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# Measure theoretical approach to almost periodicity

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**Abstract.** In this article, we consider various classes of multi-dimensional $\rho$-almost periodic type functions in general measure. We present several structural results about the introduced classes of generalized $\rho$-almost periodic functions, providing also some applications to the abstract Volterra integro-differential inclusions.

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## 1. Introduction and preliminaries

The class of almost periodic functions was first studied by H. Bohr around 1925 and later generalized by many other mathematicians, including S. Bochner, who first defined and studied the almost automorphic functions in 1955 (cf. research monographs [2, 10, 12, 30, 16, 18, 20, 23, 35, 36] for more details about the subject). Almost periodic functions and almost automorphic functions are of fundamental importance in the qualitative analysis of solutions to the abstract (nonlinear) Volterra integro-differential equations in Banach spaces.

If $(X, \|\cdot\|)$ is a complex Banach space, $I = [0, \infty)$ or $I = \mathbb{R}$, $f : I \to X$ is a continuous function and $\epsilon > 0$, then a number $\tau > 0$ is said to be an $\epsilon$-period for $f(\cdot)$ if

$$\|f(s+\tau) - f(s)\| \leq \epsilon, \quad s \in I.$$

It is said that $f(\cdot)$ is almost periodic if for each $\epsilon > 0$ the set of all $\epsilon$-period for $f(\cdot)$ is relatively dense in $[0, \infty)$, i.e., there exists $l' > 0$ such that any subinterval of $[0, \infty)$ of length $l'$ meets $\vartheta(f, \epsilon)$. By $AP(I : X)$ we denote the Banach space of all almost periodic functions $f : \mathbb{R} \to X$, equipped with the sup-norm.

Any almost periodic function $f : I \to X$ is uniformly recurrent, which means that $f(\cdot)$ is continuous and there exists a strictly increasing sequence $(\alpha_k)$ of positive real numbers such that $\lim_{k\to+\infty} \alpha_k = +\infty$ and

$$\lim_{k\to+\infty} \sup_{t\in\mathbb{R}} \big\| f(t+\alpha_k) - f(t) \big\| = 0.$$

If $p > 0$, then it is said that a function $f \in L^p_{loc}(I : X)$ is Stepanov $p$-bounded if

$$\|f\|_{S^p} := \sup_{s\in I} \int_s^{s+1} \|f(r)\|^p \, dr < +\infty;$$

furthermore, it is said that $f(\cdot)$ is Stepanov $p$-almost periodic if its Bochner transform $\hat{f} : I \to L^p([0,1] : X)$, defined by $\hat{f}(s)(r) := f(s+r)$, $s \in I$, $r \in [0,1]$, is almost periodic (cf. [16], [22] and references quoted therein).

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On the other hand, the class of almost periodic functions in the sense of the Lebesgue measure (also called the class of $m$-almost periodic functions) was introduced by W. Stepanov in 1926 ([32]) and further analyzed by S. Stoiński in [33] (1994) and [34] (1999). A Lebesgue measurable function $f : \mathbb{R} \to X$ is said to be $m$-almost periodic (almost periodic in view of the Lebesgue measure $m$) if for each real number $\epsilon,\ \eta > 0$ the set

$$\Biggl\{\tau \in \mathbb{R} : \sup_{s \in \mathbb{R}} m\Bigl(\bigl\{r \in [s, s+1] : \|f(r+\tau) - f(r)\| \geq \eta\bigr\}\Bigr) \leq \epsilon\Biggr\}$$

is relatively dense in $\mathbb{R}$. In this definition, we do not need two real numbers $\epsilon,\ \eta > 0$: a very simple argumentation shows that a Lebesgue measurable function $f : \mathbb{R} \to X$ is $m$-almost periodic if and only if for each real number $\epsilon > 0$ the set

$$\Biggl\{\tau \in \mathbb{R} : \sup_{s \in \mathbb{R}} m\Bigl(\bigl\{r \in [s, s+1] : \|f(r+\tau) - f(r)\| \geq \epsilon\bigr\}\Bigr) \leq \epsilon\Biggr\}$$

is relatively dense in $\mathbb{R}$. Furthermore, we know that if $g : \mathbb{R} \to \mathbb{C}$ is an almost periodic function and $g(\cdot)$ has a bounded analytical extension in a strip around the real axis, then the function $f : \mathbb{R} \to \mathbb{C}$, given by $f(t) := 1/g(t)$, if $g(t) \neq 0$, and $f(t) := 0$, if $g(t) = 0$, is $m$-almost periodic. In particular, the function

$$f(t) = \frac{1}{2 + \cos t + \cos(\sqrt{2}t)}, \quad t \in \mathbb{R}$$

is not bounded but $f(\cdot)$ is $m$-almost periodic; furthermore, we know that $f(\cdot)$ is not Stepanov almost periodic, i.e., Stepanov 1-almost periodic, since it is not Stepanov bounded (see [5, Example 6] given by D. Bugajewski and A. Nawrocki), as well as that $f(\cdot)$ is Stepanov $(1/4)$-almost periodic (see [5, Example 7, Theorem 8]).

It is worth noting that in [14, Example 3.3], P. Kasprzaka, A. Nawrocki and J. Signerska-Rynkowska have constructed a continuous $m$-almost periodic function $f : \mathbb{R} \to \mathbb{R}$ such that the mean value

$$M(f) := \lim_{s \to +\infty} \frac{1}{s} \int_0^s f(r)\, dr$$

does not exist; in particular, this implies that $f(\cdot)$ cannot be Besicovitch almost periodic in the sense of [16] (cf. also [14, Theorem 3.10]). We also know that there exists a bounded, uniformly continuous, Levitan almost periodic function $f : \mathbb{R} \to \mathbb{R}$ such that $f(\cdot)$ is not $m$-almost periodic, as well as that there exists a bounded, continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(\cdot)$ is $m$-almost periodic but not Levitan almost periodic; see, e.g., [28, Example 3.1, Example 3.3].

For further information concerning the class of one-dimensional $m$-almost periodic functions and their applications, the reader may consult research articles [5] by D. Bugajewski, A. Nawrocki, [4] by D. Bugajewski, K. Kasprzak, A. Nawrocki and the doctoral dissertation [29] of A. Nawrocki. Some extensions of the class of $m$-almost periodic functions have been analyzed by A. Michałowicz and S. Stoiński in [25], following the approach of M. Levitan. It is also worth noting that in [8], L. I. Danilov has considered a class of the Lebesgue measurable functions $f : \mathbb{R} \to X$ such that for each real number $\epsilon > 0$ there exists a strictly increasing sequence $(\tau_k)$ of positive real numbers such that $\lim_{k \to +\infty} \tau_k = +\infty$ and

$$\lim_{k \to +\infty} \sup_{s \in \mathbb{R}} m\Bigl(\bigl\{r \in [s, s+1] : \|f(r+\tau_k) - f(r)\| \geq \epsilon\bigr\}\Bigr) = 0.$$

Let us explain now the main ideas which go beyond earlier papers in this area. First of all, we would like to emphasize that the class of $c$-almost periodic functions in view of the Lebesgue measure has not been analyzed so far, even in the one-dimensional setting; this fact strongly influenced us to write this paper ($c \in \mathbb{C} \setminus \{0\}$). Furthermore, the class of multi-dimensional $m$-almost periodic functions has not been considered so far; in this paper, we introduce and analyze various classes of multi-dimensional almost periodic type functions in general measure. In place of the usually considered bounded linear operators $\rho = cId$, where $c \in \mathbb{C} \setminus \{0\}$ and $Id$ denotes the identity operator on the underlying Banach space, here we consider general binary relations $\rho$ (cf. our recent paper [11] by M. Fečkan et al. for further information in this direction). An interesting application is given in the study of the existence and uniqueness of $m$-almost periodic regular solutions of the wave equation in the plane, whose solutions are given by the famous d'Alembert formula (cf. Example 2).

Before going any further, we would like to emphasize that this is probably the first paper written so far which concerns almost periodic solutions of the abstract Volterra integro-differential inclusions with the Stepanov-$p$-almost periodic inhomogeneites provided that the solution operator family is only norm integrable. The main results of this paper are Theorem 2 and Theorem 3, where we have examined the convolution invariance of measure almost periodicity; cf. Subsection 4.1 for more details in this direction (for some other results established in this paper, one may refer e.g. to Proposition 1, Proposition 2, Proposition 3 and Proposition 4; cf. also Example 1 and Example 3). As a concrete application, we consider here the existence and uniqueness of asymptotically almost periodic type solutions of the following abstract Cauchy problem of non-scalar type:

$$x'(t) = Ax(t) + \int_0^t b(t-s)(A + aId)x(s)\, ds + f(t), \quad x(0) = x_0,$$

where $b(t)$ is a scalar-valued kernel, $b \in C^1([0,\infty))$, $a \in \mathbb{C}$, $f : [0,\infty) \to H$ is continuous and $A$ is a densely defined, self-adjoint closed linear operator in an infinite-dimensional complex Hilbert space $H$; cf. research article [26] by R. K. Miller and R. L. Wheeler for the first results obtained in this direction.

It should also be emphasized that this is probably the first research study which concerns the existence and uniqueness of $m$-almost periodic solutions of the semilinear Volterra integral equations. As a specific application, we consider the existence and uniqueness of solutions of the following integral equation:

$$u(\mathbf{t}) = f(\mathbf{t}) + \int_{-\infty}^{t_1}\int_{-\infty}^{t_2}\cdots\int_{-\infty}^{t_n} a(\mathbf{t}-\mathbf{s})F(\mathbf{s}; u(\mathbf{s}))\, d\mathbf{s}, \quad \mathbf{t} \in \mathbb{R}^n,$$

which belong to the space $S^\infty(\mathbb{R}^n : X)$ consisting of all bounded continuous functions which are Stepanov $p$-almost periodic for any finite exponent $p \geq 1$; here, $X$ is a finite-dimensional complex Banach space, $a \in L^1((0,\infty)^n)$ and the inhomogenity $F(\cdot;\cdot)$ enjoys certain features.

