Beyond Markovianity of heavy-tailed Pearson diffusions - fractional case

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Outline

- Fractional diffusions
- Pearson and fractional Pearson diffusions
- Transition densities of Pearson and fractional Pearson diffusions
- Fisher-Snedecor diffusion
  transition densities in non-fractional and fractional case
- Correlation structure of fractional Fisher-Snedecor diffusion

Fractional diffusion – definition

- $X_1 = (X_1(t),\, t \geq 0)$ – Markovian diffusion with transition densities $p_1(x,t;y)$
- $D = (D_t,\, t \geq 0)$ – standard stable subordinator independent of the diffusion X_1, with the Laplace transform

$$\mathbb{E}[e^{-sD_t}] = \exp\left(-ts^{\alpha}\right), \quad s \geq 0, \quad 0 < \alpha < 1$$

- $E_t = \inf\{x > 0\colon D_x > t\}$ - inverse of the α-stable subordinator D
- $(E_t,\, t \geq 0)$ – non-Markovian and non-decreasing, for every t random variable E_t has a density $f_t(\cdot)$ with the Laplace transform

$$\mathbb{E}[e^{-sE_t}] = \int_0^{\infty} e^{-sx} f_t(x)dx = \mathcal{E}_{\alpha}(-st^{\alpha}),$$

where $\mathcal{E}_{\alpha}(-st^{\alpha})$ is the Mittag-Leffler function

$$\mathcal{E}_{\alpha}(-st^{\alpha}) = \sum_{j=0}^{\infty} \frac{(-st^{\alpha})^j}{\Gamma(1+\alpha j)} \tag{1}$$

- **fractional diffusion** – non-Markovian process defined via time-change of the diffusion $X_1(t)$ by the inverse E_t of the α-stable subordinator, i.e.

$$X_{\alpha}(t) = X_1\left(E_t\right), \quad t \geq 0$$

Fractional diffusions – applications

- **hydrology** – modeling sticking and trapping of contaminant particles in a porous medium (Meerschaert et al., 2003) or a river flow (Chakraborty et al., 2009)
- **finance** – modeling delays between trades (Scalas, 2006)
- **statistical physics** – fractional time derivative appears in the equation for a continuous time random walk limit and reflects random waiting times between particle jumps (Meerschaert, 2004)

Fractional Pearson diffusion – definition

- **fractional Pearson diffusion (FPD)** – non-Markovian process

$$(X_\alpha(t),\, t \geq 0) = (X_1(E_t),\, t \geq 0)\,,$$

where $(X_1(t),\, t \geq 0)$ is the Pearson diffusion

- **Pearson diffusion (PD)** – a unique strong solution (Øksendal, Theorem 5.2.1) of the SDE

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \quad t \geq 0 \tag{2}$$

with polynomial infinitesimal parameters

$$\mu(x) = a_0 + a_1 x, \quad \sigma(x) = \sqrt{2b(x)} = \sqrt{2(b_2x^2 + b_1x + b_0)}$$

- $\mathfrak{p}(x)$ – the stationary density of the diffusion (2) belongs to the Pearson family of continuous distributions
- $\mu(x)$ and $b(x)$ are related to the polynomials in the **Pearson differential equation**

$$\frac{\mathfrak{p}'(x)}{\mathfrak{p}(x)} = \frac{(a_1 - 2b_2)x + (a_0 - b_1)}{b_2x^2 + b_1x + b_0}$$

Pearson diffusions - classification

six subfamilies of PD – according to the degree of polynomial $b(x)$ and, in the quadratic case, to the sign of b_2 and the sign of its discriminant Δ:

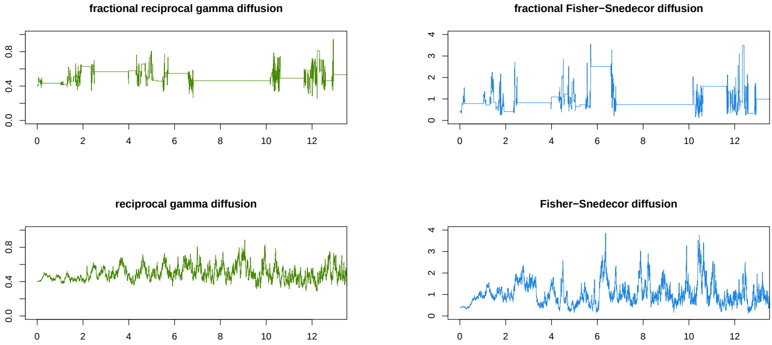
- constant $b(x)$ – OU process (Gaussian stationary distribution)
- linear $b(x)$ – CIR process (gamma stationary distribution)
- quadratic $b(x)$ with $b_2 < 0$ – Jacobi diffusion (beta stationary distribution)
- quadratic $b(x)$ with $b_2 > 0$ and $\Delta > 0$ – Fisher-Snedecor (FS) diffusion
- quadratic $b(x)$ with $b_2 > 0$ and $\Delta = 0$ – reciprocal gamma (RG) diffusion
- quadratic $b(x)$ with $b_2 > 0$ and $\Delta < 0$ – Student diffusion

