partial x_1^{(e_j)(J-l+1)}}(\theta,t)\xi_1^{(e_j)(J-l+1)}$$

and

$$\frac{\partial v_{jl}}{\partial x_1^{(e_j)}}(x_1^{(e_j)},t)=\frac{\partial v_{jl}}{\partial x_1^{(e_j)}}(0,t)+\sum_{m=1}^{J-l}\varepsilon^m\frac{1}{m!}\frac{\partial^{m+1}v_{jl}}{\partial x_1^{(e_j)(m+1)}}(0,t)\xi_1^{(e_j)m}$$
$$+\varepsilon^{J-l+1}\frac{1}{(J-l+2)!}\frac{\partial^{J-l+2}v_{jl}}{\partial x_1^{(e_j)(J-l+2)}}(\theta,t)\xi_1^{(e_j)(J-l+1)},$$

$\xi^{(e_j)}=x^{(e_j)}/\varepsilon$.

Then the result of the substitution of the first sum of (7) in the $e_{min}/2$-neighborhood of nodes and vertices O_i is finally equal to

$$f(x,t)+\sum_{l=0}^{J}\varepsilon^{l-2}F_{il}(\xi,t)+R_{J\varepsilon}(x,t), \tag{11}$$

where $\xi=(x-O_i)/\varepsilon$,

$$F_{il}(\xi,t)=-\{f(O_i,t)\{\sum_{j=1}^{M}\zeta(\frac{\xi_1^{(e_j)}}{3r})\psi(\xi_1^{(e_j)})-1\}\delta_{l2}$$
$$+2\sum_{m+p=l-1}\sum_{j=1}^{M}\Big(\frac{1}{m!}\frac{\partial^{m+1}v_{jp}}{\partial x_1^{(e_j)(m+1)}}(0,t)\xi_1^{(e_j)m}\frac{\partial}{\partial\xi_1^{(e_j)}}\zeta(\frac{\xi_1^{(e_j)}}{3r})\Big)\psi(\xi_1^{(e_j)})$$
$$+\sum_{m+p=l}\sum_{j=1}^{M}\Big(\frac{1}{mp}\frac{\partial^m v_{jl}}{\partial x_1^{(e_j)m}}(0,t)\xi_1^{(e_j)m}\frac{\partial^2}{\partial\xi_1^{(e_j)2}}\zeta(\frac{\xi_1^{(e_j)}}{3r})\Big)\psi(\xi_1^{(e_j)})\}, \tag{12}$$

where $\psi(\xi_1^{(e_j)}) = 1$ if $\xi_1^{(e_j)} \geq 0$, and $\psi(\xi_1^{(e_j)}) = 0$ if $\xi_1^{(e_j)} < 0$; by convention, all terms depending on the local variables vanish out of the cylinder $\Pi_j = \{\xi_1^{(e_j)} \in (0,+\infty), \xi^{(e_j)\prime} \in \sigma^{(e_j)}\}$; $R_{J\varepsilon}(x,t)$ is uniformly bounded by $C\varepsilon^{J-1}$, where C is a constant independent of ε, determined by the L_∞-norms of derivatives $\frac{\partial^{J-l+1} v_{jp}}{\partial x_1^{(e_j)(J-l+1)}}$ and $\frac{\partial^{J-l+2} v_{jp}}{\partial x_1^{(e_j)(J-l+2)}}$.

In order to compensate these right-hand sides, functions V_{il}^{BL} satisfy the equations

$$-\Delta V_{il}^{BL} = -F_{il}(\xi,t) - \frac{\partial V_{i,l-2}^{BL}}{\partial t}(\xi,t), \tag{13}$$

set in $\Omega_i = \omega_i \cup \Big(\cup_{j:O_i\in e_j} \Pi_j \Big)$ (here the union is taken over all j such that e_j contains O_i as an end point), with the Neumann boundary condition on $\partial\Omega_j$:

$$-\frac{\partial}{\partial n} V_{il}^{BL} = 0. \tag{14}$$

If O_i is a vertex, then on the part $\partial\Omega_i \cap \partial\omega_i$ of the boundary we set condition

$$V_{il}^{BL} = 0, \tag{15}$$

while condition (14) holds only on the part $\partial\Omega_i \backslash \partial\omega_i$ of the boundary.

Consider first the case when O_i is a node.