In this paper, we essentially employ the notion of $m$-almost periodicity and the ideas developed by S. Stoiński. We can freely say that these ideas are extremely important in the deeper analysis of Stepanov-$p$-almost periodic functions because a very simple argumentation shows that any Stepanov-$p$-almost periodic function $f : I \to Y$ is $m$-almost periodic ($p > 0$), as well as that any bounded $m$-almost periodic function $f : I \to Y$ is Stepanov-$p$-almost periodic ($p > 0$). Therefore, we can construct a great number of unbounded $m$-almost periodic functions $f : \mathbb{R} \to \mathbb{R}$ which are not locally $p$-integrable ($p > 0$); cf. [22, Example 5]. Let us also mention here that the function $\chi_{[0,1/2]}(\cdot)$, where $\chi_A(\cdot)$ denotes the characteristic function of set $A$, is bounded and equi-Weyl-$p$-almost periodic for any exponent $p > 0$ but not $m$-almost periodic, as easily approved (cf. [20] for the notion and more details).

The structure of this paper can be described as follows. After explaining the notion and terminology used, we recall the basic definitions and facts about multi-dimensional almost periodic type functions and their Stepanov generalizations (Section 2). Our main contributions are given in Section 3, where we examine multi-dimensional $\rho$-almost periodic type functions in general measure, and Section 4, where we consider the applications of the established results to the abstract Volterra integro-differential inclusions. We explore the convolution invariance of multi-dimensional $\rho$-almost periodicity in general measure in Subsection 4.1, while the applications to the semilinear Volterra integral equations are given in Subsection 4.2. In the final section of the paper, we provide several comments and final remarks about the introduced function spaces.

Before proceeding any further, we feel it is our duty to say that this paper is too specialized, addressed only to a small number of specialists. We strongly believe that the introduced classes of almost periodic functions in general measure will receive much more attention of the authors working in the field of almost periodic functions and their applications in the near future.

**Notation and terminology.** We assume that $(X, \|\cdot\|)$ and $(Y, \|\cdot\|_Y)$ are complex Banach spaces; $Id$ stands for the identity operator on $Y$, $L(Y)$ stands for the Banach space of all bounded linear operators from $Y$ into $Y$, $n \in \mathbb{N}$ is a fixed integer, $\mathbb{N}_0 := \{0, 1, 2, ..., n, ...\}$, $\mathbb{N}_n := \{1, 2, ..., n\}$ and $\lceil s \rceil := \inf\{k \in \mathbb{Z} : s \leq k\}$ $(s \in \mathbb{R})$. By $|\cdot|$ we denote the Euclidean metric in $\mathbb{R}^n$; if $r > 0$ and $\mathbf{t}_0 \in \mathbb{R}^n$, then we set

$B(\mathbf{t}_0, r) := \{\mathbf{t} \in \mathbb{R}^n : |\mathbf{t} - \mathbf{t}_0| \leq r\}$ and $L(\mathbf{t}_0, r) := \{\mathbf{t} \in \mathbb{R}^n : |\mathbf{t} - \mathbf{t}_0| < r\}$. By $\mathcal{B}$ we denote a certain collection of non-empty subsets of $X$ satisfying that for each $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B$; $m(\cdot)$ denotes the Lebesgue measure on $\mathbb{R}^n$ and $P(A)$ denotes the power set of $A$. The vector space $C_b(I : Y)$, where $\emptyset \neq I \subseteq \mathbb{R}^n$, consists of all continuous functions $u : I \to Y$ satisfying that $\sup_{\mathbf{t} \in I} \|u(\mathbf{t})\|_Y < +\infty$; equipped with the sup-norm $\| \cdot \|_\infty := \sup_{\mathbf{t} \in I} \| \cdot (\mathbf{t})\|_Y$, $C_b(I : Y)$ becomes a Banach space. For more details about the space $L^p(\Omega : Y)$, where $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ is a Lebesgue measurable set and $p > 0$, we recommend the recent research article [22] and references quoted therein. We will deal henceforth with the space $L^p_\nu(\Omega : Y) := \{u : \Omega \to Y \ ; \ u(\cdot) \text{ is Lebesgue measurable and } ||u||_p < \infty\}$, where $p > 0$, $\| \cdot \|_p := \|\nu(\mathbf{t}) \cdot (\mathbf{t})\|_{L^p(\Omega:Y)}$ and $\nu : \Omega \to (0, \infty)$ is a Lebesgue measurable function.

## 2. Multi-dimensional $\rho$-almost periodic type functions and their Stepanov generalizations

The following notion is extremely important for us:

**Definition 1.** (cf. [11, Definition 2.1] and [17, Definition 2.1]) Suppose that $\emptyset \neq I' \subseteq \mathbb{R}^n$, $\nu : I \to [0, \infty)$, $\emptyset \neq I \subseteq \mathbb{R}^n$, $F : I \times X \to Y$, $\rho$ is a binary relation on $Y$ and $I + I' \subseteq I$. Then we say that:

(i) $F(\cdot; \cdot)$ is Bohr $(\mathcal{B}, I', \rho, \nu)$-almost periodic if for every $B \in \mathcal{B}$ and $\epsilon > 0$ there exists $l' > 0$ such that for each $\mathbf{w}_0 \in I'$ there exists $\tau \in B(\mathbf{w}_0, l') \cap I'$ such that, for every $\mathbf{t} \in I$ and $x \in B$, there exists $y_{\mathbf{t};x} \in \rho(F(\mathbf{t}; x))$ with
$$\left\|F(\mathbf{t} + \tau; x) - y_{\mathbf{t};x}\right\|_Y \cdot \nu(\mathbf{t}) \leq \epsilon.$$

(ii) $F(\cdot; \cdot)$ is $(\mathcal{B}, I', \rho, \nu)$-uniformly recurrent if for every $B \in \mathcal{B}$ there exists a sequence $(\tau_k)$ in $I'$ such that $\lim_{k \to +\infty} |\tau_k| = +\infty$ and that, for every $\mathbf{t} \in I$ and $x \in B$, there exists $y_{\mathbf{t};x} \in \rho(F(\mathbf{t}; x))$ with
$$\lim_{k \to +\infty} \sup_{\mathbf{t} \in I; x \in B} \left\|F(\mathbf{t} + \tau_k; x) - y_{\mathbf{t};x}\right\|_Y \cdot \nu(\mathbf{t}) = 0.$$

We exclude the term "$\nu$" from the symbols if $\nu \equiv 1$.

Assume that $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $\emptyset \neq Z \subseteq Y^\Omega$ and $\Lambda + \Omega \subseteq \Lambda$, where $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ is a Lebesgue measurable set and $m(\Omega) > 0$. Let $P_Z \subseteq Z^\Lambda$, $0 \in P_Z$, let $(P_Z, d_{P_Z})$ be a pseudometric space, and let $\|f\|_{P_Z} := d_{P_Z}(f, 0)$, $f \in P_Z$. If $F : \Lambda \times X \to Y$, then the multi-dimensional Bochner transform $\hat{F}_\Omega : \Lambda \times X \to Y^\Omega$ is given by
$$\left[\hat{F}_\Omega(\mathbf{t}; x)\right](\mathbf{u}) := F(\mathbf{t} + \mathbf{u}; x), \quad \mathbf{t} \in \Lambda, \ \mathbf{u} \in \Omega, \ x \in X.$$

The following general notion has recently been analyzed in [22]:

**Definition 2.** Suppose that $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $F : \Lambda \times X \to Y$, R is a certain collection of sequences in $\mathbb{R}^n$ and the assumptions $\mathbf{t} \in \Lambda$, $\mathbf{b} \in$ R and $l \in \mathbb{N}$ imply $\mathbf{t} + \mathbf{b}(l) \in \Lambda$. Then it is said that the function $F(\cdot; \cdot)$ is Stepanov $(\Omega, \mathrm{R}, \mathcal{B}, \mathcal{P}_Z)$-multi-almost periodic, resp. strongly Stepanov $(\Omega, \mathrm{R}, \mathcal{B}, \mathcal{P}_Z)$-multi-almost periodic in the case that $\Lambda = \mathbb{R}^n$, if for every $B \in \mathcal{B}$ and for every sequence $(\mathbf{b}_k = (b_k^1, b_k^2, \cdots, b_k^n)) \in$ R there exist a subsequence $(\mathbf{b}_{k_l} = (b_{k_l}^1, b_{k_l}^2, \cdots, b_{k_l}^n))$ of $(\mathbf{b}_k)$ and a function $F^*_\Omega : \Lambda \times X \to Z$ such that, for every $l \in \mathbb{N}$ and $x \in B$, we have $\hat{F}_\Omega(\cdot + (b_{k_l}^1, \cdots, b_{k_l}^n); x) - F^*_\Omega(\cdot; x) \in P_Z$ and
$$\lim_{l \to +\infty} \sup_{x \in B} \left\|\hat{F}_\Omega(\cdot + (b_{k_l}^1, \cdots, b_{k_l}^n); x) - F^*_\Omega(\cdot; x)\right\|_{P_Z} = 0, \tag{1}$$
resp. $\hat{F}(\cdot + (b_{k_l}^1, \cdots, b_{k_l}^n); x) - F^*(\cdot; x) \in P_Z$, $F^*(\cdot - (b_{k_l}^1, \cdots, b_{k_l}^n); x) - \hat{F}(\cdot; x) \in P_Z$, (1) holds and
$$\lim_{l \to +\infty} \sup_{x \in B} \left\|F^*_\Omega(\cdot - (b_{k_l}^1, \cdots, b_{k_l}^n); x) - \hat{F}_\Omega(\cdot; x)\right\|_{P_Z} = 0.$$

Consider now the following conditions:

(SM-1): Let $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $\emptyset \neq \Lambda' \subseteq \mathbb{R}^n$, $\Lambda' + \Lambda \subseteq \Lambda$, $\Lambda + \Omega \subseteq \Lambda$ and $\mathbb{F} : \Lambda \to (0, \infty)$.

(SM-2): For every $\mathbf{t} \in \Lambda$, $\mathcal{P}_{\mathbf{t}} = (P_{\mathbf{t}}, d_{\mathbf{t}})$ is a pseudometric space of functions from $Y^{\mathbf{t}+\Omega}$ containing the zero function; $\|f\|_{P_{\mathbf{t}}} := d_{\mathbf{t}}(f, 0)$ for all $f \in P_{\mathbf{t}}$. We also assume that $\mathcal{P} = (P, d)$ is a pseudometric space of functions from $\mathbb{C}^{\Lambda}$ containing the zero function and set $\|f\|_P := d(f, 0)$ for all $f \in P$. The argument from $\Lambda$ will be denoted by $\cdot\cdot$ and the argument from $\mathbf{t} + \Omega$ will be denoted by $\cdot$.

