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On the ends of groups and the Veech groups of infinite-genus surfaces*

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Abstract. In this paper, we study the PSV construction, which provides a step by step method for obtaining tame translation surfaces with a suitable Veech group. In addition, we slightly modify this construction, and for each finitely generated subgroup $G < \mathrm{GL}_+(2,\mathbb{R})$ without contracting elements, we produce a tame translation surface S with infinite genus such that its Veech group is G. Furthermore, the ends space of S can be written as $\mathcal{B} \sqcup \mathcal{U}$, where $\mathcal{B}$ is homeomorphic to the ends space of the group G, and $\mathcal{U}$ is a countable, discrete, dense, and open subset of the ends space of S.

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1. Introduction

Geometrically, an *end* of a topological space is a point at infinity. In [9], Freudenthal introduced the concept of ends and explored some of its applications in group theory. One can define the ends space $\mathrm{Ends}(G)$ of a finitely generated group G as the ends space of the Cayley graph $\mathrm{Cay}(G,H)$, where H is a generating set of G (see [10, 13]). In the context of orientable surfaces, Kerékjártó [17] studied their ends and introduced the classification of non-compact orientable surfaces, which determines the topological type of any orientable surface S by its genus $g(S) \in \mathbb{N}\cup\{\infty\}$ and two closed subsets, $\mathrm{Ends}_\infty(S) \subseteq \mathrm{Ends}(S)$, of the Cantor set. These subsets are referred to as the ends space of S, and the ends of S having (infinite) genus (see [28]). Our focus is on studying surfaces with infinite genus.

Translation surfaces have naturally appeared in various contexts: dynamical systems (see [16, 15]), Teichmüller theory (see [18, 21]), Riemann surfaces (see [20, 34]), among others. Our focus is on the so-called *tame* translation surfaces. Using the charts of a translation surface S, one can pull back the standard Riemannian metric on $\mathbb{R}^2$ to equip the surface S with a flat Riemannian metric μ. This flat metric induces a distance map d on S. A translation surface S is said to be *tame* [30] if, for each point $x \in \widehat{S}$ (where $\widehat{S}$ is the metric completion of S with respect to

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d), there is a neighborhood $U_x \subset \widehat{S}$ that is isometric to either an open subset of the Euclidean plane or an open subset around a ramification point of a (finite or infinite) cyclic branched covering of the unit disk. It is worth noting that if S is a compact translation surface, then S is necessarily tame. Several authors have studied such surfaces (see, for instance, [3, 7, 8, 26, 31]), which provides strong motivation for our research.

During the 1980s, Veech [32] associated a group of matrices $\Gamma < \mathrm{GL}(2,\mathbb{R})$ to each translation surface, now commonly known as the *Veech group of S*. He proved that if the Veech group $\Gamma(S)$ of a compact translation surface S is a lattice–meaning $\Gamma(S)$ is a Fuchsian group such that the quotient space $\mathbb{H}^2/\Gamma$ has a finite hyperbolic area–then the behavior of the geodesic flow on S exhibits dynamical properties similar to those described by Weyl's theorem for the geodesic flow on the torus. This result is known as the *Veech's dichotomy*. It has since attracted the attention of many researchers (see, for example, [6, 12, 14]).

The Veech group associated to a compact translation surface is a Fuchsian group [33]. In the case of a tame translation surface, if $\Gamma(S)$ is the Veech group of the tame translation surface S, then one of the following holds [24, Theorem 1.1]:

(1) $\Gamma(S)$ is countable and without contracting elements, it means $\Gamma(S)$ is disjoint from the set $\{A \in \mathrm{GL}_+(2,\mathbb{R}) : \|Av\| < \|v\| \text{ for all } v \in \mathbb{R}^2 \setminus \{\mathbf{0}\}\}$, where $\|\ \|$ is the Euclidean norm on $\mathbb{R}^2$, or

(2) $\Gamma(S)$ is conjugated to $P := \left\{ \begin{pmatrix} 1 & t \\ 0 & s \end{pmatrix} : t \in \mathbb{R} \text{ and } s \in \mathbb{R}^+ \right\}$, or

(3) $\Gamma(S)$ is conjugated to $P' < \mathrm{GL}_+(2,\mathbb{R})$, the subgroup generated by P and $-\mathrm{Id}$, or

(4) $\Gamma(S)$ is equal to $\mathrm{GL}_+(2,\mathbb{R})$.

Our work contributes to the problem of realizing subgroups of $\mathrm{GL}_+(2,\mathbb{R})$ as Veech groups of (non-compact) tame translation surfaces. We will discuss some of the studies involved in the problem of realizing groups as symmetry groups of translation surface. In [24], the authors developed a step-by-step process referred to as the *PSV construction*, aimed at constructing, for each subgroup $G < \mathrm{GL}(2,\mathbb{R})$ without contracting elements, a tame Loch Ness monster with Veech group G. Up to homeomorphism, the *Loch Ness monster* is the only surface with infinite genus and a unique end [23]. In the case of *origamis*, translation surfaces formed by appropriately gluing unit squares, any finite group can be represented as the automorphism group of the Loch Ness monster when it is viewed as an origami [11]. The PSV construction, with slight modifications, was used in [25] to realize any subgroup $G < \mathrm{GL}_+(2,\mathbb{R})$ without contracting elements as the Veech group of a large class of tame translation surfaces of infinite genus. These results, along with those addressing the realization of Veech groups for translation surfaces with non-self-similar end spaces [22], have been extended to resolve the problem of realizing symmetry groups of infinite genus translation surfaces [2].

We have also explored and made slight modifications to the PSV construction, resulting in a theorem that establishes an explicit connection between the ends space of a tame translation surface and the ends space of its respective Veech group.

Theorem 1. *Given a finitely generated subgroup G of $\mathrm{GL}_+(2,\mathbb{R})$ without contracting elements, there exists a tame translation surface S whose Veech group is G. The ends space $\mathrm{Ends}(S)$ of S satisfies:*

(1) *If G is finite, then the surface S has as many ends as there are elements in the group G, and each end has infinite genus.*

(2) *If G is not finite, then the ends space of S can be represented as*

$$\mathrm{Ends}(S) = \mathrm{Ends}_\infty(S) = \mathcal{B} \sqcup \mathcal{U},$$

where $\mathcal{B}$ is a closed subset of $\mathrm{Ends}(S)$ homeomorphic to $\mathrm{Ends}(G)$, and $\mathcal{U}$ is a countable, discrete, dense, and open subset of $\mathrm{Ends}(S)$.

As the ends space of a finitely generated group has either zero, one, two, or infinitely many ends [10, 13], we immediately obtain the following corollary:

Corollary 1. *The ends space of the tame translation surface S is one of the following:*

(1) *If the group G has one end, then $\mathrm{Ends}(S)$ is homeomorphic to the ordinal number $\omega + 1$. In other words, the ends space of S is homeomorphic to the closure of $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$.*

(2) *If the group G has two ends, then $\mathrm{Ends}(S)$ is homeomorphic to the ordinal number $\omega \cdot 2 + 1$. This means that the ends space of S is homeomorphic to two copies of the closure of $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$.*

(3) *If the group G has infinitely many ends, then $\mathrm{Ends}(S)$ contains a subset homeomorphic to the Cantor set, with its complement being a countable, discrete, dense, and open subset of $\mathrm{Ends}(S)$.*

The paper is structured as follows: In Section 2, we collect the principal tools needed to understand the classification of non-compact surfaces theorem and explore the concept of ends on groups. Section 3 provides an introduction to the theory of tame translation surfaces and discusses the Veech group. Finally, Section 4 is dedicated to proving our main result.