The existence and uniqueness of the solution to (13), (14) with exponentially decaying at infinity gradient was studied in [10]. The solution exists iff

$$\int_{\Omega_i} \{F_{il}(\xi,t) + \frac{\partial V_{i,l-2}^{BL}}{\partial t}(\xi,t)\} d\xi = 0. \tag{16}$$

This condition yields:

$$\sum_{j:O_i\in e_j} \frac{\partial v_{jl-1}}{\partial x_1^{(e_j)}}(0,t)|\sigma^{(e_j)}| = g_{l-1}(t), \tag{17}$$

where

$$\begin{aligned} g_l(t) = &- \sum_{j:O_i\in e_j} \sum_{m+p=l, m\neq 0} \frac{1}{m!} \frac{\partial^{m+1} v_{jp}}{\partial x_1^{(e_j)(m+1)}}(0,t) \int_{\Omega_i} \xi_1^{(e_j)m} \frac{\partial}{\partial \xi_1^{(e_j)}} \zeta(\frac{\xi_1^{(e_j)}}{3r}) \psi(\xi_1^{(e_j)}) d\xi \\ &- \sum_{j:O_i\in e_j} \sum_{m+p=l+1, m\geq 2} \frac{1}{m!} \frac{\partial^{m} v_{jp}}{\partial x_1^{(e_j)m}}(0,t) \int_{\Omega_i} \xi_1^{(e_j)m} \frac{\partial^2}{\partial \xi_1^{(e_j)2}} \zeta(\frac{\xi_1^{(e_j)}}{3r}) \psi(\xi_1^{(e_j)}) d\xi \\ &+ \int_{\Omega_i} \frac{\partial V_{i,l-2}^{BL}}{\partial t}(\xi,t) d\xi \\ &- f(O_i,t) \int_{\Omega_i} \{ \sum_{j:O_i\in e_j} \zeta(\frac{\xi_1^{(e_j)}}{3r}) \psi(\xi_1^{(e_j)}) - 1\} \chi(\frac{|\xi|}{e_{min}}) d\xi \delta_{l1}. \end{aligned} \tag{18}$$

This solution tends to some constants depending on time as a parameter. Denote the constant corresponding to the outlet Π_j as $c^i_{jl}(t)$. It is known that the solution of problem (13), (14) is unique up to an additive constant (function of t). So, we determine one of these constants, say $c^i_{j_1 l}(t) = 0$. Then all other constants are uniquely defined. Edge e_{j_1} of the bundle $\mathcal{B}^{(i)}$ is called below the selected edge of the bundle.

On the other hand, it is clear that their values depend on the values of $v_{jq}(0,t)$ on the right-hand side of (13) because in (12) $v_{il}(0,t)$ are the coefficients in the last sum corresponding to $m = 0$. Notice that

$$U_{jl}(\xi,t) = -v_{jl}(0,t)\zeta(\frac{\xi_1^{(e_j)}}{3r})\psi(\xi_1^{(e_j)})$$

is a solution of the problem

$$\begin{aligned} -\Delta U_{jl} &= v_{jl}(0,t)\frac{\partial^2}{\partial \xi_1^{(e_j)2}}\zeta(\frac{\xi_1^{(e_j)}}{3r})\psi(\xi_1^{(e_j)}), \quad \xi \in \Omega_i, \\ -\frac{\partial}{\partial n}U_{jl} &= 0, \quad \partial\Omega_i, \end{aligned}$$

and this solution evidently tends to $v_{jl}(0,t)$ on every outlet Π_j. These constants also depend on the values of the derivatives of v_{jp} at $(0,t)$ with $p < l$, and so these values are known from the previous steps of induction.

Analogous problems should be solved in the infinite domains Ω_i, $i = N_1+1,...,N$ (for vertices). These domains have only one outlet to infinity, but the boundary conditions are mixed: (14), (15). In this case, there always exists a unique solution with an exponentially decaying gradient, but the solution tends at infinity to some constant $c^i_{jl}(t)$, which can be calculated as in [10].

Let us now choose the values of v_{jl} at the nodes and vertices such that all constants $c_{jl}(t)$ vanish. To this end we organize the calculus of v_{jl} and V^{BL}_{jl} by induction in the following way.