The following notion has been introduced in [19, Definition 2.2]:

**Definition 3.** Assume that (SM-1)-(SM-2) hold. By $S^{(\mathbb{F},\rho,\mathcal{P}_{\mathbf{t}},\mathcal{P})}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$ we denote the set consisting of all functions $F : \Lambda \times X \to Y$ such that, for every $\epsilon > 0$ and $B \in \mathcal{B}$, there exists a finite real number $L > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{t}_0, L) \cap \Lambda'$ such that for every $x \in B$, the mapping $\mathbf{u} \mapsto G_x(\mathbf{u}) \in \rho(F(\mathbf{u}; x))$, $\mathbf{u} \in \Omega$ is well defined, and

$$\sup_{x \in B} \Big\| \mathbb{F}(\cdot\cdot) \big\| F(\tau + \cdot; x) - G_x(\cdot) \big\|_{P_{\cdot\cdot}} \Big\|_P < \epsilon.$$

## 3. Multi-dimensional $\rho$-almost periodic type functions in general measure

We assume that $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $\nu : \Lambda \to [0, \infty)$, $m' : P(\mathbb{R}^n) \to [0, \infty]$, $m'(\emptyset) = 0$, $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ is a non-empty compact set and $\Lambda + \Omega \subseteq \Lambda$. For each $\epsilon > 0$ and for each two functions $f : \Lambda \to Y$ and $g : \Lambda \to Y$, we set

$$d_\epsilon(f, g) := \sup_{\mathbf{t} \in \Lambda} m'\Big(\big\{\mathbf{r} \in \mathbf{t} + \Omega : \|f(\mathbf{r}) - g(\mathbf{r})\|_Y \cdot \nu(\mathbf{r}) \geq \epsilon\big\}\Big); \tag{2}$$

define also $\|f\|_{P_\epsilon} := d_\epsilon(0, f)$. Then $0 \leq d_\epsilon(f, g)$ and $d_\epsilon(f, f) = 0$, so that $d_\epsilon(\cdot; \cdot)$ is a premetric on the space of all functions from $\Lambda$ into $Y$; furthermore, we have $d_\epsilon(f, g) = d_\epsilon(g, f)$ and $d_\epsilon(f, g) = d_\epsilon(f + h, g + h)$ so that $d_\epsilon(\cdot; \cdot)$ is a translation invariant pseudo-semimetric on the space of all functions from $\Lambda$ into $Y$ (according to M. M. Deza and M. Laurent [9], these features are sufficiently enough to call $d_\epsilon(\cdot; \cdot)$ a distance). Furthermore, the following assertions hold true:

(i) If $f : \Lambda \to Y$, $g : \Lambda \to Y$, $h : \Lambda \to Y$ and the assumptions $A,\ B,\ C \subseteq \mathbb{R}^n$ and $A \subseteq B \cup C$ imply $m'(A) \leq m'(B) + m'(C)$, then

$$d_\epsilon(f, h) \leq d_{\epsilon/2}(f, g) + d_{\epsilon/2}(g, h), \quad \epsilon > 0. \tag{3}$$

(ii) Assume that $\emptyset \neq \Lambda' \subseteq \mathbb{R}^n$, $\Lambda + \Lambda' \subseteq \Lambda$, $\tau \in \Lambda'$, $M > 0$, the assumption $\mathbf{v} \in \Lambda + \Omega + \tau$ implies $\nu(\mathbf{v} - \tau) \leq M\nu(\mathbf{v})$ and the assumption $A \subseteq B \subseteq \mathbb{R}^n$ implies $m'(A) \leq m'(B)$. Then we have

$$d_\epsilon(f(\cdot + \tau), g(\cdot + \tau)) \leq d_{\epsilon/M}(f, g), \tag{4}$$

for any two functions $f : \Lambda \to Y$ and $g : \Lambda \to Y$.

(iii) Assume that $T \in L(Y)$, $f : \Lambda \to Y$ and $g : \Lambda \to Y$. Then

$$d_\epsilon(Tf, Tg) \leq d_{\epsilon/\|T\|}(f, g), \tag{5}$$

where $d_{\epsilon/\|T\|}(f, g) = 0$ for $T = 0$.

(iv) Assume that $f : \Lambda \to Y$ and $g : \Lambda \to Y$. If the assumption $A \subseteq B \subseteq \mathbb{R}^n$ implies $m'(A) \leq m'(B)$, then for each $\epsilon' \in (0, \epsilon)$ we have

$$d_\epsilon(f, g) \leq d_{\epsilon'}(f, g) \quad \text{and} \quad \|f\|_{P_\epsilon} \leq \|f\|_{P_{\epsilon'}}. \tag{6}$$

(v) Assume additionally that the function $m'(\cdot)$ is not identically zero. Then the triangle inequality

$$d_\epsilon(f, h) \leq d_\epsilon(f, g) + d_\epsilon(g, h)$$

does not hold in general (choose, e.g., $f \equiv \epsilon$, $g \equiv \epsilon/2$, $h \equiv 0$ and $\nu \equiv 1$).

(vi) The supposition $d_\epsilon(f,g)=0$ does not imply $f=g$ a.e. (choose, e.g., $f\equiv\epsilon/2$, $g\equiv\epsilon/4$ and $\nu\equiv 1$).

Now we would like to introduce the following classes of functions:

**Definition 4.** Assume that $\emptyset\neq\Lambda'\subseteq\mathbb{R}^n$, $\emptyset\neq\Lambda\subseteq\mathbb{R}^n$, $F:\Lambda\times X\rightarrow Y$, $\rho$ is a binary relation on $Y$, $R(F)\subseteq D(\rho)$ and $\Lambda+\Lambda'\subseteq\Lambda$. Then it is said that:

(i) $F(\cdot;\cdot)$ is Bohr $(\mathcal{B},\Lambda',\rho,\Omega,m',\nu)$-almost periodic if for every $B\in\mathcal{B}$ and $\epsilon>0$ there exists $l'>0$ such that for each $\mathbf{w}_0\in\Lambda'$ there exists $\tau\in B(\mathbf{w}_0,l')\cap\Lambda'$ such that for every $\mathbf{t}\in\Lambda$, $x\in B$ and $\mathbf{s}\in\mathbf{t}+\Omega$, there exists $y_{\mathbf{s};x}\in\rho(F(\mathbf{s};x))$ with

$$\sup_{x\in B}\Big\|F(\cdot+\tau;x)-y_{\cdot;x}\Big\|_{P_\epsilon}\leq\epsilon.$$

(ii) $F(\cdot;\cdot)$ is $(\mathcal{B},\Lambda',\rho,\Omega,m',\nu)$-uniformly recurrent if for every $B\in\mathcal{B}$ and $\epsilon>0$ there exists a sequence $(\tau_k)$ in $\Lambda'$ such that $\lim_{k\rightarrow+\infty}|\tau_k|=+\infty$ and that for every $\mathbf{t}\in\Lambda$, $x\in B$ and $\mathbf{s}\in\mathbf{t}+\Omega$, there exists $y_{\mathbf{s};x}\in\rho(F(\mathbf{s};x))$ with

$$\lim_{k\rightarrow+\infty}\sup_{x\in B}\Big\|F(\cdot+\tau_k;x)-y_{\cdot;x}\Big\|_{P_\epsilon}=0.$$

We exclude the term "$\nu$" from the notation if $\nu\equiv 1$ and the term "$\Lambda'$" if $\Lambda'=\Lambda$; furthermore, we exclude the term "$\mathcal{B}$" from the notation if $X=\{0\}$ and the term "$\rho$" if $\rho=Id$.

*Remark* 1. The validity of condition $\Lambda+\Omega\subseteq\Lambda$ has been assumed a priori in all our previous research studies of Stepanov almost periodic type functions. This assumption is not satisfactory if we consider some specific geometrical regions in $\mathbb{R}^n$, like $\Lambda=\{(u,v)\in\mathbb{R}^n : |u-v|\leq L\}$ for some $L>0$, when we cannot simply find a compact set $\Omega$ with a positive Lebesgue measure such that $\Lambda+\Omega\subseteq\Lambda$; cf. also [11, Example 2.9]. It seems very plausible that we can investigate the multi-dimensional Stepanov almost periodic functions and the multi-dimensional almost periodic functions in general measure even if the condition $\Lambda+\Omega\subseteq\Lambda$ is neglected; we will explore this topic somewhere else.

*Remark* 2. In our approach, the function $m'(\cdot)$ is defined for all subsets in $\mathbb{R}^n$. If $m'(\cdot)$ is defined only for subsets belonging to a certain $\sigma$-algebra on $\mathbb{R}^n$, then the notion from Definition 4 can be understood only for the functions which are measurable in a certain sense. For instance, in the case of the usual Lebesgue measure $m(\cdot)$, we must additionally assume that for each $x\in X$ the function $F(\cdot;x)$ is Lebesgue measurable as well as that the function $\nu:\Lambda\rightarrow[0,\infty)$ is Lebesgue measurable. In our later investigations of $m$-almost periodic functions, we will tacitly assume these conditions.

The subsequent result follows directly from the corresponding definitions:

**Proposition 1.** *Suppose that $\emptyset\neq\Lambda'\subseteq\mathbb{R}^n$, $\emptyset\neq\Lambda\subseteq\mathbb{R}^n$, $F:\Lambda\times X\rightarrow Y$, $\rho$ is a binary relation on $Y$ and $\Lambda+\Lambda'\subseteq\Lambda$. If $F(\cdot;\cdot)$ is Bohr $(\mathcal{B},\Lambda',\rho,\nu)$-almost periodic ($(\mathcal{B},\Lambda',\rho,\nu)$-uniformly recurrent), then $F(\cdot;\cdot)$ is Bohr $(\mathcal{B},\Lambda',\rho,\Omega,m',\nu)$-almost periodic ($(\mathcal{B},\Lambda',\rho,\Omega,m',\nu)$-uniformly recurrent).*

Suppose that the requirements sufficient for the validity of estimates (3)-(4) hold, $\Lambda+\Lambda'\subseteq\Lambda$, $\tau+\Lambda=\Lambda$ for all $\tau\in\Lambda'$, $F:\Lambda\times X\rightarrow Y$, $R(F)\subseteq D(\rho)$ and $\rho(x)$ has the cardinality one for every $x\in R(F)$. Arguing as in the proof of [11, Proposition 2.2], we may deduce that $\Lambda+(\Lambda'-\Lambda')\subseteq\Lambda$ as well as that the Bohr $(\mathcal{B},\Lambda',\rho,\Omega,m',\nu)$-almost periodicity ($(\mathcal{B},\Lambda',\rho,\Omega,m',\nu)$-uniform recurrence) implies the Bohr $(\mathcal{B},\Lambda'-\Lambda',Id,\Omega,m',\nu)$-almost periodicity ($(\mathcal{B},\Lambda'-\Lambda',Id,\Omega,m',\nu)$-uniform recurrence) of $F(\cdot;\cdot)$; the statement of [11, Corollary 2.3] can be generalized in this manner as well. Furthermore, if the requirements sufficient for the validity of estimates (3)-(5) hold, then the denouements from [11, Example 2.8] can be formulated in our new framework; this can also be done with the statements established in [11, Theorem 2.11, (i)-(iv); Proposition 2.12] and [15, Proposition 2.7].