2. Ends

In this section, we shall introduce the concept of the space of ends of a topological space X in its most general context. We shall also explore the classification theorem of non-compact orientable surfaces based on their ends spaces. Finally, we shall discuss the concept of ends of groups.

Definition 1 (see [9])**.** *Let X be a locally compact, locally connected, connected, and Hausdorff space, and let $(U_n)_{n\in\mathbb{N}}$ be an infinite nested sequence $U_1 \supset U_2 \supset \dots$ of non-empty connected open subsets of X, such that the following conditions hold:*

(1) *For each $n \in \mathbb{N}$, the boundary ∂U_n of U_n is compact.*

(2) *The intersection* $\bigcap_{n\in\mathbb{N}} \overline{U_n} = \emptyset$.

(3) *For any compact subset* $K \subset X$, *there is* $m \in \mathbb{N}$ *such that* $K \cap U_m = \emptyset$.

Two nested sequences $(U_n)_{n\in\mathbb{N}}$ *and* $(U'_n)_{n\in\mathbb{N}}$ *are equivalent if for each* $n \in \mathbb{N}$, *there exist* $j, k \in \mathbb{N}$ *such that* $U_n \supset U'_j$ *and* $U'_n \supset U_k$. *The corresponding equivalence classes of these sequences are called the ends of* X. *The ends space* $\mathrm{Ends}(X)$ *of* X *is the space whose elements are the ends of* X, *and it is endowed with the following topology: for any non-empty open subset* U *of* X, *such that its boundary* ∂U *is compact, we define*

$$U^* := \{[U_n]_{n\in\mathbb{N}} \in \mathrm{Ends}(X) \mid U_j \subset U \text{ for some } j \in \mathbb{N}\}.$$

Then the set of all such U^*, *where* U *is open and has a compact boundary in* X, *forms a basis for the topology of* $\mathrm{Ends}(X)$ *(see [9, 1. Kapitel]).*

Theorem 2 (see [27])**.** *The space* $\mathrm{Ends}(X)$, *with the topology defined above, is Hausdorff, totally disconnected, and compact.*

2.1. Ends of a surface

A *surface* S is a connected 2-manifold without boundary, which may or may not be closed. In this manuscript, we shall only consider orientable surfaces. By a *subsurface* of S we mean an embedded surface, which is a closed subset of S, and whose boundary consists of a finite number of nonintersecting simple closed curves. Note that a subsurface may or may not be compact. The *reduced genus* of a compact subsurface $\tilde{S} \subset S$, with $q(\tilde{S})$ boundary curves and Euler characteristic $\chi(\tilde{S})$, is the number

$$g(\tilde{S}) = 1 - \frac{1}{2}\left(\chi(\tilde{S}) + q(\tilde{S})\right).$$

The *genus* of the surface S is the supremum of the genera of its compact subsurfaces. This genus may be a non-negative integer or ∞. The surface S is said to be *planar* if it has genus zero; in other words, S is homeomorphic to an open of the complex plane.

Remark 1. *In this case, from the definition of ends given in Definition 1, we may assume that for the sequence* $(U_n)_{n\in\mathbb{N}}$ *the closures* $\overline{U}_n$ *are subsurfaces. In this setting, an end* $[U_n]_{n\in\mathbb{N}}$ *of a surface* S *is called planar if there is* $l \in \mathbb{N}$ *such that the subsurface* $\overline{U}_l \subset S$ *is planar.*

We define the subset $\mathrm{Ends}_\infty(S)$ of $\mathrm{Ends}(S)$ to consist of all ends of S, which are not planar (*ends having infinite genus*). It follows directly from the definition that $\mathrm{Ends}_\infty(S)$ is a closed subset of $\mathrm{Ends}(S)$ (see [28, p. 261]), and the triplet $(g, \mathrm{Ends}_\infty(S), \mathrm{Ends}(S))$, where g is the genus of S, is a topological invariant.

Theorem 3 (Classification of non-compact surfaces [17, 28])**.** *Two surfaces* S_1 *and* S_2 *having the same genus are topologically equivalent if and only if there exists a homeomorphism* $f : \mathrm{Ends}(S_1) \to \mathrm{Ends}(S_2)$ *such that* $f(\mathrm{Ends}_\infty(S_1)) = \mathrm{Ends}_\infty(S_2)$.

Definition 2 (see [23]). *The Loch Ness monster is the unique, up to homeomorphism, infinite genus surface with exactly one end.*

Remark 2 (see [29]). *The surface S has m ends, for some $m \in \mathbb{N}$, if and only if for any compact subset $K \subset S$, there is a compact $K^{'} \subset S$ such that $K \subset K^{'}$ and $S \setminus K^{'}$ consists of m connected components.*

2.2. Ends of a group

Given a generating set H (closed under inverse) of a group G, the *Cayley graph of G with respect to the generating set H* is the graph $\mathrm{Cay}(G, H)$, where the vertices are the elements of G, and there is an edge between two vertices g_1 and g_2 if and only if there is $h \in H$ such that $g_1 h = g_2$. Throughout this paper, the Cayley graph $\mathrm{Cay}(G, H)$ will be the geometric realization of an abstract graph [4, p. 226].

When the set H is finite, the Cayley graph $\mathrm{Cay}(G, H)$ is a locally compact, locally connected, connected, and Hausdorff space. In this case, we define the *ends space of G* as $\mathrm{Ends}(G) := \mathrm{Ends}(\mathrm{Cay}(G, H))$.

Proposition 1 (see [19]). *Let G be a finitely generated group. The ends space of the Cayley graph of G does not depend on the choice of the finite generating set.*

Theorem 4 (see [10, 13]). *Let G be a finitely generated group. Then G has either zero, one, two, or infinitely many ends.*

3. Tame translation surfaces

An atlas $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in I}$ on the surface S is called a *translation atlas* if S, except for a subset of points $\mathrm{Sing}(S) \subset S$, can be covered by the charts from such atlas. Moreover, for any pair of charts (U_α, ϕ_α) and (U_β, ϕ_β) in $\mathcal{A}$ such that $U_\alpha \cap U_\beta \neq \emptyset$, the associated transition map

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{R}^2 \to \phi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R}^2$$

is locally the restriction of a translation. We assume that each point in $\mathrm{Sing}(S)$ is non-removable, which means the translation atlas can not be extended to any of the points in $\mathrm{Sing}(S)$. An element x in $\mathrm{Sing}(S)$ is called a *singular point of S* or *singularity*. A *translation structure* on S is a maximal translation atlas on the surface. If S admits a translation structure, it will be called a *translation surface*.

For a translation surface S, we can pull back the Euclidean (Riemannian) metric of $\mathbb{R}^2$ via its translation structure; thus we obtain a flat Riemannian metric μ on S. Let $\widehat{S}$ denote the *metric completion of S* with respect to the flat Riemannian metric μ. According to the uniformization theorem [1, p. 580], the only complete translation surfaces $S = \widehat{S}$ are the Euclidean plane, the torus, and the cylinder [5, p. 193].

Definition 3 (see [30]). *A translation surface S is said to be tame if for each point $x \in \widehat{S}$, there exists a neighborhood $U_x \subset \widehat{S}$ isometric to either:*

(1) *Some open subset of the Euclidean plane, or*

(2) *An open subset of the ramification point of a (finite or infinite) cyclic branched covering of the unit disk in the Euclidean plane.*

In the latter case, if the neighborhood U_x is isometric to the finite cyclic branched covering of finite order $m \in \mathbb{N}$, then the point x is called a finite cone angle singularity of angle $2m\pi$. If U_x is isometric to the infinite cyclic branched covering, then x is called an infinite cone angle singularity.