For $l = 0$, first we solve the problem on the graph $\mathcal{B}$

$$\begin{aligned} \frac{\partial v_{j0}((x_1^{(e_j)},t)}{\partial t} - \frac{\partial^2 v_{j0}(x_1^{(e_j)},t)}{\partial x_1^{(e_j)\ 2}} &= f_j(x_1^{(e_j)},t), \quad x_1^{(e_j)} \in (0,|e_j|), t > 0, \\ \sum_{j:O_i\in e_j} \frac{\partial v_{j0}}{\partial x_1^{(e_j)}}(0,t)|\sigma^{(e_j)}| &= 0, \\ v_{jl}(0,t) &= v_{j_1 l}(0,t),\ j : O_i \in e_j, \\ &\quad j_1 \text{ is the selected edge of } \mathcal{B}_i, i = 1,...,N_1, \\ v_{j0}(0,t) &= 0,\ i = N_1+1,...,N, \\ v_{j0}((x_1^{(e_j)},0) &= 0, \end{aligned} \tag{19}$$

and define

$$V_{i0}^{BL}(\xi,t)=\{1-\sum_{j:O_i\in e_j}\zeta(\frac{\xi_1^{(e_j)}}{3r})\psi(\xi_1^{(e_j)})\}v_{j_10}(0,t),\ i=1,...,N_1,$$
$$V_{i0}^{BL}(\xi,t)=0,\ i=N_1+1,...,N, \tag{20}$$

where e_{j_1} is the selected edge of the bundle. V_{i0}^{BL} defined in this way satisfies (13), (14) (eventually (15)) and tends to zero as $|\xi|\to+\infty$.

Assume that we have constructed v_{js} for all $s\leq l-1$, and $V_{is}^{BL}(\xi,t),\ s\leq l-1$. Consider problems (13), (14) (eventually (15)) where the expressions F_{jl} are defined by formulas (12) without the term corresponding to $m=0$ in the last sum. If we denote these new functions on the right-hand sides by Φ_{jl}, then

$$F_{il}=\Phi_{il}-\sum_{j:\ O_i\in e_j}v_{jq}(0,t)\frac{\partial^2}{\partial\xi_1^{(e_j)2}}\zeta(\frac{\xi_1^{(e_j)}}{3r})\psi(\xi_1^{(e_j)}) \tag{21}$$

Let us solve problems (13), (14) (eventually (15)) with Φ_{il} instead of F_{il} on the right-hand side. Denote by $\tilde{V}_{il}^{BL}$ its solutions. Denote by $\tilde{c}_{jl}^i(t)$ the limits of solutions $\tilde{V}_{il}^{BL}$ at the outlets corresponding to Π_j. Then consider the following problem in the graph:

$$\frac{\partial v_{jl}((x_1^{(e_j)},t)}{\partial t}-\frac{\partial^2 v_{jl}(x_1^{(e_j)},t)}{\partial x_1^{(e_j)\ 2}}=0,\quad x_1^{(e_j)}\in(0,|e_j|),t>0,$$
$$\sum_{j:O_i\in e_j}\frac{\partial v_{jl}}{\partial x_1^{(e_j)}}(0,t)|\sigma^{(e_j)}|=g_l(t),$$
$$v_{jl}(0,t)=v_{j_1l}(0,t)+\tilde{c}_{jl}^i(t),\ j:O_i\in e_j,$$
$$j_1 \text{ is the selected edge of } \mathcal{B}_i, i=1,...,N_1,$$
$$v_{jl}(0,t)=\tilde{c}_{jl}^i(t),\ i=N_1+1,...,N,$$
$$v_{jl}((x_1^{(e_j)},0)=0, \tag{22}$$

and define

$$V_{il}^{BL}(\xi,t)=\tilde{V}_{il}^{BL}(\xi,t)+\{1-\sum_{j:O_i\in e_j}\zeta(\frac{\xi_1^{(e_j)}}{3r})\psi(\xi_1^{(e_j)})\}v_{j_1l}(0,t) \tag{23}$$
$$-\sum_{j:O_i\in e_j,\ j\neq j_1}\zeta(\frac{\xi_1^{(e_j)}}{3r})\psi(\xi_1^{(e_j)})\tilde{c}_{jl}^i(t),\ i=1,...,N_1, \tag{24}$$

and

$$V_{il}^{BL}(\xi,t)=\tilde{V}_{il}^{BL}(\xi,t)-\zeta(\frac{\xi_1^{(e_j)}}{3r})\psi(\xi_1^{(e_j)})\tilde{c}_{jl}^i(t),\ i=N_1,...,N. \tag{25}$$

Note that condition (16) is satisfied because $v_{j,l-1}$ satisfy (17), see $(22)_2$.

Now $V_{il}^{BL}(\xi,t)\to 0$ as $|\xi|\to+\infty$.