The next result can also be reworded for the corresponding classes of uniformly reccurent functions in measure; the proof is rather simple (for basic details concerning $L^p$-spaces and integration theory, we may also refer to [24, Chapter 13, Appendix A]; here, $L^p_{m',\nu}(\mathbf{t}+\Omega:Y)=\{u:\mathbf{t}+\Omega\rightarrow Y;\|u(\cdot)\|_Y\nu(\cdot)\in L^p_{m'}(\mathbf{t}+\Omega:Y)\}$ is equipped with the norm $\|u\|_{L^p_{m',\nu}(\mathbf{t}+\Omega:Y)}=\|\|u(\cdot)\|_Y\nu(\cdot)\|_{L^p_{m'}(\mathbf{t}+\Omega:Y)}$ for all $p>0$ and $\mathbf{t}\in\Lambda$):

**Proposition 2.** *Suppose that $\emptyset \neq \Lambda' \subseteq \mathbb{R}^n$, $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $F : \Lambda \times X \to Y$, $\rho$ is a binary relation on $Y$ and $\Lambda + \Lambda' \subseteq \Lambda$.*

*(i) If $F \in S^{(1,\rho,\mathcal{P}_{\mathbf{t}},\mathcal{P})}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$, where $P = C_b(\Lambda : \mathbb{C})$ and $P_t = L^p_{m',\nu}(\mathbf{t} + \Omega : Y)$ for some measure $m'(\cdot)$ on $\mathbb{R}^n$ and $p > 0$ ($\mathbf{t} \in \Lambda$), then $F(\cdot;\cdot)$ is Bohr $(\mathcal{B}, \Lambda', \rho, \Omega, m', \nu)$-almost periodic.*

*(ii) If $\rho = T \in L(Y)$, $\sup_{x\in B,\mathbf{t}\in\Lambda} \|F(\mathbf{t}; x)\|_Y < \infty$ for all $B \in \mathcal{B}$ and the function $F(\cdot;\cdot)$ is Bohr $(\mathcal{B}, \Lambda', \rho, \Omega, m', \nu)$-almost periodic for some bounded function $\nu(\cdot)$ and measure $m'(\cdot)$ on $\mathbb{R}^n$ satisfying $\sup_{\mathbf{t}\in\Lambda} m'(\mathbf{t} + \Omega) < +\infty$, then $F \in S^{(1,\rho,\mathcal{P}_{\mathbf{t}},\mathcal{P})}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$ with $P = C_b(\Lambda : \mathbb{C})$ and $P_t = L^p_{m',\nu}(\mathbf{t} + \Omega : Y)$ for all $\mathbf{t} \in \Lambda$.*

Proposition 2(i) can be applied to the functions considered in [22, Example 3, Example 4], with $\nu(\cdot) \not\equiv 1$; moreover, Proposition 2(ii) extends [33, Theorem 2.7]. Furthermore, in [22, Corollary 2], we have extended the classical Bochner theorem. This result can be further extended in the following way (we define the notion of $m$-almost periodicity of $F(\cdot)$ as in the one-dimensional setting; cf. also [28, Remark 3.4], where the author has also imposed the boundedness of function $F(\cdot)$ to derive its almost periodicity):

**Proposition 3.** *Suppose that $F : \mathbb{R}^n \to Y$ is $m$-almost periodic and uniformly continuous. Then $F(\cdot)$ is almost periodic.*

*Proof.* Keeping in mind Proposition 2(ii) and [22, Corollary 2], it suffices to show that $F(\cdot)$ is bounded. To this end, fix a number $\epsilon > 0$. Then we can find three finite real numbers $c > 0$, $l' > 0$ and $0 < \delta < \epsilon/2$ such that, for every $\mathbf{w}_0 \in \mathbb{R}^n$, there exists $\tau \in B(\mathbf{w}_0, l')$ such that $m(\{\mathbf{s} \in \mathbf{t} + [0, 1]^n : \|F(\mathbf{s} + \tau) - F(\mathbf{s})\|_Y < \delta\}) \geq 1 - c\delta^n$ for all $\mathbf{t} \in \mathbb{R}^n$, $m((\mathbf{t} + [0, 1]^n) \cap B(\mathbf{s}, \delta)) > c\delta^n$, provided $\mathbf{t} \in \mathbb{R}^n$ and $\mathbf{s} \in \mathbf{t} + [0, 1]^n$, and that the assumption $|x - y| < \delta$ for some $x,\ y \in \mathbb{R}^n$ implies $\|F(x) - F(y)\|_Y \leq \epsilon/2$. If $\mathbf{t} \in \mathbb{R}^n$ is arbitrary and $\mathbf{s} \in \mathbf{t} + [0, 1]^n$, then it can be simply shown that there exists $x \in B(\mathbf{s}, \delta)$ such that $\|F(x + \tau) - F(x)\| < \delta$. Since $F(\cdot)$ is continuous, we have that there exists a finite real number $M > 0$ such that $\|F(\mathbf{t})\|_Y \leq M$ for all $\mathbf{t} \in B(0, 2l')$. Pick up $\tau \in \mathbb{R}^n$ such that $x + \tau \in B(0, 2l')$ and $m(\{\mathbf{s} \in \mathbf{t} + [0, 1]^n : \|F(\mathbf{s} + \tau) - F(\mathbf{s})\|_Y < \delta\}) \geq 1 - c\delta^n$ for all $\mathbf{t} \in \mathbb{R}^n$. Then we have $\|F(x + \tau) - F(x)\|_Y < \delta$ and $\|F(x) - F(\mathbf{s})\| \leq \epsilon/2$ so that $\|F(\mathbf{s})\| \leq M + \delta + \epsilon/2$. This ends the proof. □

Now we will illustrate Proposition 3:

**Example 1.** The function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) := \sum_{k=1}^{\infty} \frac{1}{k} \sin^2\Bigl(\frac{x}{2^k}\Bigr)\, dt, \quad x \in \mathbb{R},$$

is (Besicovitch) unbounded, uniformly continuous and uniformly recurrent (see [13, Theorem 1.1] and [18, Theorem 2.4.2]). Due to Proposition 3, $f(\cdot)$ cannot be $m$-almost periodic; on the other hand, Proposition 1 yields that $f(\cdot)$ is $m$-uniformly recurrent, with the meaning clear.

The following extension of a well-known result of S. Stoiński (see, e.g., [28, Therem 2.7]) is applicable if $\Lambda = [0, \infty)^n$ or $\Lambda = \mathbb{R}^n$; we will present all relevant details for completeness (cf. also the proofs of [6, Proposition 2.16] and [20, Proposition 2.1.8(i)]):

**Proposition 4.** *Suppose that $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $\Lambda + \Lambda \subseteq \Lambda$, $\rho = T \in L(Y)$, $F : \Lambda \times X \to Y$ is Bohr $(\mathcal{B}, \Lambda, T, \Omega, m')$-almost periodic, $\sup_{x\in B,\mathbf{t}\in\Lambda\cap K} \|F(\mathbf{t}; x)\|_Y < +\infty$ for each compact $K \subseteq \mathbb{R}^n$ and the supposition $A \subseteq B \subseteq \mathbb{R}^n$ implies $m'(A) \leq m'(B)$. If*

$$(\forall l > 0)\ (\exists \mathbf{w_0} \in \Lambda)\ (\exists k > 0)\ (\forall \mathbf{t} \in \Lambda)(\exists \mathbf{w}_0' \in \Lambda)(\forall \mathbf{w}_0'' \in B(\mathbf{w}_0', l) \cap \Lambda)\ \mathbf{t} - \mathbf{w}_0'' \in B(\mathbf{w_0}, kl') \cap \Lambda;$$

*then for each $B \in \mathcal{B}$ and each sequence of positive real numbers $(\lambda_k)$ tending to zero, we have*

$$\lim_{k\to+\infty} \sup_{\mathbf{s}\in\Lambda; x\in B} m'\Bigl(\bigl\{\mathbf{r} \in \mathbf{s} + \Omega : \lambda_k \|F(\mathbf{r}; x)\|_{Y\geq 1}\bigr\}\Bigr) = 0.$$

*Proof.* Fix $B \in \mathcal{B}$ and $\epsilon > 0$. Then there is a number $l' > 0$ such that for each $\mathbf{w}_0 \in \Lambda$ there exists $\tau \in B(\mathbf{w_0}, l') \cap \Lambda$ such that

$$\sup_{x \in B} m'\Bigl(\bigl\{\mathbf{s} \in \mathbf{t} + \Omega : \|F(\mathbf{s} + \tau; x) - TF(\mathbf{s}; x)\|_Y \geq \epsilon\bigr\}\Bigr) \leq \epsilon, \quad \mathbf{t} \in \Lambda. \tag{7}$$

Suppose that $\mathbf{w}_0 \in \Lambda$ and $k > 0$ are chosen such that (7) holds. The prescribed assumption implies that the set $\{F(\mathbf{s}; x) : \mathbf{s} \in B(\mathbf{w_0}, kl') \cap \Lambda,\ x \in B\}$ is bounded in $Y$. Let $\mathbf{t} \in \Lambda$ be fixed. Then there exists $\mathbf{w_0'} \in \Lambda$ such that, for every $\mathbf{w_0''} \in B(\mathbf{w}_0', l') \cap \Lambda$, we have $\mathbf{t} \in \mathbf{w_0''} + [B(\mathbf{w_0}, kl') \cap \Lambda]$. On the other hand, there exists $\tau = \mathbf{w_0''} \in B(\mathbf{w}_0', l') \cap \Lambda$ such that (7) holds. Clearly, $\mathbf{s} = \mathbf{t} - \tau \in B(\mathbf{w_0}, kl') \cap \Lambda$, which simply implies that

$$\sup_{x \in B} m'\Bigl(\bigl\{\mathbf{s} \in \mathbf{t} - \tau + \Omega : \|F(\mathbf{s} + \tau; x) - TF(\mathbf{s}; x)\|_Y \geq \epsilon\bigr\}\Bigr) \leq \epsilon, \quad \mathbf{t} \in \Lambda$$

and

$$\sup_{x \in B} m'\Bigl(\bigl\{\mathbf{s} \in \mathbf{t} + \Omega : \|F(\mathbf{s}; x) - TF(\mathbf{s} - \tau; x)\|_Y \geq \epsilon\bigr\}\Bigr) \leq \epsilon, \quad \mathbf{t} \in \Lambda.$$

