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Exploring the configurations $(12_4, 16_3)$ of Václav Metelka in a cubic structure

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Abstract. In this paper, we investigate four configurations $(12_4, 16_3)$ in cubic structures, introduced by Václav Metelka, and discover new unexpected extra collinearities in two Metelka's configurations that arise from our realizations in cubic structures. Next, we establish the existence of various geometric concepts in four Metelka's configuration including tangential points, inflection points, sextactic points, and quadrilaterals, and find and study significant relationships among these geometric concepts.

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1. Introduction

This study examines the configuration $(12_4, 16_3)$ within a cubic structure. A configuration (v_r, b_k) is a finite incidence structure characterized by the following properties:

(i) It consists of v points and b lines.

(ii) There are k points on each line and r lines through each point.

(iii) Two distinct lines meet at most once, and two distinct points are connected by at most one line.

The configuration $(12_4, 16_3)$ presents 12 points each contained in 4 lines, and 16 lines each containing 3 points.

The motivation for this paper arises from the findings presented in [5]. In that paper, four configurations $(12_4, 16_3)$ were obtained whose points can lie on a single cubic curve in the projective plane. These configurations are labeled S_1, S_2, S_3, and S_4, and their points are consistently labeled 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, P, and Q, and they are all different. Attempting to inscribe a specific configuration into an elliptic curve led Metelka to notice the presence of additional collinearities. Some historical facts and realizations of configurations $(12_4, 16_3)$ were discussed in [2] and [3]. Metelka's papers ([5, 6]) about configurations $(12_4, 16_3)$ had a significant impact on the first authors' decision to consider these configurations. According to [2], O. Hesse constructed the initial configuration $(12_4, 16_3)$ in 1848. Later, J. de Vries (1889), B. Bydžovský (1939), J. Metelka (1944), and M. Zacharias (1948) built four more variants. After these initial isolated structures, Czechoslovak researchers including B. Bydžovský, J. Metelka, and J. Novák constructed over 200 configurations. In [2], Gropp found that the number of non-isomorphic combinatorial configurations $(12_4, 16_3)$ is equal to 574. The question of how many of them can be realized as geometric configurations is still open. The method introduced in this paper could help answer the question of embeddings into cubic curves, i.e., more generally, into a cubic structure. A number of authors emphasize the importance of Metelka's configurations ([1, 4]). In the present paper, we explore the properties of geometric realization of four Metelka's configurations within cubic structure, and obtain several novel collinearities in the geometric realization of two of them. But firstly, we discuss some fundamental notions related to cubic structures.

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The cubic structure is introduced in [7]. Let Q be a nonempty set whose elements are called *points*, and let $[\ ] \subseteq Q^3$ be a ternary relation on Q. Such a relation and the ordered pair $(Q, [\ ])$ will be called a *cubic relation* and a *cubic structure*, respectively, provided the following conditions hold:

C1. For any two points $a, b \in Q$, there is a unique point $c \in Q$ such that $[a, b, c]$ (i.e., $(a, b, c) \in [\ ]$).

C2. The relation $[\ ]$ is totally symmetric (i.e., $[a, b, c]$ implies $[a, c, b]$, $[b, a, c]$, $[b, c, a]$, $[c, a, b]$, and $[c, b, a]$).

C3. $[a, b, c]$, $[d, e, f]$, $[g, h, i]$, $[a, d, g]$, and $[b, e, h]$ imply $[c, f, i]$, which can be written in the form of the following table:

$$\begin{array}{cc|c|} a & b & c \\ d & e & f \\ g & h & i \end{array}\ .$$

In [7], various examples of cubic structures are provided, including one noteworthy case where Q is the set of all non-singular points of a cubic curve in the plane. The notation $[a, b, c]$ indicates that points $a, b, c \in Q$ are collinear in this context. Therefore, in a general cubic structure $(Q, [\ ])$, if $[a, b, c]$ holds true, we will say that points a, b, and c form a *line*. Otherwise, we say that (a, b, c) is a *triangle*.

The concept of tangentials of points was introduced in [8]. The point a' is called the *tangential* of the point a if $[a, a, a']$ holds. If a' is the tangential of the point a, then we will also say that the point a is an antecedent of the point a'. Obviously, every point has one and only one tangential a'. The tangential a'' of the tangential a' of a point a is called its *second tangential*. Two points are said to be *corresponding* if they have the common tangential. Given a point x, we will always denote its tangential and the second tangential x' and x''. The following statements are proved in [7].