Let us calculate the result of substitution of (7) in the operator $\frac{\partial}{\partial t}-\Delta$. Taking into account (10), we get

$$\Big(\frac{\partial}{\partial t}-\Delta\Big)u_\varepsilon^{(J)}=f(x,t)+R_{J\varepsilon}(x,t)+R_{J\varepsilon}^{(1)}(x,t), \tag{26}$$

where as it was noted above

$$\|R_{J\varepsilon}\|_{L_\infty(B_\varepsilon\times(0,T))}\le C\varepsilon^{J-1},$$

and

$$\begin{aligned}R_{J\varepsilon}^{(1)}=&\varepsilon^{J-1}\sum_{i=1}^{N}\Big(\frac{\partial V_{i,J-1}^{BL}}{\partial t}(\frac{x-O_i}{\varepsilon},t)(1-\zeta(\frac{x-O_i}{e_{min}}))\\&+\varepsilon^{J}\sum_{i=1}^{N}\Big(\frac{\partial V_{i,J}^{BL}}{\partial t}(\frac{x-O_i}{\varepsilon},t)(1-\zeta(\frac{x-O_i}{e_{min}}))\Big)+R_{J\varepsilon}^{(2)},\\R_{J\varepsilon}^{(2)}=&\sum_{i=1}^{N}\Big(\frac{\partial}{\partial t}-\Delta\Big)\Big(V_i^{BL}(\frac{x-O_i}{\varepsilon},t)(1-\zeta(\frac{x-O_i}{e_{min}}))\Big)\tilde\chi(\frac{x-O_i}{e_{min}}),\end{aligned}$$

where $\tilde\chi(y)=1$ if $|y|\in[1/3,2/3]$, and $\tilde\chi(y)=0$ if $|y|<1/3$ or $|y|>2/3$.

The support of $R_{J\varepsilon}^{(2)}$ is situated in the middle third of every cylinder $B_{j\varepsilon}$, where functions V_i^{BL} as well as their derivatives $\frac{\partial}{\partial t}$, ∇,∇^2 are exponentially small inthe $L_\infty-$norm (see [10],[**?**]).

So, for $R_{J\varepsilon}^{(2)}$ (and hence for $R_{J\varepsilon}^{(1)}$ as well) we get

$$\|R_{J\varepsilon}^{(2)}\|_{L_\infty(B_\varepsilon\times(0,T))}\le C\varepsilon^{J-1},$$

and

$$\|R_{J\varepsilon}^{(1)}\|_{L_\infty(B_\varepsilon\times(0,T))}\le C\varepsilon^{J-1}.$$

Here C is a constant independent of ε.

Note that the boundary and initial conditions are satisfied by $u_\varepsilon^{(J)}$ exactly.

Applying now the a priori estimate (6), we get

$$\|u_\varepsilon^{(J)}-u_\varepsilon\|_{H^1(B_\varepsilon\times(0,T))}\le C\varepsilon^{J-1}$$

and so,

$$\|u_\varepsilon^{(J+1)}-u_\varepsilon\|_{H^1(B_\varepsilon\times(0,T))}\le C\varepsilon^{J}.$$

Comparing $u_\varepsilon^{(J)}$ and $u_\varepsilon^{(J+1)}$ we notice that

$$\|u_\varepsilon^{(J+1)}-u_\varepsilon^{(J)}\|_{H^1(B_\varepsilon\times(0,T))}\le C\varepsilon^{J} \tag{27}$$

with C independent of ε. So, from the triangle inequality we get

$$\|u_\varepsilon^{(J)}-u_\varepsilon\|_{H^{2,1}(B_\varepsilon\times(0,T))}\le C\varepsilon^{J}. \tag{28}$$

Remark 2. *The asymptotic expansion (7) can be slightly modified without loss of accuracy. Namely, the argument* $\frac{|x-O_i|}{e_{min}}$ *in the cutoff function* ζ *may be replaced by* $C_J\frac{|ln\varepsilon||x-O_i|}{e_{min}}$*, where the constant* C_J *is chosen in such a way that the absolute values of the boundary layer functions, as well as of their derivatives, are smaller than* ε^{J+2} *in the zone where the cutoff function is different from one and zero. Indeed, the boundary layer functions* V_{il}^{BL} *and their derivatives decay exponentially: there exist positive constants* $c_1,\ c_2$ *such that for* $|\xi|>r$,

$$|V_{il}^{BL}(\xi,t)|,|\frac{\partial V_{il}^{BL}(\xi,t)}{\partial\xi_j}|\leq c_1exp(-c_2|\xi|).$$

It follows from [10] and the ADN-ellipticity [1, 2] of the elliptic equations. The same estimates hold for their time derivatives of order $J-l+3$.