Since $\mathbf{s} - \tau \in \Omega + [B(\mathbf{w_0}, kl') \cap \Lambda]$ and $T \in L(Y)$, there exists a finite constant $M_\epsilon > 0$ such that $\|TF(\mathbf{s} - \tau; x)\|_Y \leq M_\epsilon$ for all $x \in B$ and $\mathbf{s} \in \mathbf{t} + \Omega$. The desired result follows from this estimate, the given assumption on $m'(\cdot)$ and the inclusions

$$\begin{aligned}\bigl\{\mathbf{s} \in \mathbf{t} + \Omega : \lambda_k \|F(\mathbf{s}; x)\|_Y \geq 1\bigr\} &\subseteq \bigl\{\mathbf{s} \in \mathbf{t} + \Omega : \|F(\mathbf{s}; x)\|_Y > M_\epsilon + \epsilon\bigr\} \\ &\subseteq \bigl\{\mathbf{s} \in \mathbf{t} - \tau + \Omega : \|F(\mathbf{s}; x) - TF(\mathbf{s} - \tau; x)\|_Y \geq \epsilon\bigr\},\end{aligned}$$

which hold for all $x \in B$ and all sufficiently large integers $k \geq k_0(\epsilon)$. □

The condition [5, (13)] and the assertion of [5, Theorem 8] can be straightforwardly extended to the multi-dimensional setting. But, it seems much more complicated to provide some sufficient conditions which would ensure that an $m$-almost periodic function $F : \mathbb{R}^n \to Y$ is equi-Weyl-$p$-almost periodic for some (all) exponents $p \geq 1$.

We continue with the following illustrative application of the d'Alembert formula:

**Example 2.** Let $a > 0$ and $|c| = 1$; then a unique regular solution of the wave equation $u_{tt} = a^2 u_{xx}$ in domain $\{(x, t) : x \in \mathbb{R},\ t > 0\}$, equipped with the initial conditions $u(x, 0) = f(x) \in C^2(\mathbb{R})$ and $u_t(x, 0) = g(x) \in C^1(\mathbb{R})$, is given by

$$u(x, t) = \frac{1}{2}\bigl[f(x - at) + f(x + at)\bigr] + \frac{1}{2a}\int_{x-at}^{x+at} g(s)\, ds, \quad x \in \mathbb{R},\ t > 0.$$

If we assume that the function $x \mapsto (f(x), g^{[1]}(x))$, $x \in \mathbb{R}$ is $(cId, [0,1], m)$-almost periodic, where $g^{[1]}(\cdot) \equiv \int_0^{\cdot} g(s)\, ds$, then the solution $u(x, t)$ can be extended to the whole real line in $t$ and this solution is $(cId, [0,1]^2, m_2)$-almost periodic in $(x, t) \in \mathbb{R}^2$; in order not to make any confusion, $m(\cdot)$ denotes here the Lebesgue measure in $\mathbb{R}$ and $m_2(\cdot)$ denotes the Lebesgue measure in $\mathbb{R}^2$. To show this, fix $\epsilon > 0$. Then there exists a finite real number $l' > 0$ such that any subinterval $I$ of $\mathbb{R}$ of length $l'$ contains a point $\tau \in I$ such that

$$\sup_{t \in \mathbb{R}} m\bigl(\{x \in [t, t+1] : |f(x + \tau) - cf(x)| < \epsilon\}\bigr) \geq 1 - \epsilon, \quad t \in \mathbb{R} \tag{8}$$

and (8) is valid with the function $f(\cdot)$ replaced by the function $g^{[1]}(\cdot)$ therein. Furthermore, one has $(x',\ t',\ \tau_1,\ \tau_2 \in \mathbb{R})$:

$$\begin{aligned}&\Bigl|u\bigl(x' + \tau_1, t' + \tau_2\bigr) - cu(x', t')\Bigr| \\ &\leq \frac{1}{2}\Bigl|f\bigl((x' - at') + (\tau_1 - a\tau_2)\bigr) - cf(x' - at')\Bigr| \\ &+ \frac{1}{2}\Bigl|f\bigl((x' + at') + (\tau_1 + a\tau_2)\bigr) - cf([x' + at' + (\tau_1 + a\tau_2)] - (\tau_1 + a\tau_2))\Bigr| \\ &+ \frac{1}{2a}\Bigl|g^{[1]}\bigl((x' - at') + (\tau_1 - a\tau_2)\bigr) - cg^{[1]}(x' - at')\Bigr| \\ &+ \frac{1}{2a}\Bigl|g^{[1]}\bigl((x' + at') + (\tau_1 + a\tau_2)\bigr) - cg^{[1]}(x' + at')\Bigr|.\end{aligned} \tag{9}$$

Set now $X := x' - at'$ and $Y := x' + at'$. Then $x = (X+Y)/2$, $y = (-X+Y)/2a$, the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, given by $T(X,Y) = ((X+Y)/2, (-X+Y)/2a)$ for all $(X,Y) \in \mathbb{R}^2$, is an isomorphism, and $\det([T]) = 1/2a$. Suppose now that $(x',t') \in [s_1, s_1+1] \times [s_2, s_2+1]$ for some $s_1,\ s_2 \in \mathbb{R}$. Then it can be simply shown that the transformation $T^{-1}(\cdot)$ maps the rectangle $[s_1, s_1+1] \times [s_2, s_2+1]$ onto the closed quadrilateral $P$ with the vertices $A(s_1 - as_2, s_1 + as_2)$, $B(s_1 - as_2 - a, s_1 + as_2 + a)$, $C(s_1 + 1 - as_2, s_1 + 1 + as_2)$ and $D(s_1 + 1 - as_2 - a, s_1 + 1 + as_2 + a)$; clearly, $m_2(P) = 2a$. Define $X_\epsilon := \{x \in [s_1 - as_2 - a, s_1 - as_2 + 1] : \max(|f(x+\tau) - cf(x)|, |g^{[1]}(x+\tau) - cg^{[1]}(x)|) < \epsilon\}$ and $Y_\epsilon := \{x \in [s_1 + as_2, s_1 + as_2 + a + 1] : \max(|f(x+\tau) - cf(x)|, |g^{[1]}(x+\tau) - cg^{[1]}(x)|) < \epsilon\}$. Then $m(X_\epsilon) \geq a+1-\lceil a+1\rceil\epsilon$, $m(Y_\epsilon) \geq a+1-\lceil a+1\rceil\epsilon$ and $m_2(\{(X,Y) \in P : X \in X_\epsilon,\ Y \in Y_\epsilon\}) \geq 2a - (\lceil a+1\rceil\epsilon)^2$. Therefore,

$$m_2\Big(\Big\{(x,t) \in [s_1, s_1+1] \times [s_2, s_2+1] : X \in X_\epsilon,\ Y \in Y_\epsilon\Big\}\Big) \geq \frac{1}{2a}\Big(2a - \big(\lceil a+1\rceil\epsilon\big)^2\Big).$$

Keeping this estimate in mind and (9), the final conclusion follows similarly as in [18, Example 2, pp. XXXIV-XXXV]. Let us finally note that the possible applications can also be given to the Kirchhoff formula and the Poisson formula; see [18].

Before proceeding further, let us only mention that [18, Example 6.1.13, Example 6.1.16] can be reworded in our new setting; these examples show the importance of considerations of general regions $\Lambda$ and $\Lambda'$ in our analysis. Now we will state and prove the following extension of [33, Theorem 6] (cf. also [20, Theorem 2.1.12(v)], the notion of metric space $\mathcal{X}$ introduced in [33], and [5, Definition 7]):

**Theorem 1.** *Suppose that $M > 0$, $\emptyset \neq \Lambda' \subseteq \mathbb{R}^n$, $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $\rho = T \in L(Y)$ and $\Lambda + \Lambda' \subseteq \Lambda$. Suppose, further, that for each $k \in \mathbb{N}$ we have that $F_k : \Lambda \times X \to Y$ is a Bohr $(\mathcal{B}, \Lambda', \rho, \Omega, m', \nu)$-almost periodic ($(\mathcal{B}, \Lambda', \rho, \Omega, m', \nu)$-uniformly recurrent) function and for each $\epsilon > 0$ and $B \in \mathcal{B}$ one has:*

$$\lim_{k\to+\infty} \sup_{x\in B} \big\| F_k(\cdot;x) - F(\cdot;x) \big\|_{P_\epsilon} = 0.$$

*Then $F(\cdot;\cdot)$ is Bohr $(\mathcal{B}, \Lambda', \rho, \Omega, m', \nu)$-almost periodic ($(\mathcal{B}, \Lambda', \rho, \Omega, m', \nu)$-uniformly recurrent), provided that the suppositions $A,\ B,\ C \subseteq \mathbb{R}^n$ and $A \subseteq B \cup C$ imply $m'(A) \leq m'(B) + m'(C)$, and the supposition $\mathbf{v} \in \Lambda + \Omega + \tau$ for some $\tau \in \Lambda'$ implies $\nu(\mathbf{v} - \tau) \leq M\nu(\mathbf{v})$.*

*Proof.* The proof is basically a simple consequence of the following estimates ($k \in \mathbb{N}$, $x \in X$, $\tau \in \Lambda'$):

$$\begin{aligned}
\big\|F(\cdot+\tau;x) - TF(\cdot;x)\big\|_{P_\epsilon} &\leq \big\|F(\cdot+\tau;x) - F_s(\cdot+\tau;x)\big\|_{P_{\epsilon/2}} + \big\|F_s(\cdot+\tau;x) - TF(\cdot;x)\big\|_{P_{\epsilon/2}} \\
&\leq \big\|F(\cdot+\tau;x) - F_s(\cdot+\tau;x)\big\|_{P_{\epsilon/2}} + \big\|F_s(\cdot+\tau;x) - TF_l(\cdot;x)\big\|_{P_{\epsilon/4}} \\
&\quad + \big\|TF_l(\cdot;x) - TF(\cdot;x)\big\|_{P_{\epsilon/4}} \\
&\leq \big\|F(\cdot;x) - F_s(\cdot;x)\big\|_{P_{\epsilon/2M}} + \big\|F_s(\cdot+\tau;x) - TF_s(\cdot;x)\big\|_{P_{\epsilon/4}} \\
&\quad + \big\|F_s(\cdot;x) - F(\cdot;x)\big\|_{P_{\epsilon/4\|T\|}}.
\end{aligned}$$

Keeping this in mind, we can apply estimates (3)-(5). □

We proceed further with the observation that it would not be so easy to formulate a satisfactory analogue of Theorem 1 for the class of (strongly) $(\mathrm{R}, \mathcal{B}, \Omega, L, m', \nu)$-multi-almost periodic functions; see also the proof of [17, Proposition 2.6] and the estimates given in (6).