- **important references:**

  Kolmogorov (1931), Wong (1964), Forman & Sørensen (2008), Avram et al. (2013a, 2013b)

PD and FPD – sample paths

Sample paths of fractional and non-fractional RG and FS diffusions with parameters $\theta = 0.01$ and $\alpha = 0.7$ based on 10000 points, starting from 0.4



Non-heavy-tailed Pearson diffusions

- **OU, CIR and Jacobi diffusions**

- **transition densities** – $p(x,t;y) = \frac{\partial}{\partial x} P(X_t \leq x | X_0 = y)$
  - closed-form expressions

    S. Karlin and H.M. Taylor (1981) A Second Course in Stochastic Processes, Academic Press, New York

  - spectral representations of transition densities – given in terms of the pure-point spectrum of the infinitesimal generator and the corresponding eigenfunctions (Hermite, Laguerre and Jacobi polynomials, respectively)

  - spectral analysis – overview of existing results given in

    N.N. Leonenko, M.M. Meerschaert and A. Sikorskii (2013) Fractional Pearson diffusions, Journal of Mathematical Analysis and Applications, **403***(2): 532–546*

Heavy-tailed Pearson diffusions

- **reciprocal gamma, Fisher-Snedecor and Student diffusions**
- **transition densities** – representable in terms of the spectrum of the corresponding infinitesimal generator and related functions
- **infinitesimal generator** of heavy-tailed Pearson diffusion

$$\mathcal{G}f(x) = \mu(x)f'(x) + \frac{\sigma^2(x)}{2}f''(x) \tag{3}$$

$\mu(x)$ - linear; $\sigma^2(x)$ - quadratic, with positive leading coefficient

- **spectrum** of the Sturm-Liouville operator $(-\mathcal{G})$
  - discrete spectrum $\sigma_d \subset [0, \Lambda\rangle$ - finite set of eigenvalues
    eigenfunctions are finite systems of orthogonal polynomials (Bessel, Fisher-Snedecor and Romanovski polynomials, respectively)
  - absolutely continuous spectrum $\sigma_{ac}(\mathcal{G})$ in $\langle\Lambda, \infty\rangle$
    functions related to the $\sigma_{ac}(\mathcal{G})$ - confluent (RG) and Gauss (FS, Student) hypergeometric functions

Fisher-Snedecor diffusion

- Fisher-Snedecor diffusion (FSD) **SDE**

$$dX_1(t) = -\theta \left(X_1(t) - \frac{\beta}{\beta - 2} \right) dt + \sqrt{\frac{4\theta}{\gamma(\beta-2)} X_1(t) \left(\gamma X_1(t) + \beta\right)} \, dW(t), \tag{4}$$

where $t \geq 0$ and $\theta > 0$ (autocorrelation parameter)

- **stationary density**

$$\mathfrak{fs}(x) = \frac{\beta^{\frac{\beta}{2}}}{B\left(\frac{\gamma}{2}, \frac{\beta}{2}\right)} \frac{(\gamma x)^{\frac{\gamma}{2}-1}}{(\gamma x + \beta)^{\frac{\gamma}{2}+\frac{\beta}{2}}} \gamma \, \mathrm{I}_{\langle 0,\infty\rangle}(x), \quad \gamma > 0, \quad \beta > 2$$

- **transition density** – spectral representation

$$p_1(x,t;y) = p_d(x,t;y) + p_c(x,t;y) \tag{5}$$

derived in

*F. Avram, N.N. Leonenko and N.Š. (2013) Spectral representation of transition density of Fisher-Snedecor diffusion, Stochastics, **85**(2): 346–369*

FSD – discrete part of transition density

- transition density – **discrete part**

$$p_d(x,t;y) = \mathfrak{fs}(x) \sum_{n=0}^{\left\lfloor \frac{\beta}{4} \right\rfloor} e^{-\lambda_n t} \, F_n(y) \, F_n(x) \tag{6}$$