Lemma 1. *If a', b' and c' are the tangentials of the points a, b, and c, then $[a, b, c]$ implies $[a', b', c']$.*

Lemma 2. *If $[a, b, c]$, $[a, e, f]$, $[b, f, d]$, and $[c, d, e]$, then a, d; b, e; and c, f are pairs of corresponding points, and the associated tangentials a', b', and c' satisfy $[a', b', c']$.*

For any integer n greater than 1, the nth tangential of a point is defined as the tangential of its $(n-1)$th tangential, with the first tangential of the point a denoted a'. An *inflection* point in a cubic structure is defined as a point a that satisfies $[a, a, a]$, indicating that the point a is self-tangential. A *sextactic* point is defined as the antecedent of an inflection point.

The points a, b, c, d, e, f are said to form a *quadrilateral* $\{a, d; b, e; c, f\}$ if there exist lines $[a, b, c]$, $[a, e, f]$, $[d, b, f]$, and $[d, e, c]$. We say that the points from each pair of points a, d; b, e; and c, f are *opposite*.

2. Configuration S_1

Metelka's configuration S_1 has the following sixteen lines:

$$\begin{gathered} [0,1,7],\ [0,2,4],\ [0,3,P],\ [0,5,9],\ [1,5,Q],\ [1,6,9],\ [1,8,P],\ [2,3,9], \\ [2,6,8],\ [2,P,Q],\ [3,4,8],\ [3,7,Q],\ [4,5,P],\ [4,6,Q],\ [5,6,7],\ [7,8,9]. \end{gathered} \tag{1}$$

It follows from the table

$$\begin{array}{cc|c|} 5 & 9 & 0 \\ 6 & 2 & 8 \\ 7 & 3 & Q \end{array}$$

that there exists a line $[0, 8, Q]$, which is not present in the previous configuration, i.e., this is an "extra" line. Since there are lines $[0, 1, 7]$, $[0, 5, 9]$, $[6, 1, 9]$, and $[6, 5, 7]$, as well as the lines $[0, 4, 2]$, $[0, 8, Q]$, $[6, 4, Q]$, and $[6, 8, 2]$, there exist quadrilaterals $\{0, 6; 1, 5; 7, 9\}$ and $\{0, 6; 4, 8; 2, Q\}$. According to Lemma 2, the pairs of points $0, 6$; $1, 5$; $7, 9$; $4, 8$; and $2, Q$ have the same tangentials, i.e., the equalities $0' = 6'$, $1' = 5'$,

$7' = 9'$, $4' = 8'$, and $2' = Q'$ hold. What conclusions may be drawn about these tangentials, and what can we conclude about the tangentials $3'$ and P' related to the points 3 and P?

From the tables

$$\begin{array}{cc|c|} \cline{3-3} 1 & 7 & 0 \\ P & 3 & 0 \\ 8 & Q & 0 \\ \cline{3-3} \end{array}\ , \qquad \begin{array}{cc|c|} \cline{3-3} 2 & 8 & 6 \\ Q & 4 & 6 \\ P & 3 & 0 \\ \cline{3-3} \end{array}$$

we get $[0,0,0]$ and $[6,6,0]$, i.e., the equality $0' = 6' = 0$ holds true, which means that the point 0 is an inflection point, and the point 6 is a sextactic point. From tables

$$\begin{array}{cc|c|} \cline{3-3} 1 & Q & 5 \\ 6 & 7 & 5 \\ 9 & 3 & 2 \\ \cline{3-3} \end{array}\ , \qquad \begin{array}{cc|c|} \cline{3-3} 1 & 6 & 9 \\ 8 & 7 & 9 \\ P & 5 & 4 \\ \cline{3-3} \end{array}$$

we obtain the equalities $1' = 5' = 2$ and $7' = 9' = 4$, while tables

$$\begin{array}{cc|c|} \cline{3-3} 0 & P & 3 \\ 9 & 2 & 3 \\ 5 & Q & 1 \\ \cline{3-3} \end{array}\ , \qquad \begin{array}{cc|c|} \cline{3-3} 1 & 8 & P \\ 5 & 4 & P \\ Q & 3 & 7 \\ \cline{3-3} \end{array}$$

prove the equalities $3' = 1$ and $P' = 7$. Let us denote the common tangentials of the points 4, 8, and 2, Q, U and V, respectively. This is shown in Table 1:

Point	0	1	2	3	4	5	6	7	8	9	P	Q
Tangential	0	2	V	1	U	2	0	4	U	4	7	V

Table 1: Points and their tangentials

Therefore, according to Lemma 1, we get the implications:

$$\begin{gathered} [0,2,4] \Rightarrow [0,V,U],\ [1,8,P] \Rightarrow [2,U,7],\ [2,3,9] \Rightarrow [V,1,4], \\ [2,P,Q] \Rightarrow [V,7,V] \Rightarrow V' = 7 \Rightarrow V'' = 4 \Rightarrow V''' = U, \\ [3,4,8] \Rightarrow [1,U,U] \Rightarrow U' = 1 \Rightarrow U'' = 2 \Rightarrow U''' = V. \end{gathered}$$

From the tables

$$\begin{array}{cc|c|} \cline{3-3} 0 & P & 3 \\ 4 & Q & 6 \\ 2 & 2 & V \\ \cline{3-3} \end{array}\ , \qquad \begin{array}{cc|c|} \cline{3-3} 2 & 8 & 6 \\ 0 & 3 & P \\ 4 & 4 & U \\ \cline{3-3} \end{array}$$

we get $[3,6,V]$ and $[6,P,U]$. Thus, we proved the following theorem.

Theorem 1. *In the geometric realization within a cubic structure of Metelka's configuration S_1 with lines* (1)*, there is an extra line* $[0,8,Q]$*. The tangentials of its points are displayed in Table 1, where points U and V satisfy* $[0,U,V]$, $[2,7,U]$, $[6,P,U]$, $[1,4,V]$, *and* $[3,6,V]$. *Each of these two points is the third tangential of the other one. The point* 0 *is an inflection point, the point* 6 *is a sextactic point and there are quadrilaterals* $\{0,6;1,5;7,9\}$ *and* $\{0,6;4,8;2,Q\}$.

Remark 1. In the combinatorial configuration, the set of lines is fixed by definition. However, when such a structure is realized geometrically in the plane, additional incidences may arise. Some triples of points not intended to be collinear in the abstract incidence structure may nevertheless lie on a common line. Such lines are referred to as extra lines, and they belong to the geometry of the realization rather than to the combinatorial configuration itself. So, in the geometric realization of Metelka's S_1 configuration, within a cubic structure, we find an extra line.

3. Configuration S_2

Metelka's configuration S_2 has the following sixteen lines:

$$\begin{gathered}[0,1,Q],\ [0,2,8],\ [0,5,P],\ [0,6,7],\ [1,5,8],\ [1,6,P],\ [1,7,9],\ [2,3,7],\\ [2,6,Q],\ [2,4,9],\ [3,4,8],\ [3,5,9],\ [3,P,Q],\ [4,5,6],\ [4,7,Q],\ [8,9,P].\end{gathered} \tag{2}$$

From the tables

$$\begin{array}{cc|c|} \cline{3-3} 1 & Q & 0 \\ 8 & 2 & 0 \\ 5 & 6 & 4 \\ \cline{3-3} \end{array}, \quad \begin{array}{cc|c|} \cline{3-3} 0 & Q & 1 \\ 6 & P & 1 \\ 7 & 3 & 2 \\ \cline{3-3} \end{array}, \quad \begin{array}{cc|c|} \cline{3-3} 4 & 8 & 3 \\ 5 & 9 & 3 \\ 6 & P & 1 \\ \cline{3-3} \end{array}$$

we get the equalities $0' = 4$, $1' = 2$, and $3' = 1$. Now we will repeatedly apply Lemma 1. From $[0,1,Q]$ we get $[4,2,Q']$, which due to $[4,2,9]$ implies $Q' = 9$. From $[3,P,Q]$ we get $[1,P',9]$, which together with $[1,7,9]$ results in $P' = 7$. From $[1,6,P]$ we derive $[2,6',7]$, and due to $[2,3,7]$ we obtain $6' = 3$. From $[2,6,Q]$ we get $[2',3,9]$, which due to $[5,3,9]$ implies $2' = 5$. From $[0,2,8]$ we get $[4,5,8']$, which due to $[4,5,6]$ implies $8' = 6$. From $[0,5,P]$ we get $[4,5',7]$, which because of $[4,Q,7]$ implies $5' = Q$. $[8,9,P]$, $[4,5,6]$, and $[0,6,7]$ imply $[6,9',7]$, $[4',Q,3]$, and $[4,3,7']$ and due to $[6,0,7]$, $[P,Q,3]$, and $[4,3,8]$ we get $9' = 0$, $4' = P$, $7' = 8$. This proves:

Theorem 2. *In the geometric realization within a cubic structure of Metelka's configuration S_2 with lines (2), there is a cycle of successive tangentials*

$$0, 4, P, 7, 8, 6, 3, 1, 2, 5, Q, 9.$$

4. Configuration S_3

Metelka's configuration S_3 has the following sixteen lines:

$$\begin{gathered}[0,2,P],\ [0,3,5],\ [0,6,8],\ [0,9,Q],\ [1,2,3],\ [1,4,P],\ [1,5,8],\ [1,6,9],\\ [2,4,5],\ [2,6,7],\ [3,4,9],\ [3,7,8],\ [4,7,Q],\ [5,6,Q],\ [7,9,P]\ ,[8,P,Q].\end{gathered} \tag{3}$$

From the tables

$$\begin{array}{cc|c|} \cline{3-3} 2 & P & 0 \\ 6 & 8 & 0 \\ 7 & Q & 4 \\ \cline{3-3} \end{array} \quad \begin{array}{cc|c|} \cline{3-3} 2 & 3 & 1 \\ P & 4 & 1 \\ 0 & 9 & Q \\ \cline{3-3} \end{array} \quad \begin{array}{cc|c|} \cline{3-3} 1 & 3 & 2 \\ P & 0 & 2 \\ 4 & 5 & 2 \\ \cline{3-3} \end{array} \quad \begin{array}{cc|c|} \cline{3-3} 4 & 9 & 3 \\ 2 & 1 & 3 \\ 5 & 6 & Q \\ \cline{3-3} \end{array}$$

one gets the equalities $0' = 4$, $1' = Q$, $2' = 2$, and $3' = Q$. It follows that $Q' = 2$ since by Lemma 1 we get $[Q,2,Q]$ from [1, 2, 3]. Similarly, from $[0,2,P]$, $[0,3,5]$, and $[0,9,Q]$, we obtain $[4,2,P']$, $[4,Q,5']$, and $[4,9',2]$, which compared with the statements $[4,2,5]$, $[4,Q,7]$, and $[4,5,2]$, give $P' = 5$, $5' = 7$, and $9' = 5$, respectively. From $[1,6,9]$, $[2,4,5]$, and $[8,P,Q]$, we derive $[Q,6',5]$, $[2,4',7]$, and $[8',5,2]$. When compared with $[Q,6,5]$, $[2,6,7]$, and $[4,5,2]$, these lead to the equalities $6' = 6$, $4' = 6$, and $8' = 4$. Finally, we deduce that $[7,9,P]$ implies $[7',5,5]$, which leads to the conclusion that $7' = 5' = 7$, and we obtain Table 2:

Point	0	1	2	3	4	5	6	7	8	9	P	Q
Tangential	4	Q	2	Q	6	7	6	7	4	5	5	2

Table 2: Points and their tangentials

Thus, the points 2, 6, and 7 are three inflection points on a single line $[2,6,7]$, while the points Q, 4, and 5 are sextactic points, for which these inflection points, in that order, serve as tangentials. However, these sextactic points are not on a single line; instead, we have lines $[2,4,5]$, $[Q,6,5]$, and $[Q,4,7]$. This indicates the presence of the quadrilateral $\{2,Q;6,4;7,5\}$, proving

Theorem 3. *In the geometric realization within a cubic structure of Metelka's configuration S_3 with lines (3), the points 2, 6, and 7 are inflection points lying on a single line, while the points Q, 4, and 5 are sextactic points. Together with the inflection points, they form the quadrilateral $\{2,Q;6,4;7,5\}$. The pairs of points $1,3$; $0,8$; and $9,P$ have for common tangentials, these sextactic points Q, 4, and 5, respectively.*

Theorem 4. *In the geometric realization within a cubic structure of Metelka's configuration S_3, there exist three extra lines:* $[2, 8, 9]$*,* $[0, 1, 7]$*, and* $[3, 6, P]$*.*

Proof. The statement follows from the tables

$$\begin{array}{cc|c|}\cline{3-3} 2 & 2 & 2 \\ 3 & 7 & 8 \\ 1 & 6 & 9 \\ \cline{3-3}\end{array}\,, \quad \begin{array}{cc|c|}\cline{3-3} 2 & P & 0 \\ 6 & 9 & 1 \\ 7 & 7 & 7 \\ \cline{3-3}\end{array}\,, \quad \begin{array}{cc|c|}\cline{3-3} 0 & 5 & 3 \\ 6 & 6 & 6 \\ 8 & Q & 5 \\ \cline{3-3}\end{array}\,. \qquad \square$$

The facts we proved here regarding the configuration S_3 were also established by V. Metelka in the context of the cubic curve in [6], albeit with different point notations.