In connection with the statement of [28, Theorem 2.9], we will present the following illustrative example, but here we do not use the boundedness of the analytical extension on the strips around the real axes:

**Example 3.** Assume that $G : \mathbb{R}^n \to \mathbb{R}$ is an almost periodic function, $G(\mathbf{t}) \neq 0$ for all $\mathbf{t} \in \mathbb{R}^n$ and there exist real numbers $a$ and $b$ such that $a < 0 < b$ and the function $G(\cdot)$ can be analytically extended to the region $\{(z_1, ..., z_n) \in \mathbb{C}^n : \Re z_i \in (a,b) \text{ for } 1 \leq i \leq n\}$. Then the evidence contained in the proof of [23, Theorem 5.3.1] (cf. also [18, Example 6.2.9]) shows that $\lim_{\delta\to 0+} m(\{\mathbf{s} \in \mathbf{t} + [0,1]^n : |G(\mathbf{s})| < \delta\}) = 0$, uniformly in $\mathbf{t} \in \mathbb{R}^n$, so that

$$\big(\forall \epsilon > 0\big)\big(\exists \delta_0 > 0\big)\big(\forall \delta \in (0,\delta_0)\big)\big(\forall \mathbf{t} \in \mathbb{R}^n\big)\, m\big(\{\mathbf{s} \in \mathbf{t} + [0,1]^n : |G(\mathbf{s})| \geq \delta\}\big) \geq 1 - \epsilon.$$

Suppose now that $\epsilon > 0$ is given, $\delta \in (0, \min(1/\epsilon, \delta_0/2))$ and $\tau \in \mathbb{R}^n$ is a $(\delta/2)$-almost period of the function $G(\cdot)$. If $\mathbf{s} \in \mathbf{t} + [0,1]^n$ for some $\mathbf{t} \in \mathbb{R}^n$ and $|G(\mathbf{s})| \geq \delta$, then we cannot have $|F(\mathbf{s}+\tau) - F(\mathbf{s})| \geq \epsilon$ because this would imply

$$\delta/2 > |G(\mathbf{s}+\tau) - G(\mathbf{s})| \geq \epsilon \cdot |G(\mathbf{s})| \cdot |G(\mathbf{s}+\tau)| \geq \epsilon \cdot \delta \cdot (\delta/2),$$

which is a contradiction. Hence,

$$m\big(\{\mathbf{r} \in \mathbf{s} + [0,1]^n : |F(\mathbf{r}+\tau) - F(\mathbf{r})| < \epsilon\}\big) \geq 1 - \epsilon, \quad \mathbf{s} \in \mathbb{R}^n$$

and

$$m\big(\{\mathbf{r} \in \mathbf{t} + [0,1]^n : |F(\mathbf{r}+\tau) - F(\mathbf{r})| \geq \epsilon\}\big) \leq \epsilon, \quad \mathbf{s} \in \mathbb{R}^n.$$

This implies that $F(\cdot)$ is $m$-almost periodic. In particular, the function

$$F\big(t_1, ..., t_n\big) = \frac{1}{2 + \cos t_1 + \cos(\sqrt{2}t_1)} \cdot ... \cdot \frac{1}{2 + \cos t_n + \cos(\sqrt{2}t_n)}, \ \mathbf{t} = \big(t_1, ..., t_n\big) \in \mathbb{R}^n$$

is $m$-almost periodic.

Finally, we would like to recall that we have already asked in [20, Example 6.4.10(i)] whether the function $f(\cdot)$ is Besicovitch-$p$-bounded for some finite exponent $p \geq 1$. We strongly believe that this is not the case as well as that this is a simple example of a continuous, $m$-almost periodic function $f : \mathbb{R} \to \mathbb{R}$ such that $M(f)$ does not exist; cf. also [27].

Furthermore, if $\delta > 0$, then we set $\Lambda(\delta) := \{h \in \mathbb{R}^n : |h| \leq \delta \text{ and } \mathbf{t} + h \in \Lambda \text{ for all } \mathbf{t} \in \Lambda\}$. We say that a function $F : \Lambda \times X \to Y$ is $(\mathcal{B}, \Omega, m')$-continuous if, for every $\epsilon > 0$ and $B \in \mathcal{B}$, there exists $\delta > 0$ such that, for every $h \in \Lambda(\delta)$, we have

$$\sup_{\mathbf{s} \in \Lambda; x \in B} m'\Big(\big\{\mathbf{r} \in \mathbf{s} + \Omega : \|F(\mathbf{r}+h; x) - F(\mathbf{r}; x)\|_Y \geq \epsilon\big\}\Big) \leq \epsilon.$$

Immediately from the definition, it follows that any function $F : \Lambda \times X \to Y$, which is uniformly continuous on the sets of the form $\Lambda \times B$, where $B \in \mathcal{B}$, is $(\mathcal{B}, \Omega, m')$-continuous. Moreover, if the function $F : \Lambda \times X \to Y$ is continuous on the sets of the form $\Lambda \times B$, where $B \in \mathcal{B}$, the region $\Lambda$ has certain geometrical properties (cf. [6, Proposition 2.21] for more details; we can always take $\Lambda = \mathbb{R}^n$ here) and $F(\cdot; \cdot)$ is Bohr $(\mathcal{B}, \Lambda, Id, \Omega, m')$-almost periodic, then $F(\cdot; \cdot)$ is $(\mathcal{B}, \Omega, m')$-continuous. To the best knowledge of the authors, it is not clear how one can prove that the statements of [29, Twierdzenie 1.8, Lemat 1.8, Lemat 1.9, Uwaga 1.9; pp. 23-24] hold if the function under consideration is not continuous (it is also worth noting that the argumentation of H. Bohr, which has particularly been used in the proof of [29, Lemat 1.9], is inapplicable in the multi-dimensional setting).

## 4. Applications

In this section, we will provide several important applications of the introduced notion to the abstract Volterra integro-differential inclusions and present some new important contributions to the theory of Stepanov almost periodic functions. We will divide the material of this section into two separate subsections.

### 4.1. Convolution invariance of measure almost periodicity

The convolution invariance of Bohr $(\mathcal{B}, \Lambda', \rho, \Omega, m', \nu)$-almost periodicity and $(\mathcal{B}, \Lambda', \rho, \Omega, m', \nu)$-uniform recurrence is an extremely delicate theme. We start this subsection by recalling that the convolution of an $m$-almost periodic function $f : \mathbb{R}^n \to Y$ with a function $h \in L^1(\mathbb{R}^n)$ does not have to exists; furthermore, if the value $(h * f)(\mathbf{t}) = \int_{-\infty}^{\infty} h(\mathbf{t} - \mathbf{s}) f(\mathbf{s})\, d\mathbf{s}$ exists for all $\mathbf{t} \in \mathbb{R}^n$, then the convolution $(h * f)(\cdot)$ is not $m$-almost periodic in general (cf. [5, Example 3, Example 4] for the one-dimensional setting).

Now we will state the following result (for the sake of convenience, we consider here the situation in which $\nu(\cdot) \equiv 1$; $\mathbf{t} > \mathbf{0}$ means than any component of $\mathbf{t} \in \mathbb{R}^n$ is positive):

**Theorem 2.**

(i) *Let* $(R(\mathbf{t}))_{\mathbf{t}>\mathbf{0}} \subseteq L(X)$ *be a strongly continuous operator family such that* $\int_{(0,\infty)^n} \|R(\mathbf{t})\|\, d\mathbf{t} < \infty$. *If* $f : \mathbb{R}^n \to X$ *is bounded and Bohr* $(\Lambda', \rho, [0,1]^n, m)$*-almost periodic* ($(\Lambda', \rho, [0,1]^n, m)$*-uniformly recurrent), where* $\rho = T \in L(X)$, *then the function* $F : \mathbb{R}^n \to X$, *given by*

$$F(\mathbf{t}) := \int_{-\infty}^{t_1} \int_{-\infty}^{t_2} \cdots \int_{-\infty}^{t_n} R(\mathbf{t}-\mathbf{s}) f(\mathbf{s})\, d\mathbf{s}, \quad \mathbf{t} \in \mathbb{R}^n, \tag{10}$$

*is bounded, continuous and Bohr* $(\Lambda', \rho, [0,1]^n)$*-almost periodic* ($(\Lambda', \rho, [0,1]^n)$*-uniformly recurrent), provided that* $R(\mathbf{t})T = TR(\mathbf{t})$, $\mathbf{t} > \mathbf{0}$.