We denote by $\mathrm{Sing}(\widehat{S})$ the set of all finite and infinite cone angle singularities of $\widehat{S}$. An element of $\mathrm{Sing}(\widehat{S})$ is called a *cone angle singularity of* $\widehat{S}$, or simply a *cone point*.

3.1. Saddle connection and markings

A *saddle connection* γ on a tame translation surface S is a geodesic interval joining two cone points and not having cone points in its interior. In the translation structure of S, we can find a chart (U, φ) such that the open U contains the saddle connection γ, excluding its endpoints. The map φ sends γ to a straight line segment in $\mathbb{R}^2$. This straight line segment can be oriented in two possible directions denoted by $[\theta], [-\theta] \in \mathbb{R}/2\pi\mathbb{Z}$, for some $\theta \in \mathbb{R}$. Then we can associate to γ two oppositely oriented vectors $\{v, -v\} \subset \mathbb{R}^2$, corresponding to directions $[\theta]$ and $[-\theta]$, respectively. Moreover, the norm of these vectors is equal to the length of γ measured with respect to the flat Riemannian metric μ on S. Each of these vectors is called a *holonomy vector of* γ. Clearly, the holonomy vectors of γ are well-defined, that is, they do not depend on the choice of the chart (U, φ).

A *marking* m on the tame translation surface S is a finite length geodesic not having cone points inside it. Similarly to the case of saddle connection, we can associate to the marking m two *holonomy vectors* $\{v, -v\} \subset \mathbb{R}^2$. Two markings are said to be *parallel* if their respective holonomy vectors are also parallel. It does not matter if the markings are on different surfaces [24, Definition 3.4].

Definition 4 (see [25])**.** *Let m_1 and m_2 be two parallel markings having the same length on translation surfaces S_1 and S_2, respectively. We cut S_1 and S_2 along m_1 and m_2, respectively, turning S_1 and S_2 into the surfaces with boundary $\tilde{S}_1$ and $\tilde{S}_2$, respectively. Each of their boundaries is formed by two straight line segments. Now, we consider the union $\tilde{S}_1 \cup \tilde{S}_2$ and identify (glue) such (four) segments using translations to obtain a connected tame translation surface S (see Figure 1). This gluing relation of these segments will be denoted as $m_1 \sim_{glue} m_2$, and called the operation of gluing the markings m_1 and m_2. Then the surface S will be written in the following form:*

$$S := (S_1 \cup S_2)/m_1 \sim_{glue} m_2.$$

We say that S is obtained from S_1 and S_2 by regluing along m_1 and m_2.

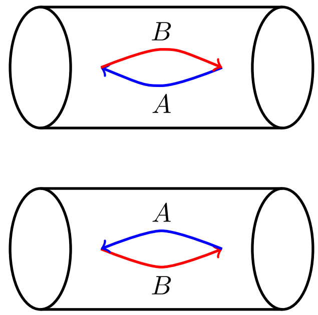

Figure 1: *Gluing markings*

3.2. Veech group

Let S be a tame translation surface. A homeomorphism $T : \widehat{S} \to \widehat{S}$ is called an *affine diffeomorphism* if it satisfies the following properties:

(1) It sends cone points to cone points.

(2) The function T is an affine map in the local coordinates of the translation atlas on S.

We denote by $\mathrm{Aff}_+(S)$ the group of all affine orientations preserving diffeomorphism from the tame translation surface S to itself.

Given a tame translation surface S and a map $T \in \mathrm{Aff}_+(S)$, then using the translation structure on S, we hold that the differential $dT(p)$ of T at any point $p \in S$ is a constant matrix A that belongs to $\mathrm{GL}_+(2, \mathbb{R})$. We then define the map

$$D : \mathrm{Aff}_+(S) \to \mathrm{GL}_+(2, \mathbb{R}),$$

where $D(T)$ is the differential matrix of T. Using the chain rule, it is easy to verify that D is a group homomorphism.

Definition 5 (see [32])**.** *The image of D, denoted by $\Gamma(S)$, is called the Veech group of S.*

The group $\mathrm{GL}_+(2, \mathbb{R})$ acts on the set of all translation surfaces by postcomposition on charts. More precisely, this action sends the couple (g, S) to the translation surface S_g, which is called *the affine copy of S*. The translation structure on S_g is obtained by postcomposing each chart on S by the affine transformation associated to the matrix g. Further, this action defines an affine diffeomorphism $f_g : S \to S_g$, where the differential $df_g(p)$ of f_g at any point $p \in S$ is the matrix g.

4. Proof of Theorem 1

Let G be a finitely generated subgroup of $\mathrm{GL}_+(2, \mathbb{R})$ without contracting elements, and let H be a finite generating set of G. The set H can be written as $H = \{h_j :$

$j \in \{1, \ldots, J\}\}$, for some $J \in \mathbb{N}$. We shall obtain the surface S using the PSV construction, which will be briefly outlined below. Afterward, we shall prove that S is a tame translation surface with Veech group G. Finally, we will describe the ends space of S.

4.1. PSV construction

For each countable subgroup G of $\mathrm{GL}_+(2,\mathbb{R})$ without contracting elements, Przytycki, Weitze-Schmithüsen, and Valdez, in [24, 4. Countable Veech group], described a method to construct a tame translation surface homeomorphic to the Loch Ness monster, with Veech group G. We refer to this method as the *PSV construction.* From a metric spaces point of view, the process is as follows:

Step 1. The decorated surface

We build a *suitable* tame Loch Ness monster S_{dec} using copies of the Euclidean plane and a cyclic branched covering of the Euclidean plane, which are appropriately attached via gluing markings. The resulting surface S_{dec} is referred to as *decorated.* For each $h_j \in H$, we mark S_{dec} with two infinite families of (suitable) markings

$$h_j\check{M}^{-j} := \left\{h_j\check{m}_i^{-j} : \forall i \in \mathbb{N}\right\} \text{ and } M^{-j} := \left\{m_i^{-j} : \forall i \in \mathbb{N}\right\}.$$

Step 2. The puzzle associated to the triplet $(1, G, H)$

For each $g \in G$, we take the affine copy S_g of the decorated surface S_{dec}. We then define two families of markings on S_g:

$$gh_j\check{M}^{-j} := \left\{gh_j\check{m}_i^{-j} : \forall i \in \mathbb{N}\right\} \text{ and } gM^{-j} := \left\{gm_i^{-j} : \forall i \in \mathbb{N}\right\}.$$

These families corresponded to the image of $h_j\check{M}^{-j}$ and M^{-j} on S_{dec} (respectively) under the diffeomorphism $f_g : S_{\text{dec}} \to S_g$. Thus, we define the *puzzle associated to the triplet* $(1, G, H)$ as

$$\mathfrak{P}(1, G, H) := \{S_g : g \in G\},$$

as defined in [25, Definition 3.1]. The term 1 means that the decorated surface has only one end.

Step 3. The assembled surface S to the puzzle $\mathfrak{P}(1, G, H)$.

We define the *assembled surface to the puzzle* $\mathfrak{P}(1, G, H)$ (see [25, Definition 3.1]) as follows:

$$S := \bigcup_{g \in G} S_g \Big/ \sim,$$

where $\sim$ is the equivalence relation given by the following gluing of the markings: for each edge (g, gh_j) of the Cayley graph $\mathrm{Cay}(G, H)$, the marking $gh_j\check{m}_i^{-j}$ on S_g is glued to the marking $gh_jm_i^{-j}$ on S_{gh_j}, for each $i \in \mathbb{N}$.