5. Configuration S_4

The configuration S_4 includes the following sixteen lines:

$$\begin{gathered}[0, 2, P],\ [0, 3, 5],\ [0, 6, 8],\ [0, 9, Q],\ [1, 2, 3],\ [1, 4, P],\ [1, 5, 8],\ [1, 6, 9],\\ [2, 4, 7],\ [2, 5, 6],\ [3, 4, 9],\ [3, 7, 8],\ [4, 5, Q]\ [6, 7, Q],\ [7, 9, P],\ [8, P, Q].\end{gathered} \tag{4}$$

Let us examine the tables

$$\begin{array}{cc|c|}\cline{3-3} 2 & P & 0 \\ 6 & 8 & 0 \\ 5 & Q & 4 \\ \cline{3-3}\end{array}\,, \quad \begin{array}{cc|c|}\cline{3-3} 2 & 3 & 1 \\ P & 4 & 1 \\ 0 & 9 & Q \\ \cline{3-3}\end{array}\,, \quad \begin{array}{cc|c|}\cline{3-3} 1 & 2 & 3 \\ 5 & 0 & 3 \\ 8 & P & Q \\ \cline{3-3}\end{array}\,, \quad \begin{array}{cc|c|}\cline{3-3} 4 & 5 & Q \\ P & 8 & Q \\ 1 & 1 & Q \\ \cline{3-3}\end{array}\,.$$

First, we establish the equalities $0' = 4$, $1' = Q$, and $3' = Q$ from the first three of these tables. Then, using the fourth table, we derive $Q' = Q$. Next, by applying Lemma 1 to $[1, 2, 3]$, we get $[Q, 2', Q]$, which gives $2' = Q' = Q$. From $[0, 3, 5]$ we obtain $[4, Q, 5']$, which, when combined with $[4, Q, 5]$, results in $5' = 5$. We also have that $[4, 5, Q]$ implies $[4', 5, Q]$, which leads to $4' = 4$, and then $[1, 4, P]$ implies $[Q, 4, P']$, which, when combined with $[4, 5, Q]$, results in $P' = 5$. $[2, 5, 6]$, $[2, 4, 7]$, $[1, 5, 8]$, and $[1, 6, 9]$ imply $[Q, 5, 6']$, $[Q, 4, 7']$, $[Q, 5, 8']$, and $[Q, 4, 9']$, and due to $[Q, 4, 5]$, we get $6' = 4$, $7' = 5$, $8' = 4$, $9' = 5$, see Table 3.

Point	0	1	2	3	4	5	6	7	8	9	P	Q
Tangential	4	Q	Q	Q	4	5	4	5	4	5	5	Q

Table 3: Points and their tangentials for configuration S_4

As a result, the points Q, 4, and 5 are three inflection points on a single line $[Q, 4, 5]$, while the remaining nine points $1, 2, 3$; $0, 6, 8$; and $7, 9, P$ are sextactic points, for which these inflection points, in that order, serve as tangentials. Since there are lines $[Q, 4, 5]$, $[Q, 8, P]$, $[1, 4, P]$, $[1, 8, 5]$; $[Q, 4, 5]$, $[Q, 6, 7]$, $[2, 4, 7]$, $[2, 6, 5]$; as well as lines $[Q, 4, 5]$, $[Q, 0, 9]$, $[3, 4, 9]$, $[3, 0, 5]$, there exist quadrilaterals $\{Q, 1; 4, 8; 5, P\}$, $\{Q, 2; 4, 6; 5, 7\}$, and $\{Q, 3; 4, 0; 5, 9\}$. So, we proved:

Theorem 5. *In the geometric realization within a cubic structure of Metelka's configuration S_4 with lines (4), the points Q,* 4*, and* 5 *are inflection points lying on a single line, while the remaining nine points are sextactic points. The triples of points* $1, 2, 3$*;* $0, 6, 8$*; and* $7, 9, P$ *have for their common tangentials inflection points Q,* 4*, and* 5*, respectively. Nine sextactic points lie on the lines* $[1, 2, 3]$, $[8, 6, 0]$, $[P, 7, 9]$, $[2, 0, P]$, $[1, 6, 9]$, *and* $[3, 8, 7]$. *Inflection and sextastic points form the quadrilaterals* $\{Q, 1; 4, 8; 5, P\}$, $\{Q, 2; 4, 6; 5, 7\}$*, and* $\{Q, 3; 4, 0; 5, 9\}$.

6. Conclusions

This study aims to investigate four $(12_4, 16_3)$ Metelka's configurations within a cubic structure. Certain publications emphasize Metelka's substantial contribution to the implementation of configurations $(12_4, 16_3)$. We discovered novel, unforeseen collinearities in the geometrical realizations of two of these configurations. Findings in this paper provide novel insights into the configuration $(12_4, 16_3)$ that improve our understanding of geometric characteristics and interrelations inside cubic structures.

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