- **eigenvalues** of the SL operator $(-\mathcal{G})$

$$\lambda_n = \frac{\theta}{\beta - 2} \, n(\beta - 2n), \quad n \in \{0, 1, \ldots, \lfloor \beta/4 \rfloor\}, \quad \beta > 2 \tag{7}$$

- **eigenfunctions** of the SL operator $(-\mathcal{G})$ – Fisher-Snedecor polynomials

$$F_n(x) = K_n \, x^{1-\frac{\gamma}{2}} \, (\gamma x + \beta)^{\frac{\gamma}{2}+\frac{\beta}{2}} \, \frac{d^n}{dx^n} \left\{ 2^n \, x^{\frac{\gamma}{2}+n-1} \, (\gamma x + \beta)^{n-\frac{\gamma}{2}-\frac{\beta}{2}} \right\} \tag{8}$$

FSD – continuous part of transition density

- transition density – **continuous part**

$$p_c(x,t;y) = \mathfrak{fs}(x)\,\frac{1}{\pi} \int_{\Lambda=\frac{\theta\beta^2}{8(\beta-2)}}^{\infty} e^{-\lambda t}\, a(\lambda)\, f_1(y,-\lambda) f_1(x,-\lambda)\, d\lambda \qquad (9)$$

- function f_1 – solution of the SL equation $\mathcal{G}f(x) = -\lambda f(x)$ for $\lambda > \Lambda$

$$f_1(x,-\lambda) = {}_2F_1\left(-\frac{\beta}{4} + ik(\lambda), -\frac{\beta}{4} - ik(\lambda); \frac{\gamma}{2}; -\frac{\gamma}{\beta}\,x\right), \qquad (10)$$

$$k(\lambda) = -i\sqrt{\frac{\beta^2}{16} - \frac{\lambda(\beta-2)}{2\theta}}$$

- normalizing constant

$$a(\lambda) = k(\lambda) \left| \frac{B^{\frac{1}{2}}\left(\frac{\gamma}{2}, \frac{\beta}{2}\right)\, \Gamma\left(-\frac{\beta}{4} + ik(\lambda)\right) \Gamma\left(\frac{\gamma}{2} + \frac{\beta}{4} + ik(\lambda)\right)}{\Gamma\left(\frac{\gamma}{2}\right)\, \Gamma\left(1 + 2ik(\lambda)\right)} \right|^2 \qquad (11)$$

Fractional FSD – transition density

- **fractional FS diffusion** – $(X_\alpha(t),\, t \geq 0)$, where $X_\alpha(t) = X_1(E_t)$, $t \geq 0$
  - $(X_1(t),\, t \geq 0)$ – FS diffusion given by the SDE (4)
  - $(E_t,\, t \geq 0)$, where $E_t = \inf\{x > 0\colon D_x > t\}$
    inverse of the α-stable subordinator, $0 < \alpha < 1$

- **transition density** – defined as

$$P(X_\alpha(t) \in B | X_\alpha(0) = y) = \int_B p_\alpha(x, t; y) dx \tag{12}$$

for any Borel set B from $\mathcal{B}_{(0,\infty)}$

Fractional FSD – transition density

- transitions density $p = p_\alpha(x, t; y)$ of the FS diffusion satisfies the following equations:
  - fractional forward (Fokker-Planck) equation

$$\frac{\partial^\alpha p}{\partial t^\alpha} = \frac{\partial}{\partial x}\left(-\mu(x)p\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\sigma^2(x)}{2}p\right)$$

    with the point-source initial condition $p(x, 0; y) = \delta(x - y)$
  - fractional backward equation

$$\frac{\partial^\alpha p}{\partial t^\alpha} = \mu(y)\frac{\partial p}{\partial y} + \frac{\sigma^2(y)}{2}\frac{\partial^2 p}{\partial y^2}$$

  - $\partial^\alpha/\partial t^\alpha$ – **Caputo fractional derivative** of order $0 < \alpha < 1$

$$\frac{d^\alpha f(x)}{dx^\alpha} = \frac{1}{\Gamma(1-\alpha)}\int_0^\infty \frac{d}{dx} f(x-y) y^{-\alpha}\, dy$$