4.2. We employ PSV construction to obtain the surface S

Step 1. The decorated surface

The following auxiliary construction is necessary to obtain the decorated surface.

Construction 1 (Buffer surface). *For each $j \in \{1, \ldots, J\}$, we consider $\mathbb{E}(j,1)$ and $\mathbb{E}(j,2)$ copies of the Euclidean plane, which are endowed with a fixed origin $\boldsymbol{0}$ and an orthogonal basis $\beta = \{e_1, e_2\}$. We define markings on these surfaces, which are described by their endpoints. On $\mathbb{E}(j,1)$, we draw the families of markings:*

$$\check{M}^j := \left\{\check{m}_i^j := (4ie_1, (4i+1)e_1) : \forall i \in \mathbb{N}\right\}, \text{ and}$$
$$L := \{l_i := ((4i+2)e_1, (4i+3)e_1) : \forall i \in \mathbb{N}\}.$$

On $\mathbb{E}(j,2)$ we take the family of markings:

$$L^{'} := \left\{l_i^{'} := ((2i+1)e_2,\ e_1 + (2i+1)e_2) : \forall i \in \mathbb{N}\right\},$$

and the marking:

$$h_j\check{m}^{-j} := (2e_2,\ e_1 + 2e_2).$$

Finally, the marking $l_i \in L$ on $\mathbb{E}(j,1)$ and the marking $l_i^{'} \in L^{'}$ on $\mathbb{E}(j,2)$ are glued, for each $i \in \mathbb{N}$. Thus, we obtain a tame Loch Ness monster

$$S(Id, h_j), \tag{1}$$

which is called the buffer surface associated to the element h_j of H (see Figure 2).

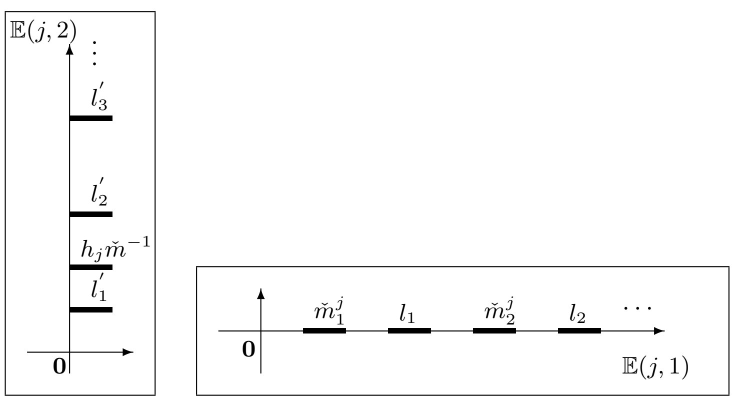

Figure 2: *Buffer surface $S(Id, h_j)$*

Remark 3. *The buffer surface $S(Id, h_j)$ is a modification of the surface appearing in Construction 4.4 in [24]. We emphasize that the family of markings $\check{M}^j$ and the marking $h_j\check{m}^{-j}$ on $S(Id, h_j)$ have not been glued yet. In addition, the set of singular points of $S(Id, h_j)$ consists of infinitely many cone angle singularities of angle 4π.*

Construction 2 (Decorated surface)**.** *We take $\mathbb{E}$, the Euclidean plane, endowed with a fixed origin $\overline{\mathbf{0}}$, and an orthogonal basis $\beta = \{e_1, e_2\}$. Analogously, we shall define markings on this surface, described by their endpoints. For each $j \in \{1, \dots, J\}$, on $\mathbb{E}$ we define the families of markings:*

$$M^j := \left\{ m_i^j := ((2i-1)e_1 + je_2,\, 2ie_1 + je_2) : \forall i \in \mathbb{N} \right\}, \text{ and}$$
$$M := \{ m_i := ((4i-1)e_1, 4ie_1) : \forall i \in \mathbb{N} \}.$$

Now, we recursively draw new markings on $\mathbb{E}$. For $j = 1$, we choose two suitable real numbers $x_1 > 0$ and $y_1 < 0$ and define the marking:

$$m^{-1} := (x_1e_1 + y_1e_2,\, x_1e_1 + h_1^{-1}e_1 + y_1e_2)$$

on $\mathbb{E}$, such that m^{-1} is disjoint from the families of markings M and M^j for each $j \in \{1, \dots, J\}$.

For $n \leq J$, we choose two suitable real numbers $x_n > 0$ and $y_n < 0$ and define the marking:

$$m^{-n} := (x_ne_1 + y_ne_2,\, x_ne_1 + h_n^{-1}e_1 + y_ne_2)$$

on $\mathbb{E}$, such that m^{-n} is disjoint from the families of markings M and M^j for each $j \in \{1, \dots, J\}$. Moreover, the marking m^{-n} is also disjoint from the markings $m^{-1}, \dots, m^{-(n-1)}$ defined in the previous steps.

Let $\pi : \tilde{\mathbb{E}} \to \mathbb{E}$ be the three fold cyclic covering of $\mathbb{E}$, branched over the origin. Then we denote as

$$\tilde{M} := \{\tilde{m}_i : \forall i \in \mathbb{N}\}$$

one of the three sets of markings on $\tilde{\mathbb{E}}$ defined by $\pi^{-1}(M)$. Now, we take on $\mathbb{E}$ the markings $t_1 := (e_2, 2e_2)$ and $t_2 := (-e_2, -2e_2)$, which will be used to generate new markings on $\tilde{\mathbb{E}}$. Then we denote as $\tilde{t_1}$ and $\tilde{t_2}$ one of the three markings on $\tilde{\mathbb{E}}$ defined by $\pi^{-1}(t_1)$ and $\pi^{-1}(t_2)$, respectively, such that they are on the same fold of $\tilde{\mathbb{E}}$ as $\tilde{M}$.

Finally, we take the union of surfaces $\mathbb{E} \cup \tilde{\mathbb{E}} \bigcup_{j\in\{1,\dots,J\}} S(Id, h_j)$ (see equation (1)*), and glue markings as follows:*

(1) *The markings $\tilde{t_1}$ and $\tilde{t_2}$ on $\tilde{\mathbb{E}}$ are glued.*

(2) *The marking m_i on $\mathbb{E}$ is glued to the marking $\tilde{m}_i$ on $\tilde{\mathbb{E}}$, for each $i \in \mathbb{N}$.*

(3) *The marking m_i^j on $\mathbb{E}$ is glued to the marking $\check{m}_i^j$ on $S(Id, h_j)$, for each $i \in \mathbb{N}$ and each $j \in \{1, \dots, J\}$.*

Thus, we obtain the tame Loch Ness monster

$$S_{\text{dec}}, \tag{2}$$

which is called a decorated surface (see Figure 3).