Therefore, if $|x-O_i|\geq C_J\varepsilon|\ln\varepsilon|e_{min}/3$*, then*

$$|V_{il}^{BL}(\frac{x-O_i}{\varepsilon},t)|\leq c_1exp\{-c_2C_J|\ln\varepsilon|e_{min}/3\}=c_1\varepsilon^{c_2C_Je_{min}/3}.$$

Choose C_J *such that*

$$c_2C_Je_{min}/3\geq J+2. \tag{29}$$

Then for V_{il}^{BL} *and its derivatives we get the estimate* $c_1\varepsilon^{J+2}$*. So, the difference between*

$$\zeta(\frac{|x-O_i|}{e_{min}})V_{il}^{BL}(\frac{x-O_i}{\varepsilon},t)$$

and

$$\zeta(\frac{|ln\varepsilon||x-O_i|}{e_{min}})V_{il}^{BL}(\frac{x-O_i}{\varepsilon},t)$$

can be estimated by

$|V_{il}^{BL}(\frac{x-O_i}{\varepsilon},t)|\leq c_1\varepsilon^{J+2}$ *in the domain*

$$supp\{\zeta(\frac{|x-O_i|}{e_{min}})-\zeta(\frac{|ln\varepsilon||x-O_i|}{e_{min}})\},$$

where $\frac{|ln\varepsilon||x-O_i|}{C_Je_{min}}\geq 1/3$.

In the same way we get a similar estimate for the derivatives of this difference. It means that the change of the argument $\frac{|x-O_i|}{e_{min}}$ *by* $\frac{|ln\varepsilon||x-O_i|}{C_Je_{min}}$ *in* ζ *gives an additional residual of order* ε^J *(the factor* ε^{-2} *appears after two derivations in* x *variable), and so it does not lead to any loss of accuracy.*

Denote by $u_{a\varepsilon}^{(J)}$ *expansion (7) modified in such way. So,*

$$\|u_{a\varepsilon}^{(J)}-u_\varepsilon\|_{H^1(B_\varepsilon\times(0,T))}\leq C\varepsilon^J. \tag{30}$$

6. Asymptotic partial decomposition of the domain for the heat equation

In this section, we apply the method of partial asymptotic decomposition of the domain assuming that f_j are $C^{J+4}-$smooth functions.

Let us describe the algorithm of the method of asymptotic partial domain decomposition (MAPDD) for the heat equation set in a tube structure B_ε. Let δ be a small positive number much greater than ε (it will be chosen of order $\varepsilon|ln\varepsilon|$). For any edge $e = \overline{O_iO_j}$ of the graph of the structure introduce two hyperplanes orthogonal to this edge and crossing it at the distance δ from its ends. Denote the cross-sections of the cylinder $B_\varepsilon^{(e)}$ containing e by these two hyperplanes, by$S_{i,j}$ (at the distance δ from O_i) and $S_{j,i}$ (at the distance δ from O_j), respectively, and denote part of the cylinder $B_\varepsilon^{(e)}$ between these two cross-sections by $B_{ij}^{dec,\varepsilon}$. Denote by$B_i^{\varepsilon,\delta}$ the connected truncated by cross-sections $S_{i,j}$, part of B_ε containing the vertex or the node O_i. Denote by $e_{ij}^{dec,\delta}$ part of the edge $\overline{O_iO_j}$ concluded between cross-sections $S_{i,j}$ and $S_{j,i}$.

Define subspace $H^1_{\gamma 0}(B_\varepsilon \times (0,T), \delta)$ $(H^1_{\gamma 0}(B_\varepsilon, \delta))$ of the space $H^1_{\gamma 0}(B_\varepsilon \times (0,T))$ (i.e., $H^1_{\gamma 0}(B_\varepsilon)$, such that its elements have vanishing transversal derivatives $\nabla'_{x^{(e)}}$ on every truncated cylinder $B_{ij}^{dec,\varepsilon}$. Define

$$H^{1,0}_{\gamma 0}(B_\varepsilon \times (0,T), \delta) = \{v \in H^{1,0}_{\gamma 0}(B_\varepsilon \times (0,T)); \nabla'_{x^{(e)}} v = 0 \forall B_{ij}^{dec,\varepsilon}\}.$$

