Singular two-parameter eigenvalue problems

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The *two-parameter eigenvalue problem* has the form

$$\begin{aligned} A_1x_1 &= \lambda B_1x_1 + \mu C_1x_1, \\ A_2x_2 &= \lambda B_2x_2 + \mu C_2x_2, \end{aligned} \tag{1}$$

where A_i, B_i, and C_i are given $n_i \times n_i$ complex matrices, $\lambda, \mu \in \mathbb{C}$, and $x_i \in \mathbb{C}^{n_i}$ for $i = 1, 2$. A pair (λ, μ) is an *eigenvalue* if it satisfies (1) for nonzero vectors x_1, x_2, and the corresponding eigenvector is $x_1 \otimes x_2$.

On the tensor product space we can define $n_1n_2 \times n_1n_2$ matrices

$$\begin{aligned} \Delta_0 &= B_1 \otimes C_2 - C_1 \otimes B_2, \\ \Delta_1 &= A_1 \otimes C_2 - C_1 \otimes A_2, \\ \Delta_2 &= B_1 \otimes A_2 - A_1 \otimes B_2. \end{aligned}$$

The two-parameter eigenvalue problem (1) is *nonsingular* if its operator determinant Δ_0 is invertible. Atkinson showed [1] that a nonsingular two-parameter eigenvalue problem is equivalent to the joint generalized eigenvalue problems

$$\begin{aligned} \Delta_1 z &= \lambda \Delta_0 z, \\ \Delta_2 z &= \mu \Delta_0 z, \end{aligned} \tag{2}$$

where $z = x_1 \otimes x_2$. Many theoretical results and numerical methods for nonsingular two-parameter eigenvalue problems are based on this relation.

However, if all linear combinations of Δ_0, Δ_1, and Δ_2 are singular, then we say that (1) is singular. Just recently, some of the above relations were generalized to singular two-parameter eigenvalue problems in [3], where it is shown that the simple finite regular eigenvalues of (1) and (2) agree. We will present a numerical method from [2] that can solve a singular two-parameter eigenvalue problem by computing the common regular eigenvalues of the associated system of two singular generalized eigenvalue problems.

As possible applications that lead to singular two-parameter eigenvalue problems we will present a numerical method for the quadratic two-parameter eigenvalue problem and a numerical method for a system of two bivariate polynomials.

References

[1] F. V. Atkinson, *Multiparameter eigenvalue problems*, Academic Press, New York, 1972.

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[3] A. Muhič and B. Plestenjak, On the quadratic two-parameter eigenvalue problem and its linearization, Linear Algebra Appl. 432 (2010), pp. 2529–2542.