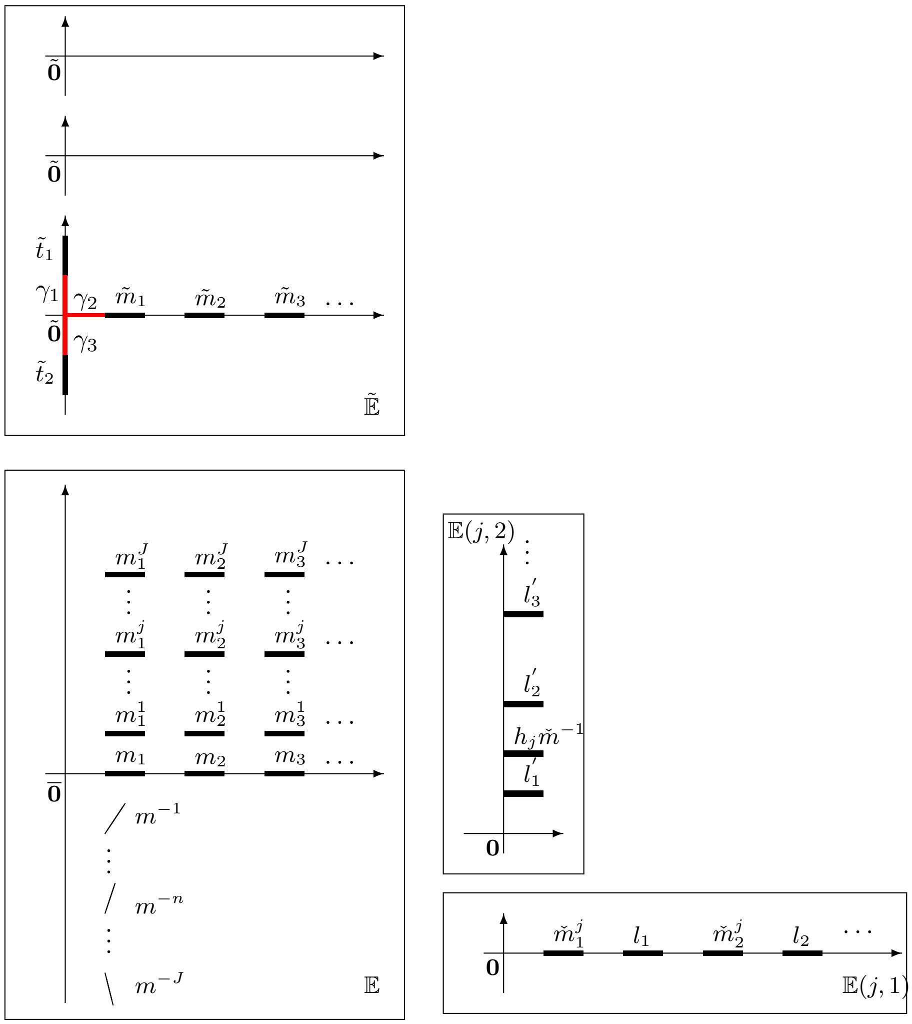

Figure 3: *Decorated surface S_{dec}*

Remark 4. *For each $j \in \{1, \ldots, J\}$, the markings $h_j\check{m}^{-j}$ and m^{-j} on the decorated surface S_{dec} have not been glued yet. Moreover, the surface S_{dec} has the following properties:*

(1) *Its set of singular points consists of infinitely many cone angle singularities of angle 4π, and only one cone angle singularity of angle 6π, which is denoted by $\tilde{\mathbf{0}}$.*

(2) *There are only three saddle connections γ_1, γ_2, and γ_3, such that each one of them has the singularity $\tilde{\mathbf{0}}$ as one of their endpoints (see Figure 3). The holonomy vectors of these saddle connections are $\{\pm e_1, \pm e_2\}$.*

The surface S_{dec} is a slight modification of the surface appearing in Construction 4.6 in [24]. In that construction, the authors introduced a tame Loch Ness monster with infinitely many markings on it. Nevertheless, in our case, we consider the same surface but with only a subset of these markings. Additionally, the decorated surfaces appearing in [25] cover different ends spaces; however, each of them has drawn an infinite family of markings for each element of H. This implies that our decorated surface S_{dec} is not studied in the aforementioned article.

Step 2. The puzzle associated to the triplet $(1, G, H)$

Let S_g be the affine copy of the decorated surface S_{dec}, for each $g \in G$. We denote by $gh_j\check{m}^{-j}$ and gm^{-j} (respectively) the markings on S_g, which are the images of the markings $h_j\check{m}^{-j}$ and m^{-j} (respectively) via the affine diffeomorphism $f_g : S_{\text{dec}} \to S_g$, where $j \in \{1, \ldots, J\}$. Thus, we define the *puzzle associated to the triplet* $(1, G, H)$ as

$$\mathfrak{P}(1, G, H) := \{S_g : g \in G\}.$$

The following lemma will be used to prove the tameness of our surface S.

Lemma 1 (see [24]). *For every $g \in G$, the distance in S_g between the families of markings $\{gh_j\check{m}^{-j} : j \in \{1, \ldots, J\}\}$ and $\{gm^{-j} : j \in \{1, \ldots, J\}\}$ is at least $1/\sqrt{2}$.*

Step 3. The assembled surface S **to the puzzle** $\mathfrak{P}(1, G, H)$

We consider the union $\bigcup_{g \in G} S_g$ and glue markings as follows: given the edge (g, gh_j) of the Cayley graph $Cay(G, H)$, we glue the marking $gh_j\check{m}^{-j}$ on S_g to the marking gh_jm^{-j} on S_{gh_j}.

We remark that, by construction, the markings $gh_j\check{m}^{-j}$ and gh_jm^{-j} are parallel, so the gluing is well-defined. Thus, the *assembled surface to the puzzle* $\mathfrak{P}(1, G, H)$ obtained from the above gluing is a translation surface denoted by

$$S := \bigcup_{g \in G} S_g \Big/ \sim.$$

4.3. The surface S is a tame translation surface and its Veech group is the subgroup $G < \mathrm{GL}_+(2, \mathbb{R})$

One can use several of the ideas described in [25, Theorem 3.7] to easily prove the following lemmas.

Lemma 2. *The translation surface S is tame.*

Proof. We must show that S is a complete metric space with respect to its natural flat metric d, and its set of singularities is discrete in S. Let $(\widehat{S}, \widehat{d})$ be the metric completion space of (S, d). For each $g \in G$, we define the connected open subset

$$S'_g := S_g \setminus \left\{gh_j\check{m}^{-j}, gm^{-j} : j \in \{1, \ldots, J\}\right\} \subset S_g, \tag{3}$$

which is obtained from S_g (see equation (2)) by removing the markings $gh_j\check{m}^{-j}$ and gm^{-j} for each $j \in \{1, \ldots, J\}$. Using the inclusion map, the open subset $S'_g \subset S_g$ can be considered as a connected open subset of S. Then, the closure $\overline{S'_g}$ of S'_g in S is complete. If we take a Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in S and the real number $\varepsilon = \frac{1}{2\sqrt{2}}$, then there is a positive integer $N(\varepsilon) \in \mathbb{N}$ such that for all natural numbers $m, n \geq N(\varepsilon)$, the terms x_m, x_n satisfy $\widehat{d}(x_m, x_n) < \varepsilon$. By Lemma 1, there is $g \in G$ such that the open ball $B_\varepsilon(x_{N(\varepsilon)})$ is contained in $\overline{S'_g}$. Since $\overline{B_\varepsilon(x_{N(\varepsilon)})} \subset \overline{S'(g)}$ is complete, the Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ converges within $\overline{B_\varepsilon(x_{N(\varepsilon)})}$. The discreteness of the singularities follows immediately from Lemma 1. □

Lemma 3. *The Veech group of S is G.*

Proof. Given that the group G acts on $\mathfrak{P}(1, G, H) := \{S_g : g \in G\}$ by post-composition on charts, then if we fix a matrix $\tilde{g} \in G$, for each $g \in G$, there exists a natural affine diffeomorphism $f_{\tilde{g}g} : S_g \to S_{\tilde{g}g}$, satisfying the following properties:

(1) The differential of $f_{\tilde{g}g}$ is the matrix $\tilde{g}$.

(2) The map $f_{\tilde{g}g}$ sends parallel markings to parallel markings.