The MAPDD replaces problem (1) by its projection on $H^1_{\gamma 0}(B_\varepsilon \times (0,T), \delta)$: find $u_{\varepsilon,\delta,dec} \in H^1_{\gamma 0}(B_\varepsilon \times (0,T), \delta)$ such that for almost all $t \in (0,T)$,

$$\int_{B_\varepsilon} \Big(\frac{\partial u_{\varepsilon,\delta,dec}}{\partial t} v + \nabla u_{\varepsilon,\delta,dec} \cdot \nabla v\Big) dx = \int_{B_\varepsilon} f v dx, \quad v \in H^1_{\gamma,0}(B_\varepsilon, \delta), \tag{31}$$

and satisfying

$$u_{\varepsilon,\delta,dec}|_{t=0} = 0, \tag{32}$$

which implies:

$$\begin{aligned} \int_{B_\varepsilon \times (0,T)} \Big(\frac{\partial u_{\varepsilon,\delta,dec}}{\partial t} v + \nabla u_{\varepsilon,\delta,dec} \cdot \nabla v\Big) dxdt \\ = \int_{B_\varepsilon \times (0,T)} f v dxdt, \quad v \in H^{1,0}_{\gamma 0}(B_\varepsilon \times (0,T), \delta), \qquad (33) \\ u_{\varepsilon,\delta,dec}|_{t=0} = 0. \qquad (34) \end{aligned}$$

This identity will be used in Section 7.

Theorem 3. *There exists a unique solution of this partially decomposed problem.*

The proof of this theorem repeats the proof of Theorem 1, where the Galerkin base is constructed in the space $H^1_{\gamma,0}(B_\varepsilon, \delta)$ instead of $H^1_{\gamma,0}(B_\varepsilon)$.

Theorem 4. *The estimate holds*

$$\|u_{\varepsilon,\delta,dec}\|_{H^1(B_\varepsilon\times(0,T))} \leq C_{PF}\|f\|_{L_2(B_\varepsilon\times(0,T))}, \tag{35}$$

where the constant C_{PF} is independent of ε and δ.

Indeed, such an estimate holds for the Galerkin's approximations, and thus for their limit.

Remark 3. *This estimate* (35) *holds in the case if the right-hand side is any function of $L_2(B_\varepsilon \times (0,T))$ free of the above regularity restrictions (and so it can depend on all components of x).*

Theorem 5. *Let δ satisfy the following inequality*

$$\delta \geq C_{J+1}\varepsilon|ln(\varepsilon)|, \tag{36}$$

where C_{J+1} is chosen according to (29)*. Then function $u_{a\varepsilon}^{J+1}$ belongs to the space $H^1_{\gamma 0}(B_\varepsilon \times (0,T),\delta)$ and the estimate holds for the difference $u_{a\varepsilon}^{J+1} - u_{\varepsilon,\delta,dec}$:*

$$\|u_{a\varepsilon}^{(J+1)} - u_{\varepsilon,\delta,dec}\|_{H^1(B_\varepsilon\times(0,T))} \leq C\varepsilon^J, \tag{37}$$

where constant C is independent of ε.

Proof. $u_{a\varepsilon}^{J+1}$ belongs to the space $H^1_{\gamma 0}(B_\varepsilon \times (0,T),\delta)$ by construction, see Remark 2. Moreover, $u_{a\varepsilon}^{J+1}$ satisfies equation $(1)_1$ with the residual evaluated by $C\varepsilon^J$ in the $L_\infty-$norm, and it satisfies the boundary and initial conditions exactly. So, the difference $u_{a\varepsilon}^{J+1} - u_{\varepsilon,\delta,dec}$ belongs to the space $H^1_{\gamma 0}(B_\varepsilon \times (0,T),\delta)$ and satisfies the integral identity (33) with the right-hand side f replaced by a function of order $O(\varepsilon^J)$ in the $L_\infty-$norm. Applying the Galerkin method argument as before (see Remark 1) in Theorems 3 and Theorem 4 we get estmate (37) for the difference $u_{a\varepsilon}^{J+1} - u_{\varepsilon,\delta,dec}$. □

Now comparing (30), (28) and (37) and applying the triangle inequality, we get

Theorem 6. *Let δ satisfy the following inequality*

$$\delta \geq C_{J+1}\varepsilon|ln(\varepsilon)|, \tag{38}$$

where C_{J+1} is chosen according to (29)*. Then the estimate holds for the difference $u_\varepsilon - u_{\varepsilon,\delta,dec}$:*

$$\|u_\varepsilon - u_{\varepsilon,\delta,dec}\|_{H^1(B_\varepsilon\times(0,T))} \leq C\varepsilon^J, \tag{39}$$

where constant C is independent of ε.

This estimate justifies the method of asymptotic partial decomposition of the domain for the heat equation.

