Low rank approximations of infinite-dimensional Lyapunov equations

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Abstract

We analyze the convergence properties of an explicitly constructed sequence of low rank approximations to the solution of an infinite dimensional operator Lyapunov equation. As an application of our abstract theory we consider approximations of linear control system in the context of model order reduction by balanced truncation.

In particular we are interested in systems governed by the heat equation with both distributed as well as boundary control. In a recent study by Opmeer et.al the authors have presented an ADI-based algorithm in infinite dimensional setting which was used to construct low rank approximations of the solutions of a Lyapunov equation.

Let now A and B be unbounded operators. The formal expression $AX + XA' = -BB'$, where A' and B' are appropriate operator duals, is called an abstract Lyapunov equation. We analyze the approximation properties of solutions of abstract Lyapunov equations in the setting of a scale of Hilbert spaces associated to an unbounded diagonalizable operator which satisfies the Kato's square root theorem. We call an (unbounded) operator A diagonalizable if there exists a bounded operator Q, with a bounded inverse, such that the (unbounded) operator $Q^{-1}AQ$ is a normal operator with a compact resolvent.

We assume that A generates an exponentially stable analytic semigroup and we construct—using sinc-quadrature techniques—a rank $(2k + 1) \times \text{Ran}(B)$ approximation X_{2k} to the operator X. We also give conditions under which an estimate

$$\sqrt{\sum_{i=2k+2}^{\infty} \sigma_i^2(X)} \leq \|X - X_{2k}\|_{HS} \leq O(\exp^{-\pi\sqrt{2k}}) , \qquad (1)$$

holds. Here $\|X\|_{HS} = \sqrt{\text{trace}(X^*X)}$ denotes the Hilbert-Schmidt norm of X and we note that our technique allows us to give the constant in the $O(\cdot)$ notation explicitly in terms of the weighted norms of A, B and Q and an estimate on the spectrum of A.

In the case of a (more strongly) unbounded control operator B, e.g. an operator only bounded in a weighted Hilbert space, we obtain the same type of convergence estimates in an associated weighted norm on (the subspace of) the space of compact operators.

Based on our convergence estimates we also discuss ramifications of this analysis for the design of adaptive finite element methods including the analysis of the influence of linear algebra approximations on the overall process.