Hence, the map $f : \bigcup_{g\in G} S_g \to \bigcup_{g\in G} S_{\tilde{g}g}$ defined by $f|_{S_g} := f_{\tilde{g}g}$, is a gluing markings-preserving map. This yields an affine diffeomorphism in the quotient $F_{\tilde{g}} : S \to S$ with differential matrix $\tilde{g}$. Thus, we conclude that $G < \Gamma(S)$. Conversely, we consider $f : S \to S$ an affine orientation preserving diffeomorphism different from the identity. From Remark 4, for each $g \in G$, the surface S_g has one singularity of angle 6π, which is denoted by $\tilde{\mathbf{0}}_g$. There are only three saddle connections γ_1^g, γ_2^g, and γ_3^g such that each one of them has that singularity as one of their endpoints. The holonomy vectors associated to these saddle connections are $\{\pm g \cdot e_1, \pm g \cdot e_2\}$. The function f sends the singularity $\tilde{\mathbf{0}}_{\mathrm{Id}}$ to the singularity $\tilde{\mathbf{0}}_g$ for some $g \in G$, and the differential matrix df of f must map $\{\pm e_1, \pm e_2\}$ to $\{\pm g \cdot e_1, \pm g \cdot e_2\}$. The only possibility is that $df = g$. Thus, we conclude that $\Gamma(S) < G$. □

4.4. Ends space of the surface S

The description of the ends space of S, as stated in Theorem 1, follows from the following lemmas.

Lemma 4. *If G is finite, then the surface S has as many ends as there are elements in the group G, and each end has infinite genus.*

Lemma 5. *If G is not finite, then the ends space of S can be represented in the form*

$$\mathrm{Ends}(S) = \mathrm{Ends}_\infty(S) = \mathcal{B} \sqcup \mathcal{U},$$

where $\mathcal{B}$ is a closed subset of $\mathrm{Ends}(S)$ *homeomorphic to* Ends(G)*, and $\mathcal{U}$ is a countable, dense, and open subset of* $\mathrm{Ends}(S)$*.*

Proof of Lemma 4

The group G has cardinality k for some $k \in \mathbb{N}$. Let K be a compact subset of S; we must prove that there exists a compact subset $K' \subset S$ such that $K \subset K'$, and $S \setminus K'$ consists of k open connected components, each one of them having infinite genus.

For each $g \in G$, the affine copy S_g is homeomorphic to the Loch Ness monster (see equation (2)). Since the generating set H of G is finite, the set of markings

$$\left\{gh_j\check{m}^{-j}, gm^{-j} : j \in \{1, \dots, J\}\right\}$$

on the affine copy S_g is finite. We consider the connected subsurface S'_g of S_g as in equation (3), which has the following properties:

(1) This subsurface S'_g has infinite genus, and via the inclusion map, it can be considered as a connected subsurface of S with infinite genus.

(2) The boundary $\partial S'_g$ of S'_g in S is compact because it is conformed by a finitely many disjoint closed curves.

As G is finite, from the preceding properties we hold that the set

$$S \setminus \bigcup_{g\in G} \partial S'_g = \bigcup_{g\in G} S'_g$$

consists of k open connected components, and each one of them has infinite genus.

On the other hand, let K_g be the closure of the set $K \cap S'_g$ in S_g for each $g \in G$. As K_g is a compact subset of S_g,0 there exists a compact subset $K'_g \subset S_g$ such that

$$K_g \cup \left\{gh_j\check{m}^{-j}, gm^{-j} : j \in \{1, \dots, J\}\right\} \subset K'_g,$$

and $S_g \setminus K'_g$ consist of an open connected with infinite genus. We take K' to be the closure of

$$\bigcup_{g\in G} \left(K'_g \setminus \{gh_j\check{m}^{-j}, gm^{-j} : j \in \{1, \dots, J\}\}\right)$$

in S. As G is finite, then K' is a compact subset of S. By construction, we hold that $K \subset K'$, and the set

$$S \setminus K' = \bigcup_{g\in G} (S_g \setminus K'_g) \subset \bigcup_{g\in G} S'_g$$

consists of k open connected components and each one of them having infinite genus. □

Proof of Lemma 5

The sketch of the proof is the following. We begin by defining the set $\mathcal{U}$ from the ends of the affine copies S_g, and we will prove that it is a countable, discrete, and open subset of Ends(S). Then, we shall give an appropriate embedding i_* from

Ends(G) to Ends(S), where the image of Ends(G) under i_* will be denoted by $\mathcal{B}$. By using an embedding from the Cayley graph Cay(G,H) to the surface S, we shall establish the equality

$$\text{Ends}(S) = \text{Ends}_\infty(S) = \mathcal{B} \sqcup \mathcal{U},$$

where $\mathcal{B}$ is closed, and $\mathcal{U}$ is a dense and open subset of Ends(S).

Step 1. The set $\mathcal{U}$

For each $g \in G$, we take the subsurface $S'_g \subset S_g$ defined in equation (3). Recall that the boundary $\partial S'_g$ of the subsurface S'_g is compact because it consists of finitely many disjoint closed curves. Let $[U(g)_n]_{n\in\mathbb{N}}$ be the unique end of the Loch Ness monster S_g. Without loss of generality, we can assume that $U(g)_n \subset S'_g$ for each $n \in \mathbb{N}$. From the inclusion map, the surface S'_g can be considered as a subsurface of S. Then the sequence $(U(g)_n)_{n\in\mathbb{N}}$ of S_g defines an end with infinite genus of the surface S.

Remark 5. *For any two different $g \neq \tilde{g} \in G$, the subsurfaces S'_g and $S'_{\tilde{g}}$ of S are disjoint.*

From the previous remark, we obtain the countable set $\mathcal{U}$ conformed by different ends of S given by

$$\mathcal{U} := \{[U(g)_n]_{n\in\mathbb{N}} \in \text{Ends}(S) : g \in G\} \subset \text{Ends}(S). \tag{4}$$

Let us note that the subset $\mathcal{U} \subset \text{Ends}(S)$ is both discrete and open. This is a consequence of the following fact. For each $g \in G$, the open subset $U(g)_1$ of S has a compact boundary $\partial U(g)_1$ in S. Thus, we define the open subset $(U(g)_1)^*$ of Ends(S), which satisfies

$$(U(g)_1)^* \cap \mathcal{U} = \{[U(g)_n]_{n\in\mathbb{N}}\}.$$

Step 2. The embedding $i_* : \text{Ends}(G) \hookrightarrow \text{Ends}(S)$

Let $\overline{S'_g}$ be the closure in S of the surface S'_g (see equation (3)). Given a non-empty connected open subset W of Cay(G,H) with compact boundary ∂W, we can suppose, without loss of generality, that the boundary $\partial W \subset V(\text{Cay}(G,H)) = G$. We then define the subset $\tilde{W} \subset S$ given by

$$\tilde{W} := \text{Int}\left(\bigcup_{g\in G\cap(W\cup\partial W)} \overline{S'_g}\right) \subset S. \tag{5}$$

This set $\tilde{W}$ is a non-empty, connected, and open subset of S with a compact boundary. Moreover, it is a subsurface of S with infinite genus. In the following remark, we state two properties of this object, which can be easily deduced.

Remark 6. *Given that W and V are two non-empty, connected, and open subsets of* Cay(G,H)*, each one having compact boundaries ∂W and ∂V, respectively, such that $\partial W, \partial V \subset G$, then*

(1) *If* $W \supset V$, *then* $\tilde{W} \supset \tilde{V}$.