Notice that the integration by parts in the variational formulation (31) gives the differential version of the partially decomposed problem. Namely, by denoting $\hat{u}$ the restriction of u on the part $e_{ij}^{dec,\delta}$ of the edge e we have

$$\begin{aligned}
\frac{\partial u_{\varepsilon,\delta,dec}}{\partial t} - \Delta u_{\varepsilon,\delta,dec} &= f(x,t), \quad x \in B_i^{\varepsilon,\delta}, \ i=1,...,N, t \in (0,T),\\
\frac{\partial \hat{u}_{\varepsilon,\delta,dec}}{\partial t} - \frac{\partial^2 \hat{u}_{\varepsilon,\delta,dec}}{\partial x_1^{(e)2}} &= \hat{f}(x_1^{(e)},t), \quad x \in e_{ij}^{dec,\delta}, \ \forall e; t \in (0,T),\\
\frac{\partial u_{\varepsilon,\delta,dec}}{\partial n} &= 0, \quad x \in (\partial B_i^{\varepsilon,\delta} \cap \partial B^{\varepsilon}) \backslash \gamma_{\varepsilon}, \ i=1,...,N, t \in (0,T),\\
u_{\varepsilon,\delta,dec} &= 0, \quad x \in \gamma_{\varepsilon}, t \in (0,T),\\
u_{\varepsilon,\delta,dec}(x,0) &= 0, x \in B_{\varepsilon}
\end{aligned} \tag{40}$$

with the junction condition at sections S_{ij} corresponding to the value $x_1^{(e)} = \delta$ for the local variable, which are the same as in [11]:

$$\begin{aligned}
u_{\varepsilon,\delta,dec}(x,t)|_{x_1^{(e)}=\delta} &= \hat{u}_{\varepsilon,\delta,dec}(\delta,t),\\
\frac{1}{|S_{ij}|}\int_{S_{ij}} \frac{\partial u_{\varepsilon,\delta,dec}}{\partial x_1^{(e)}} dx^{(e)\prime}|_{x_1^{(e)}=\delta} &= \frac{\partial \hat{u}_{\varepsilon,\delta,dec}}{\partial x_1^{(e)}}(\delta,t).
\end{aligned} \tag{41}$$

It means that we keep the n-dimensional in space setting $(40)_1$ for the heat equation within small pieces $B_i^{\varepsilon,\delta}, \ i=1,...,N$, (their diameters are of order $\varepsilon|ln(\varepsilon)|$), reduce the dimension to one and consider the heat equation $(40)_2$ on the pieces $e_{ij}^{dec,\delta}$ of edges e and add the junction conditions (41) between the n-dimensional and one dimensional parts. This reduction allows us to reduce the mesh $\frac{1}{\varepsilon|ln(\varepsilon)|}$ times and keep exponential precision of the computations.

Note that conditions (41) are "dissipative" in the following sense. Assume that the right-hand side f vanishes for all $t \in [t_1,t_2], \quad t_1 < t_2$. Then with $v = u_{\varepsilon,\delta,dec}$ (33) yields:

$$\int_{B_\varepsilon} u^2_{\varepsilon,\delta,dec}(x,t_2)dx \le \int_{B_\varepsilon} u^2_{\varepsilon,\delta,dec}(x,t_1)dx.$$

7. General scheme of the MAPDD in the non-steady case

Consider the general scheme of the method of asymptotic partial decomposition of the domain. Let H_ε be a Hilbert space and $\tilde{H}_\varepsilon$ its subspace. Let b_ε be a mapping from $\tilde{H}_\varepsilon \times H_\varepsilon$ to R, such that

$$\forall w_1, w_2 \in \tilde{H}_\varepsilon, |b_\varepsilon(w_1, w_1 - w_2) - b_\varepsilon(w_2, w_1 - w_2)| \ge c_1 \varepsilon^r \|w_1 - w_2\|^{1+\alpha}, \tag{42}$$

$\|.\|$ is the norm in H_ε, $\alpha > 0$, $c_1 > 0$ independent of ε.

Consider the problem

- find $u_\varepsilon \in \tilde{H}_\varepsilon$ such that

$$b_\varepsilon(u_\varepsilon, w) = (f, w), \quad \forall w \in H_\varepsilon, \tag{43}$$

where $(f,.)$ is a linear bounded functional on H_ε. Assume that there exists a unique solution to this problem.

Let $H_{\varepsilon,dec}$ be a subspace of $\tilde{H}_\varepsilon$.

Let u_ε^a be an asymptotic solution such that

(i) $u_\varepsilon^a \in H_{\varepsilon,dec}$ and

(ii) there exists $\psi_\varepsilon \in H_\varepsilon^*$ such that $\|\psi_\varepsilon\| \le c_2$, where c_2 is independent of ε and such that

$$b_\varepsilon(u_\varepsilon^a, w) = (f, w) + \varepsilon^J(\psi_\varepsilon, w) \ \ \forall w \in H_\varepsilon, \tag{44}$$

where $J > r$.