(ii) *Let* $a(\cdot)$ *be Lebesgue measurable and let* $\int_{(0,\infty)^n} |a(\mathbf{t})|\, d\mathbf{t} < \infty$. *If* $f : \mathbb{R}^n \to X$ *is bounded and Bohr* $(\Lambda', \rho, [0,1]^n, m)$*-almost periodic* ($(\Lambda', \rho, [0,1]^n, m)$*-uniformly recurrent), where* $\rho = T \in L(X)$, *then the function* $F : \mathbb{R}^n \to X$, *given by*

$$F(\mathbf{t}) := \int_{-\infty}^{t_1} \int_{-\infty}^{t_2} \cdots \int_{-\infty}^{t_n} a(\mathbf{t}-\mathbf{s}) f(\mathbf{s})\, d\mathbf{s}, \quad \mathbf{t} \in \mathbb{R}^n, \tag{11}$$

*is bounded, uniformly continuous and Bohr* $(\Lambda', \rho, [0,1]^n)$*-almost periodic* ($(\Lambda', \rho, [0,1]^n)$*-uniformly recurrent).*

*Proof.* We will show part (i) only for the class of one-dimensional, bounded, Bohr $(\Lambda', T, [0,1], m)$-almost periodic functions; cf. [6, Theorem 2.53, Theorem 2.54] for the multi-dimensional setting. It is clear that the function $F(\cdot)$ is well-defined and bounded; the continuity is a simple consequence of the dominated convergence theorem, the boundedness of the function $f(\cdot)$ and the strong continuity of $(R(t))_{t>0}$. Let $\epsilon > 0$ be fixed. Then there exists $k \in \mathbb{N}$ such that $\int_k^{+\infty} \|R(r)\|\, dr \leq \epsilon/(2(1+\|T\|)\|f\|_\infty)$. This implies

$$\int_k^{+\infty} \|R(r)\| \cdot \|f(s+\tau-r) - Tf(s-r)\|\, dr \leq \epsilon/2, \quad s \in \mathbb{R}. \tag{12}$$

Let $0 < \epsilon' < \epsilon/(4k(1+\int_0^\infty \|R(r)\|\, dr))$ and let $0 < \epsilon' < \delta/2$, where $\delta > 0$ satisfies that the assumption $m(A) < \delta$ for some Lebesgue measurable set $A \subseteq (0,\infty)$ implies $\int_A \|R(r)\|\, dr < \epsilon/4k$. Then we know that there exists $l' > 0$ such that for each $w_0 \in \Lambda'$ there exists $\tau \in B(w_0, l') \cap \Lambda'$ such that, for every $t \in \mathbb{R}$ and $s \in [t, t+1]$, we have $\|f(\cdot+\tau) - Tf(\cdot)\|_{P_{\epsilon'}} \leq \epsilon'$. Suppose that $s \in \mathbb{R}$ is arbitrary. We will show that $\|F(s+\tau) - TF(s)\|_Y < \epsilon$. Assume the contrary, i.e., $\|F(s+\tau) - TF(s)\|_Y \geq \epsilon$. Then an elementary argumentation involving (12) and the commutivity assumption $R(t)T = TR(t)$, $t > 0$ shows that

$$\sum_{j=0}^{k-1} \int_j^{j+1} \|R(r)\| \cdot \|f(s+\tau-r) - Tf(s-r)\|\, dr = \int_0^k \|R(r)\| \cdot \|f(s+\tau-r) - Tf(s-r)\|\, dr \geq \epsilon/2.$$

If $j \in \mathbb{N}_0$, $0 \leq j \leq k-1$ and $r \in [j, j+1]$, then $s - r \in [s-j-1, s-j]$ so that

$$m\Big(\big\{r \in [j, j+1] : \|f(s+\tau-r) - Tf(s-r)\| \geq \epsilon'\big\}\Big) \leq \epsilon'$$

and

$$\int_j^{j+1} \|R(r)\| \cdot \|f(s+\tau-r) - Tf(s-r)\|\, dr < \epsilon'(1+\|T\|)\|f\|_\infty \epsilon/4k + \epsilon' \int_j^{j+1} \|R(r)\|\, dr.$$

The last estimate and the choice of $\epsilon' > 0$ implies

$$\int_0^k \|R(r)\| \cdot \|f(s+\tau-r) - Tf(s-r)\|\, dr < \epsilon/2,$$

which contradicts (4.1). Part (ii) basically follows from the evidence given in the proof of [1, Proposition 1.3.2 c)]. ◻

Similarly, we can prove the next result (by $Id$ we denote the identity operator on $X$):

**Theorem 3.**

(*i*) *Let* $(R(\mathbf{t}))_{\mathbf{t}>\mathbf{0}} \subseteq L(X,Y)$ *be a strongly continuous operator family such that* $\int_{(0,\infty)^n} \|R(\mathbf{t})\| \, d\mathbf{t} < \infty$. *If* $f : \mathbb{R}^n \to X$ *is bounded and Bohr* $(\Lambda', \rho, [0,1]^n, m)$*-almost periodic* ($(\Lambda', \rho, [0,1]^n, m)$*-uniformly recurrent), where* $\rho = cId \in L(X)$*, then the function* $F : \mathbb{R}^n \to Y$*, given by* (10)*, is bounded, continuous and Bohr* $(\Lambda', \rho_Y, [0,1]^n)$*-almost periodic* ($(\Lambda', \rho_Y, [0,1]^n)$*-uniformly recurrent), where* $\rho_Y = cId \in L(Y)$.

(*ii*) *Let* $a(\cdot)$ *be Lebesgue measurable and let* $\int_{(0,\infty)^n} |a(\mathbf{t})| \, d\mathbf{t} < \infty$. *If* $f : \mathbb{R}^n \to Y$ *is bounded and Bohr* $(\Lambda', \rho, [0,1]^n, m)$*-almost periodic* ($(\Lambda', \rho, [0,1]^n, m)$*-uniformly recurrent), where* $\rho = cId \in L(Y)$*, then the function* $F : \mathbb{R}^n \to Y$*, given by* (11)*, is bounded, uniformly continuous and Bohr* $(\Lambda', \rho, [0,1]^n)$*-almost periodic* ($(\Lambda', \rho, [0,1]^n)$*-uniformly recurrent).*

Now we will provide some notes about the last two results:

*Remark* 3. If the function $f(\cdot)$ is uniformly continuous in part (i) of Theorem 2 or Theorem 3, then the resulting function $F(\cdot)$ will also be uniformly continuous, which directly follows from [1, Proposition 1.3.5 c)]. But, in the operator-valued setting, it is not clear how to prove that the boundedness of $f(\cdot)$ implies the uniform continuity of $F(\cdot)$.

*Remark* 4.

(i) Suppose that the requirements of Theorem 2(i) or Theorem 3(i) hold. If $f(\cdot)$ has a relatively compact range, then $F(\cdot)$ has a relatively compact range as well.

(ii) Suppose that the requirements of Theorem 2(ii) or Theorem 3(ii) hold. If $f(\cdot)$ has a relatively compact range, then $F(\cdot)$ has a relatively compact range as well.

In order to see that (i) holds, observe that $F(\mathbf{t}) := \int_{(0,\infty)^n} R(\mathbf{s}) f(\mathbf{t}-\mathbf{s}) \, d\mathbf{s}$, $\mathbf{t} \in \mathbb{R}^n$. If $\epsilon > 0$ is given, then there exists a finite set $\{x_i : 1 \le i \le k\} \subseteq X$ such that $\overline{R(f)} \subseteq \bigcup_{1\le i\le k} L(x_i, \epsilon)$, where $L(x_i, \epsilon)$ denotes the open ball in $X$ with the center $x_i$ and the radius $\epsilon$. Then we have

$$\overline{R(F)} \subseteq \bigcup_{1\le i\le k} L\Bigg( \int_{(0,\infty)^n} R(\mathbf{s}) x_i \, d\mathbf{s}, \epsilon \int_{(0,\infty)^n} \|R(\mathbf{s})\| \, d\mathbf{s} \Bigg),$$

which simply implies the required. We can similarly show (ii).

*Remark* 5. It is not clear whether we can replace the boundedness of function $f(\cdot)$ in the formulation of Theorem 2 or Theorem 3 by some weaker conditions, for example, by its Stepanov $p$-boundedness for some $p > 0$. In connection with this issue, we would like to stress that such attempts for the usual infinite convolution product have already been analyzed by G. Bruno and A. Pankov in 2000 (see [3, Lemma 2]). Unfortunately, the proof of this result is not correct because the authors have not shown the (absolute) convergence of the integral $\int_{\mathbb{R}} \varphi(t) u(x-t) \, dt$ in a proper way ($x \in \mathbb{R}$); here, we use the same terminology from [3].

*Remark* 6. As the referee of the former version of this paper has noticed, the assumption $R(\mathbf{t})T = TR(\mathbf{t})$, $\mathbf{t} > \mathbf{0}$ is crucial and the proof of Theorem 2 does not work if this condition is neglected. For example, suppose that $n = 1$, $X := L^1(\mathbb{R})$, $T \in L(X)$ is given by $[Tg](t) := g(-t)$, $t \in \mathbb{R}$, $g \in L^1(\mathbb{R})$ and $f(\cdot)$ is the 2-periodic extension of the function $f_0 : [0,2) \to \mathbb{R}$, given by $f_0(t) := 1$ for $0 \le t < 1$ and $f_0(t) := 0$ for $1 \le t < 2$, to the whole real line. Then we have $f(t+1) = Tf(t)$ for all $t \in \mathbb{R}$ and an arduous computation yields that $\|F(t+1) - TF(t)\|_{L^1(\mathbb{R})} \ge 2/(1+e)$ for all $t \in [0,1]$.

In order to provide certain applications of Theorem 2(i), we will first revisit paper [26] by R. K. Miller, R. L. Wheeler, and our previous analysis from [18, Example 3.3.32]:

**Example 4.** Let $Y = H$ be an infinite-dimensional Hilbert space with inner product $\langle \cdot, \cdot \rangle$. In [26], R. K. Miller and R. L. Wheeler analyzed the well-posedness of the following abstract Cauchy problem of non-scalar type:

$$x'(t) = Ax(t) + \int_0^t b(t-s)(A + aId)x(s)\,ds + f(t), \quad x(0) = x_0, \tag{13}$$

where $b(t)$ is a scalar-valued kernel, $b \in C^1([0,\infty))$, $a \in \mathbb{C}$, $f : [0,\infty) \to H$ is continuous and $A$ is a densely defined, self-adjoint closed linear operator in $H$. We know that the validity of asuppositions [26, (A1)-(A5)] with $\alpha = \beta_0 = \beta_1 = 0$ and the validity of assumption [26, (A6)] with $B\sigma(L) \neq \emptyset$ (cf. [26, p. 273] for the notion) imply by [26, Theorem 8] that there exists a unique residual resolvent $(R(t))_{t\geq 0}$ for (13) such that $\|R(\cdot)\| \in L^q([0,\infty))$ for $1 \leq q < \infty$; then, [26, Theorem 2] yields that the unique solution of (13) for all $x_0 \in D(A)$ and $f \in C^1([0,\infty) : X)$ is given by

$$x(t) = R(t)x_0 + \int_0^t R(t-s)f(s)\,ds, \quad t \geq 0.$$

In the sequel, let us emphasize that the assumption $\|R(\cdot)\| \in L^q([0,\infty))$ for $1 \leq q < \infty$ on the resolvent solution family $(R(t))_{t\geq 0}$ does not directly imply that

$$\sum_{k=0}^{\infty} \|R(\cdot)\|_{L^q[k,k+1]} < +\infty \ \text{ for some } \ q \in (1, +\infty], \tag{14}$$

which would be also very difficult to prove using the methods proposed in [26]. If we assume that the forcing term $f \in C^1([0,\infty) : X)$ satisfies that there exists a function

$$f_1 \in C^1(\mathbb{R} : H) \cap \left[ C_b(\mathbb{R} : H) \bigcap_{p\geq 1} S^p AP(\mathbb{R} : H) \right]$$

such that $f_1(t) = f(t)$ for all $t \geq 0$, then a simple argumentation involving the decomposition

$$x(t) = R(t)x_0 + \int_{-\infty}^t R(t-s)f(s)\,ds - \int_{-\infty}^0 R(t-s)f(s)\,ds, \quad t \geq 0,$$

Theorem 2(i) and Proposition 2 show that the solution $x(\cdot)$ belongs to the space

$$\bigcap_{p\geq 1} L^p([0,\infty) : H) + C_0([0,\infty) : H) + AP(\mathbb{R} : H).$$

We continue by noticing that the integrability of resolvent operator families for the abstract Volterra integral equations was considered by J. Prüss in [31, Part III, Section 10]. For further applications of Theorem 2(i), it is important to say that we have not been able to locate any relevant result examining the question whether a norm integrable resolvent family $(R(t))_{t\geq 0}$ fulfills the estimate (14); see, e.g., [31, Theorem 10.1, Corollary 10.1, pp. 262–263] for some sufficient conditions ensuring that $\int_0^{+\infty} \|R(s)\|\,ds < +\infty$. Therefore, the asymptotical almost periodicity of the corresponding abstract Volterra integral equations, with the forcing terms of the same type, follows from the consideration given in Example 4 and Theorem 2(i); [16, Proposition 2.6.11] is generally not applicable here.