(2) *If* $W \cap V = \emptyset$, *then* $\tilde{W} \cap \tilde{V} = \emptyset$.

From the above remark, the end $[W_n]_{n\in\mathbb{N}}$ of the group G naturally defines the end $[\tilde{W}_n]_{n\in\mathbb{N}}$ of the surface S, which has infinite genus. Hence, we obtain a well-defined map $i_* : \mathrm{Ends}(G) \to \mathrm{Ends}(S)$ given by

$$[W_n]_{n\in\mathbb{N}} \mapsto [\tilde{W}_n]_{n\in\mathbb{N}}. \tag{6}$$

Claim 1. *The map* i_* *is an embedding.*

Proof. We must show that i_* is *injective*. Let $[W_n]_{n\in\mathbb{N}}$ and $[V_n]_{n\in\mathbb{N}}$ be two different ends of G. Then, there is $l \in \mathbb{N}$ such that $W_l \cap V_l = \emptyset$. By item (2) of Remark 6, it follows that $\tilde{W}_l \cap \tilde{V}_l = \emptyset$. It proves that the ends $i_*([W_n]_{n\in\mathbb{N}}) = [\tilde{W}_n]_{n\in\mathbb{N}}$ and $i_*([V_n]_{n\in\mathbb{N}}) = [\tilde{V}_n]_{n\in\mathbb{N}}$ in $\mathrm{Ends}(S)$ are different.

Continuity. We consider an end $[W_n]_{n\in\mathbb{N}}$ of the group G and an open subset $V \subset S$ with a compact boundary such that $i_*([W_n]_{n\in\mathbb{N}}) = [\tilde{W}_n]_{n\in\mathbb{N}} \in V^* \subset \mathrm{Ends}(S)$. We must prove that there is a neighborhood $Z^* \subset \mathrm{Ends}(G)$ of $[W_n]_{n\in\mathbb{N}}$ such that $i_*(Z^*) \subset V^*$. Given that $[\tilde{W}_n]_{n\in\mathbb{N}} \in V^*$, there exists some $k \in \mathbb{N}$ such that

$$\tilde{W}_k \subset V.$$

We take the open subset W_k of the Cayley graph $\mathrm{Cay}(G, H)$, which defines the open subset $\tilde{W}$ (see equation (5)), and consider the open

$$Z^* := (W_k)^*$$

of $\mathrm{Ends}(G)$, which is a neighborhood of $[W_n]_{n\in\mathbb{N}}$. To ensure that $i_*(Z^*) \subset V^*$, we consider any end $[U_n]_{n\in\mathbb{N}} \in \mathrm{Ends}(G)$ such that $[U_n]_{n\in\mathbb{N}} \in Z^* = (W_k)^*$, and check that $i_*([U_n]_{n\in\mathbb{N}}) = [\tilde{U}_n]_{n\in\mathbb{N}} \in V^*$. Since $U_m \subset W_k$ for some $m \in \mathbb{N}$, it follows from item (1) of Remark 6 that

$$\tilde{U}_m \subset \tilde{W}_k.$$

As $\tilde{W}_k \subset V$, we conclude that $\tilde{U}_m \subset V$, which implies that $i_*([U_n]_{n\in\mathbb{N}}) = [\tilde{U}_n]_{n\in\mathbb{N}} \in V^*$.

Finally, the map i_* is *closed* because any continuous map from a compact space to a Hausdorff space is closed. Therefore, i_* is an embedding. □

We denote the image of the map i_* as

$$\mathcal{B} := i_*(\mathrm{Ends}(G)).$$

From the definition of the set $\mathcal{U}$ given in equation (4), we conclude that $\mathcal{B} \cap \mathcal{U} = \emptyset$, and $\mathcal{B} \sqcup \mathcal{U} \subset \mathrm{Ends}(S)$.

Step 3. The embedding $i : \mathrm{Cay}(G, H) \hookrightarrow S$

We now describe the image of each vertex and edge of $\mathrm{Cay}(G, H)$ under the map i.

For each $g \in G$, let $\overline{\mathbf{0}}_g$ denote the point in the affine copy S_g that corresponds to the image of the point $\overline{\mathbf{0}}$ (see equation (2)) in the decorated surface S_{dec} via the affine diffeomorphism $f_g : S_{\mathrm{dec}} \to S_g$. Then the surface S'_g described in equation (3) contains the point $\overline{\mathbf{0}}_g$. Thus, we define the map $h : V(\mathrm{Cay}(G, H)) = G \to S$ given by

$$g \mapsto \overline{\mathbf{0}}_g. \tag{7}$$

On the other hand, for each $j \in \{1, \ldots, J\}$, there is a simple polygonal path $\beta_j : [0, 1] \to S$ satisfying the following properties:

(1) The initial and terminal points of β_j are $\overline{\mathbf{0}}_{\mathrm{Id}}$ and $\overline{\mathbf{0}}_{h_j}$, respectively. See Figure 4.

(2) For each $i \neq j \in \{1, \ldots, J\}$, the intersection $\beta_i([0, 1]) \cap \beta_j([0, 1]) = \{\overline{\mathbf{0}}_{\mathrm{Id}}\}$.

Since the edge (Id, h_j) of the Cayley graph $\mathrm{Cay}(G, H)$ is homeomorphic to the open interval $(0, 1)$, we can suppose, without loss of generality, that the curve β_j is defined from $[Id, h_j]$ to S such that $\beta_j(\mathrm{Id}) = \overline{\mathbf{0}}_{\mathrm{Id}}$ and $\beta_j(h_j) = \overline{\mathbf{0}}_{h_j}$. Given that the Veech group of the surface S is G, for each $g \in G$, there is an affine diffeomorphism $f_g : S \to S$ whose differential is $df_g = g$. Thus, we get the composition path

$$f_g \circ \beta_j : [0, 1] \to S, \tag{8}$$

satisfying the following properties:

(1) The initial and terminal points of $f_g \circ \beta_j$ are $\overline{\mathbf{0}}_g$ and $\overline{\mathbf{0}}_{gh_j}$, respectively.

(2) For each $i \neq j \in \{1, \ldots, J\}$, the intersection $f_g \circ \beta_i([0, 1]) \cap f_g \circ \beta_j([0, 1]) = \{\overline{\mathbf{0}}_g\}$.

Similarly, since the edge (g, gh_j) of the Cayley graph $\mathrm{Cay}(G, H)$ is homeomorphic to the open interval $(0, 1)$, we can suppose, without loss of generality, that the composition path $f_g \circ \beta_j$ is defined from $[g, gh_j]$ to S such that $f_g \circ \beta_j(g) = \overline{\mathbf{0}}_g$ and $f_g \circ \beta_j(gh_j) = \overline{\mathbf{0}}_{gh_j}$.

From equations (7) and (8), we obtain the embedding

$$i : \mathrm{Cay}(G, H) \hookrightarrow S, \tag{9}$$

such that $i_{|G} := h$ and $i_{|[g,gh_j]} := f_g \circ \beta_j$ for each $g \in G$ and $j \in \{1, \ldots, J\}$.

