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Some special cellular subdivisions of simplicial complexes with an application to shape theory

Abstract.

Homotopy theory, an essential part of algebraic topology, is devoted to the problem of classifying mappings between "locally nice spaces" (e.g., polyhedra) up to homotopy. Shape theory extends homotopy theory to arbitrary spaces. The basic technique of shape theory consists in approximating spaces by polyhedra, i.e., in replacing spaces X, Y by particular inverse systems of polyhedra $\boldsymbol{X}, \boldsymbol{Y}$, called resolutions. The role of mappings $f\colon X \to Y$ is now taken up by homotopy mappings $\boldsymbol{f}\colon \boldsymbol{X} \to \boldsymbol{Y}$ between polyhedral resolutions of X and Y. These are homotopy commutative "ladders", where "rows" are just the resolutions $\boldsymbol{X}, \boldsymbol{Y}$ and "vertical arrows" f_i are mappings between members of $\boldsymbol{X}, \boldsymbol{Y}$, which form $\boldsymbol{f}$.

In strong shape theory, which takes an intermediate position between homotopy and usual shape, in the "ladders" one must fix homotopies f_{ij}, $i \leq j$, which realize the homotopy commutativity of the "ladder". For three indices $i \leq j \leq k$, one must have homotopies of dimension 2, i.e., mappings $f_{ijk}\colon X_k \times \Delta^2 \to Y_i$ which relate f_{ij}, f_{jk} and f_{ik}. This process continuous to involve homotopies of all dimensions. It yields the notion of a coherent homotopy mapping $\boldsymbol{f}\colon \boldsymbol{X} \to \boldsymbol{Y}$.

In the year 2003 I constructed a standard resolution $\boldsymbol{Y}$ for the Cartesian product $X \times P$ of a compactum X with a (possibly infinite) polyhedron P. The system $\boldsymbol{Y}$ consists of some quotient spaces Y_μ of disjoint sums of Cartesian products $X_i \times \sigma$, where X_i belongs to $\boldsymbol{X}$ and σ is a simplex of a triangulation K of P. Now I succeeded in extending that notion to a functor, i.e., I associated with every coherent homotopy mapping $\boldsymbol{f}\colon \boldsymbol{X} \to \boldsymbol{X}'$ a homotopy mapping $\boldsymbol{g}\colon \boldsymbol{Y} \to \boldsymbol{Y}'$ between the standard resolutins of $X \times P$ and $X' \times P$ such that $\boldsymbol{f}'\boldsymbol{f} = \boldsymbol{f}''$ implies that $\boldsymbol{g}'\boldsymbol{g}$ is homotopic to $\boldsymbol{g}''$.

In order to define $\boldsymbol{g}$ one needs a particuar cellular subdivision $L(K)$ of the simplicial complex K. It subdivides a 1-simplex in three 1-cells, a 2-simplex in ten 2-cells, a 3-simplex in forty-one 3-cells, etc. The "vertical arrows" of $\boldsymbol{g}$ are induced by mappings explicitly defined on Cartesian products of the form $X_i \times c$, where $c \in L(\sigma)$. The definition of $\boldsymbol{g}'\boldsymbol{g}$ depends on a further subdivision $L'(K)$ of $L(K)$. On the other hand, the definition of $\boldsymbol{g}''$ depends on a subdivision $N'(K)$ of $L(K)$ obtained by a totally different procedure. Nevertheless, it turns out that the cellular complexes $L'(K)$ and $N'(K)$ are isomorphic. This is the main reason why $\boldsymbol{g}'\boldsymbol{g}$ and $\boldsymbol{g}''$ are homotopic. Consequence: If X and X' are compacta of the same shape, then so are the producst $X \times P$ and $X' \times P$.