Subtracting (43) from (44) we get

$$b_\varepsilon(u_\varepsilon^a, w) - b_\varepsilon(u_\varepsilon, w) = \varepsilon^J(\psi_\varepsilon, w) \ \ \forall w \in H_\varepsilon, \tag{45}$$

i.e., for $w = u_\varepsilon^a - u_\varepsilon$ we have

$$\begin{aligned} c_1\varepsilon^r\|u_\varepsilon^a - u_\varepsilon\|^{1+\alpha} &\le \varepsilon^J\|\psi_\varepsilon\|\|u_\varepsilon^a - u_\varepsilon\|, \\ \|u_\varepsilon^a - u_\varepsilon\|^\alpha &\le \frac{c_2}{c_1}\varepsilon^{J-r}, \\ \|u_\varepsilon^a - u_\varepsilon\| &\le \left(\frac{c_2}{c_1}\right)^{1/\alpha}\varepsilon^{(J-r)/\alpha}. \end{aligned} \tag{46}$$

Let u_ε^d be a solution of aa partially decomposed problem, i.e., of the identity (43) restricted to the subspace $\tilde{H}_{\varepsilon,dec}$: find $u_\varepsilon^d \in \tilde{H}_{\varepsilon,dec}$ such that

$$b_\varepsilon(u_\varepsilon^d, w) = (f, w), \quad \forall w \in H_{\varepsilon,dec}, \tag{47}$$

where $H_{\varepsilon,dec}$ is a subspace of H_ε, and $\tilde{H}_{\varepsilon,dec}$ is a subspace of $H_{\varepsilon,dec} \cap \tilde{H}_\varepsilon$.

As above, we assume that the subspace $\tilde{H}_{\varepsilon,dec}$ has a simpler structure than $\tilde{H}_\varepsilon$. Let us subtract this identity from (44) written for any $w \in H_{\varepsilon,dec}$.

Then we get

$$b_\varepsilon(u_\varepsilon^a, w) - b_\varepsilon(u_\varepsilon^d, w) = \varepsilon^J(\psi_\varepsilon, w) \ \ \forall w \in H_{\varepsilon,dec}, \tag{48}$$

i.e., for $w = u_\varepsilon^a - u_\varepsilon^d$ we obtain as before

$$\|u_\varepsilon^a - u_\varepsilon^d\| \le \left(\frac{c_2}{c_1}\right)^{1/\alpha}\varepsilon^{(J-r)/\alpha}, \tag{49}$$

Comparing estimates (44) and (47) we get:

$$\|u_\varepsilon - u_\varepsilon^d\| \le \left(\frac{c_2}{c_1}\right)^{1/\alpha}\varepsilon^{(J-r)/\alpha}. \tag{50}$$

In particular, in the previous section

$$H_\varepsilon = H_{\gamma,0}^{1,0}(B_\varepsilon \times (0,T)), \tilde{H}_\varepsilon = \{v \in H_{\gamma,0}^1(B_\varepsilon \times (0,T)), v|_{t=0} = 0\},$$

$$H_{\varepsilon,dec} = H^{1,0}_{\gamma,0}(B_\varepsilon \times (0,T),\delta), \tilde{H}_{\varepsilon,dec} = \{v \in H^1_{\gamma 0}(B_\varepsilon \times (0,T),\delta), v|_{t=0} = 0\},$$

$$b_\varepsilon(u,v) = \int_{B_\varepsilon \times (0,T)} \Big(\frac{\partial u_\varepsilon}{\partial t} v + \nabla u_\varepsilon \cdot \nabla v\Big) dxdt,\ (f,v) = \int_{B_\varepsilon \times (0,T)} fvdxdt,$$

$r = 0$, $\alpha = 1$. So, Theorem 6 can be proved as a corollary of estimate (50).

So, the main result of the paper is the formulation and justification of the MAPDD in the case of the non-steady heat equation set in a thin structure. It allows to reduce dimension in the main part of the domain keeping the $n-$dimensional "zooms" near the nodes and vertices and gluing these models of different dimension by the special junction conditions (see problem (40), (41)). Justification of this method is based on the construction of an asymptotic solution to problem (1) (Section 5) and a projection of (1) on the subspace of functions independent of the transversal space variables out of some $\varepsilon|ln\varepsilon|-$neighborhoods of the nodes and vertices. This method allows to reduce considerably the computational cost of problem (1).

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