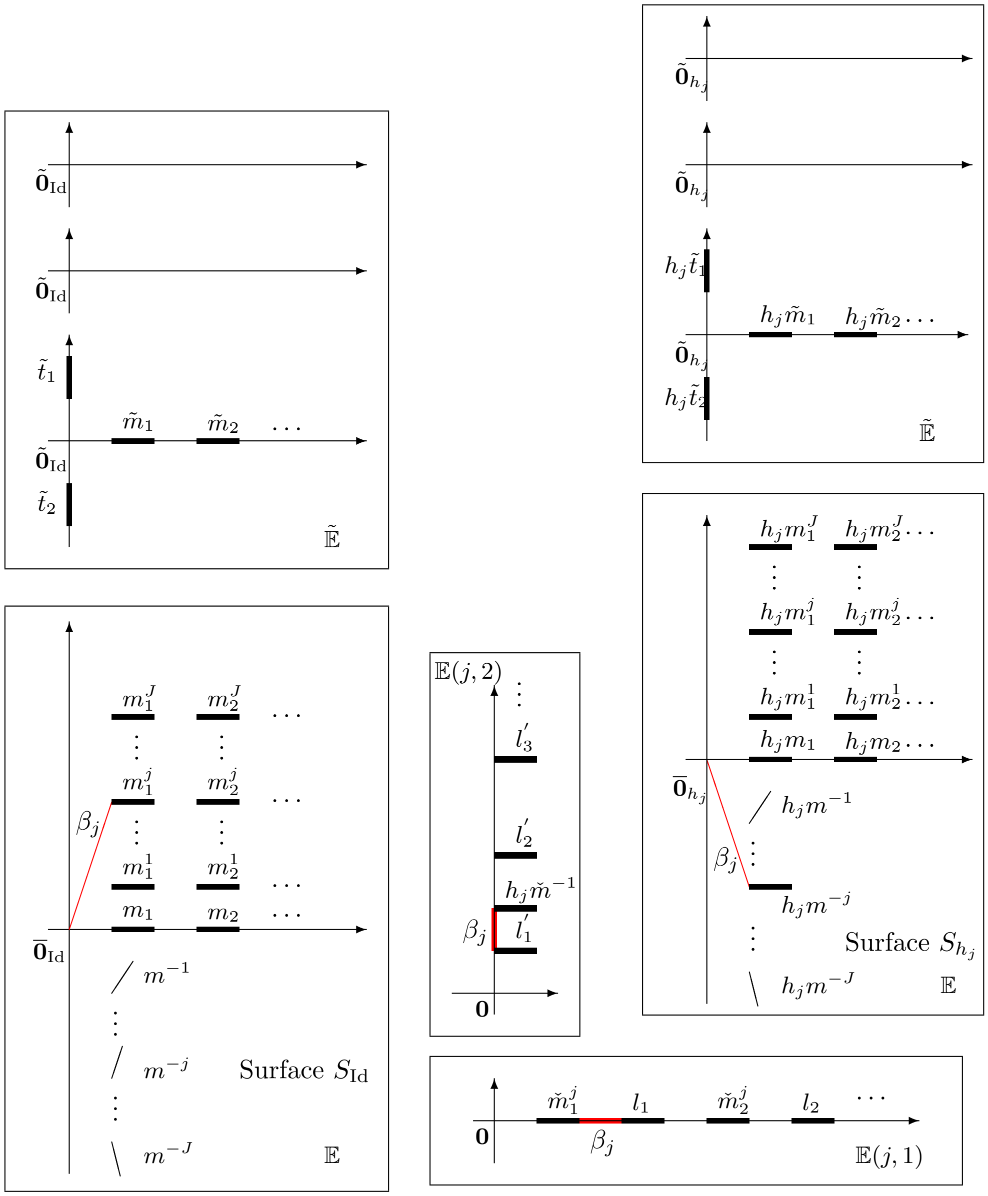

Figure 4: *Image of* β_j

Step 4. The equality $\mathrm{Ends}(S) = \mathcal{B} \sqcup \mathcal{U}$

We must prove that $\mathrm{Ends}(S) \subset \mathcal{B} \sqcup \mathcal{U}$. Let $[U_n]_{n\in\mathbb{N}}$ be an end of S. Since $S = \bigcup_{g\in G} \overline{S'_g}$, for each $n \in \mathbb{N}$, we consider the subset

$$G(n) = \left\{ g \in G : \overline{S'_g} \cap U_n \neq \emptyset \right\} \subset G,$$

and define the open subset

$$Z_n := \mathrm{Int}\left(\bigcup_{g \in G(n)} \overline{S'_g}\right) \subset S,$$

which has the following properties:

(1) Since U_n is a non-empty, connected, and open subset of S with a compact boundary, the set Z_n is also a connected and open subset of S with a compact boundary for each $n \in \mathbb{N}$.

(2) As $U_n \supset U_{n+1}$, it follows that $Z_n \supset Z_{n+1}$ for each $n \in \mathbb{N}$.

Using the definition of an end and the construction of Z_n, it is easy to show that the sequences $(Z_n)_{n\in\mathbb{N}}$ and $(U_n)_{n\in\mathbb{N}}$ define the same end of S. In other words, $[U_n]_{n\in\mathbb{N}} = [Z_n]_{n\in\mathbb{N}}$. We shall now prove that the end $[Z_n]_{n\in\mathbb{N}}$ belongs to $\mathcal{B} \sqcup \mathcal{U}$. We notice that one of the following cases must occur:

Case 1. There is $N \in \mathbb{N}$ such that $G(N)$ is finite. Then there exists $g \in G$ such that for all $m \geq N$ we hold

$$Z_m \subset S'_g.$$

This implies that the sequences $(Z_n)_{n\in\mathbb{N}}$ and $(U(g)_n)_{n\in\mathbb{N}}$ must be equivalent (see equation (4)). Thus, $[U_n]_{n\in\mathbb{N}} \in \mathcal{U}$.

Case 2. Otherwise, for each $n \in \mathbb{N}$, the subset $G(n) \subset G$ is infinite. As the embedding i described in equation (9) is a continuous map, the inverse image

$$\hat{Z}_n := i^{-1}\left(Z_n \cap i(\mathrm{Cay}(G, H))\right)$$

is a connected and open subset of $\mathrm{Cay}(G, H)$ with a compact boundary for each $n \in \mathbb{N}$. Moreover, the sequence $(\hat{Z}_n)_{n\in\mathbb{N}}$ defines an end of the group G. By the construction of the sequence $(Z_n)_{n\in\mathbb{N}}$ of S, the embedding i_* defined in (6) sends the end $[\hat{Z}_n]_{n\in\mathbb{N}}$ of G to the end $[Z_n]_{n\in\mathbb{N}}$ of S. This implies that $[Z_n]_{n\in\mathbb{N}}$ belongs to $\mathcal{B}$. Thus, we conclude that $\mathrm{Ends}(S) = \mathcal{B} \sqcup \mathcal{U}$.

Step 5. The set $\mathcal{B}$ is closed and the set $\mathcal{U}$ is dense and open

Since $\mathcal{U}$ is an open subset of $\mathrm{Ends}(S)$, its complement $\mathrm{Ends}(S) \setminus \mathcal{U} = \mathcal{B}$ is a closed subset of $\mathrm{Ends}(S)$. We shall prove that $\mathcal{U}$ is dense. Let $[Z_n]_{n\in\mathbb{N}}$ be an end of $\mathcal{B}$. We must show that this end belongs to the closure of $\mathcal{U}$.

Let U be a non-empty, connected, and open subset of S with a compact boundary such that the open subset $U^* \subset \mathrm{Ends}(S)$ contains the end $[Z_n]_{n\in\mathbb{N}}$. There exists $\tilde{g} \in \{g \in G : \overline{S'_g} \cap U \neq \emptyset\}$ such that $S'_{\tilde{g}} \subset U$. This condition implies that the end $[U(\tilde{g})_n]_{n\in\mathbb{N}}$ of $\mathcal{U}$ belongs to U^*. Therefore, the end $[Z_n]_{n\in\mathbb{N}}$ is in the closure of $\mathcal{U}$. □

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