Now we will provide the following simple application of Theorem 2(ii):

**Example 5.** We can simply construct a great number of kernels $a \in L^1((0,\infty)^n)$ which are not locally $q$-integrable at zero for any exponent $q \in (1,\infty]$. If the function $f(\cdot)$ is bounded and Stepanov-$p$-almost periodic for some $p \in [1,\infty)$, then we cannot apply [18, Theorem 6.2.36] to see that the resulting function $F(\cdot)$ is almost periodic. But, the almost periodicity of $F(\cdot)$ can be proved using Theorem 2(ii).

### 4.2. Semilinear Volterra integral equations

First of all, let us define $S^\infty(\mathbb{R}^n : X) := C_b(\mathbb{R}^n : X) \bigcap_{p\geq 1} S^p AP(\mathbb{R}^n : X)$; equipped with the sup-norm, $S^\infty(\mathbb{R}^n : X)$ is a Banach space. By $S^\infty(\mathbb{R}^n \times X : Y)$ we denote the set of all continuous functions $F : \mathbb{R}^n \times X \to Y$ such that $\sup_{\mathbf{t}\in\mathbb{R}^n, x\in K} \|F(\mathbf{t}; x)\|_Y < +\infty$ for each compact set $K \subseteq \mathbb{R}^n$ and the Bochner transform $\hat{F}(\cdot;\cdot) : \mathbb{R}^n \times X \to L^p([0,1]^n : Y)$ is Bohr $\mathcal{B}$-almost periodic for every finite exponent $p \geq 1$, where $\mathcal{B}$ contains all compact subsets of $X$.

Consider now the following integral equation:

$$u(\mathbf{t}) = f(\mathbf{t}) + \int_{-\infty}^{t_1}\int_{-\infty}^{t_2}\cdots\int_{-\infty}^{t_n} a(\mathbf{t}-\mathbf{s})F(\mathbf{s}; u(\mathbf{s}))\, d\mathbf{s}, \quad \mathbf{t}\in\mathbb{R}^n, \tag{15}$$

where $X = Y$ is a finite-dimensional complex Banach space, $a \in L^1((0,\infty)^n)$ and $f \in S^\infty(\mathbb{R}^n : X)$. The mapping $\Psi : S^\infty(\mathbb{R}^n : X) \to S^\infty(\mathbb{R}^n : X)$, given by

$$(\Psi u)(\mathbf{t}) := f(\mathbf{t}) + \int_{-\infty}^{t_1}\int_{-\infty}^{t_2}\cdots\int_{-\infty}^{t_n} a(\mathbf{t}-\mathbf{s})F(\mathbf{s}; u(\mathbf{s}))\, d\mathbf{s}, \quad \mathbf{t}\in\mathbb{R}^n,\ \mathbf{u}\in S^\infty(\mathbb{R}^n : X),$$

is well-defined due to [7, Theorem 4.4] and Theorem 2(ii). Moreover, $\Psi(\cdot)$ is a contraction provided that there exists a finite real constant $L > 0$ such that $\|F(\mathbf{t}; x) - F(\mathbf{t}; y)\| \leq L\|x - y\|$ for all $\mathbf{t}\in\mathbb{R}^n$, $x,\ y \in X$ and $L\int_{(0,\infty)^n} |a(\mathbf{s})|\, d\mathbf{s} < 1$ so that the integral equation (15) has a unique solution which belongs to the space $S^\infty(\mathbb{R}^n : X)$. Let us finally notice that the solution $u(\cdot)$ will be almost periodic if the function $f(\cdot)$ is almost periodic.

Let us emphasize now that Theorem 2 can be reformulated for the usual convolution product

$$(\mathbf{t}, x) \mapsto \int_{\mathbb{R}^n} h(\mathbf{t}-\mathbf{s})F(\mathbf{s}; x)\, d\mathbf{s}, \quad \mathbf{t}\in\mathbb{R}^n,\ x\in X,$$

where $h \in L^1(\mathbb{R}^n)$ and $F(\cdot;\cdot)$ is bounded, Bohr $(\mathcal{B}, \Lambda', \rho, [0,1]^n, m)$-almost periodic ($(\mathcal{B}, \Lambda', \rho, [0,1]^n, m)$-uniformly recurrent), where $\rho = T \in L(Y)$. The resulting function has the same properties as $F(\cdot;\cdot)$ and this can be applied in the analysis of the existence and uniqueness of bounded, Bohr $(\Lambda', \rho, [0,1]^n, m)$-almost periodic ($(\Lambda', \rho, [0,1]^n, m)$-uniformly recurrent) solutions of the abstract semilinear integral equation

$$u(\mathbf{t}) = f(\mathbf{t}) + \int_{\mathbb{R}^n} h(\mathbf{t}-\mathbf{s})F(\mathbf{s}; u(\mathbf{s}))\, d\mathbf{s}, \quad \mathbf{t}\in\mathbb{R}^n,$$

where $f(\cdot)$ is bounded, Bohr $(\Lambda', \rho, [0,1]^n, m)$-almost periodic ($(\Lambda', \rho, [0,1]^n, m)$-uniformly recurrent). Unfortunately, the applications to the heat equation in $\mathbb{R}^n$ are really confined because the Gaussian kernel rapidly decays at infinity and the constructed solutions are always almost periodic ([18]).

## 5. Conclusions and final remarks

In this paper, we have analyzed various classes of multi-dimensional $\rho$-almost periodic type functions in general measure. We have extended many statements known in the one-dimensional setting and provided some noteworthy applications of our results. We can similarly analyze the notion of $(\omega, \rho)$-periodicity and $(\omega_j, \rho_j)_{j\in\mathbb{N}_n}$-periodicity in general measure; cf. [11, Section 3] for more details about the subject.

Finally, we would like to stress the following: Suppose that $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $P_\epsilon \subseteq Y^\Lambda$, the space of all functions from $\Lambda$ into $Y$, the zero function belongs to $P_\epsilon$ and $\mathcal{P}_\epsilon = (P_\epsilon, d_\epsilon)$ is a premetric space ($\epsilon > 0$); if $f \in P_\epsilon$, then we set $\|f\|_{P_\epsilon} := d_\epsilon(f, 0)$. It is worth noting that the properties of a pseudo-semimetric given by (2) can give us the idea to generalize the notion from [17, Definition 2.1, Definition 2.2, Definition 3.1] following the method proposed for introducing the notion in Definition 4. In the general case, the quantities $\|\cdot\|_{P_\epsilon}$ and $\|\cdot\|_{P_\eta}$ are not comparable if $\epsilon < \eta$, as in the case of our former analysis of $m$-almost periodicity, so we can also introduce the following notion: $F(\cdot;\cdot)$ is said to be Bohr $(\mathcal{B}, \Lambda', \rho, \mathcal{P})_{\epsilon,\eta}$-almost periodic if for every $B \in \mathcal{B}$ and $\epsilon,\ \eta > 0$ there exists $l' > 0$ such that for each $\mathbf{w}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{w}_0, l') \cap \Lambda'$

such that, for every $\mathbf{t} \in \Lambda$ and $x \in B$, there exists $y_{\mathbf{t};x} \in \rho(F(\mathbf{t};x))$ such that $F(\cdot + \tau; x) - y_{\cdot;x} \in P_\eta$ for all $x \in B$ and

$$\sup_{x\in B}\Big\| F(\cdot + \tau; x) - y_{\cdot;x} \Big\|_{P_\eta} \le \epsilon.$$

It is very difficult to say anything relevant about the introduced notion if the premetric spaces under our consideration are not pseudometric spaces. Because of that, we can freely say that the validity of the triangle inequality is almost inevitable in any serious research on the metrical almost periodicity and its generalizations. However, some statements like [17, Proposition 2.3, Proposition 3.14], assertions (i) and (iii) clarified on pp. 234-235 of [17] and assertions (i)-(ii) clarified on p. 246 of [17] can be formulated with general premetric spaces; on the other hand, it seems that the assertions of [17, Proposition 2.6, Theorem 2.7, Proposition 3.7] cannot be properly reformulated if the triangle inequality does not hold in our framework.

The generalized almost automorphic type functions have recently been considered in the research study [21]. We close the paper by proposing the following open problem (cf. also [18, Theorem 2.1.26], [7, Theorem 2.15] and [11, Theorem 2.28]):

PROBLEM. Suppose that $(\mathbf{u}_1, \cdots, \mathbf{u}_n)$ is a basis of $\mathbb{R}^n$,

$$\Lambda' = \Lambda = \Big\{ \alpha_1 \mathbf{u}_1 + \cdots + \alpha_n \mathbf{u}_n : \alpha_i \ge 0 \text{ for all } i \in \mathbb{N}_n \Big\}$$

and $F : \Lambda \to Y$ is an unbounded Bohr $(\Omega, m)$-almost periodic function, where $\Omega = [-1, 1]^n \cap \Lambda$. Is there a (unique) Bohr $(\Omega, m)$-almost periodic function $\tilde{F} : \mathbb{R}^n \to Y$ such that $\tilde{F}(\mathbf{t}) = F(\mathbf{t})$ for all $\mathbf{t} \in \Lambda$?

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