Fractional FSD – transition density

Theorem

The transition density of fractional FS diffusion is given by

$$p_\alpha(x,t;y) = \mathfrak{fs}(x) \sum_{n=0}^{\lfloor \frac{\beta}{4} \rfloor} F_n(y)\, F_n(x)\, \mathcal{E}_\alpha(-\lambda_n t^\alpha) + \tag{13}$$

$$+ \frac{\mathfrak{fs}(x)}{\pi} \int\limits_{\frac{\theta\beta^2}{8(\beta-2)}}^{\infty} \mathcal{E}_\alpha(-\lambda t^\alpha)\, a(\lambda)\, f_1(y,-\lambda), f_1(x,-\lambda)\, d\lambda,$$

where F_n are FS polynomials given by (8)*, f_1 is the solution of the non-fractional SL problem given by* (10)*, $a(\lambda)$ is given by* (11) *and $\mathcal{E}_\alpha(-\lambda t^\alpha)$ is the Mittag-Leffler function given by* (1)*.*

- detailed proof could be found in

*N.N. Leonenko, I. Papić, A. Sikorskii and N.Š. (2017) Heavy-tailed fractional Pearson diffusions, Stochastic Processes and their Applications, **127**(11), 3512-3535*

Fractional FSD transition density – sketch of the proof

$$
\begin{aligned}
P(X_\alpha(t) \in B | X_\alpha(0) = y) &= \int_0^\infty P(X_1(\tau) \in B | X_1(0) = y)\, f_t(\tau)\, d\tau \\
&= \int_0^\infty \int_B p_1(x, \tau; y)\, f_t(\tau)\, dx\, d\tau \\
&= \int_B \int_0^\infty (p_d(x, \tau; y) + p_c(x, \tau; y))\, f_t(\tau)\, d\tau\, dx =
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
&= \int_B \mathfrak{fs}(x) \left(\int_0^\infty \sum_{n=0}^{\lfloor \frac{\beta}{4} \rfloor} F_n(y) F_n(x) e^{-\lambda_n \tau} f_t(\tau) d\tau + \frac{1}{\pi} \int_0^\infty \int_\Lambda^\infty e^{-\lambda \tau} f_t(\tau) a(\lambda) f_1(y, -\lambda) f_1(x, -\lambda) d\lambda d\tau \right) dx \\
&= \int_B \mathfrak{fs}(x) \left(\sum_{n=0}^{\lfloor \frac{\beta}{4} \rfloor} F_n(y)\, F_n(x)\, \mathcal{E}_\alpha(-\lambda_n t^\alpha) + \frac{1}{\pi} \int_\Lambda^\infty \mathcal{E}_\alpha(-\lambda t^\alpha)\, a(\lambda)\, f_1(y, -\lambda)\, f_1(x, -\lambda)\, d\lambda \right) dx
\end{aligned} \tag{15}
$$

Fractional FSD transition density – sketch of the proof

- change of the order of integration in (14)
  follows from the non-negativity of p_1 and f_t (Fubini-Tonelli theorem)

- change of the order of integration in (15)
  follows by the Fubini theorem since

$$\int_{\Lambda}^{\infty} \int_{0}^{\infty} \left| e^{-\lambda\tau} \, f_t(\tau) \, a(\lambda) \, f_1(y, -\lambda) \, f_1(x, -\lambda) \right| \, d\tau \, d\lambda < \infty$$

  (for bounds regarding the Gauss hypergeometric functions we refer to Erdelyi, Equation 17, page 77)

Fractional FSD - limiting density

Theorem

Let $X_\alpha(t)$ be the fractional FS diffusion and let $p_\alpha(x,t)$ be the density of $X_\alpha(t)$. Assume that $X_\alpha(0)$ has a twice continuously differentiable density f that vanishes at infinity. Then

$$p_\alpha(x,t) \to \mathfrak{fs}(x) \text{ as } t \to \infty,$$

where $\mathfrak{fs}(x)$ is the stationary density of the non-fractional FS diffusion.

- detailed proof - Leonenko et al. (2017)

Fractional FSD - correlation structure

- fractional Pearson diffusion $(X_\alpha(t),\, t \geq 0)$, $X_\alpha(t) = X_1(E_t)$, is in the steady state if it starts from its stationary distribution with the density $\mathfrak{fs}(\cdot)$
- the autocorrelation function of $(X_\alpha(t),\, t \geq 0)$ is given by

$$\mathrm{Corr}\,(X_\alpha(t),\, X_\alpha(s)) = \mathcal{E}_\alpha(-\theta t^\alpha) + \frac{\theta \alpha t^\alpha}{\Gamma(1+\alpha)} \int_0^{s/t} \frac{\mathcal{E}_\alpha(-\theta t^\alpha (1-z)^\alpha)}{z^{1-\alpha}} dz \quad (16)$$

for $0 < s \leq t$

- detailed proof could be found in

Leonenko, N.N., Meerschaerd, M.M. and Sikorskii, A. (2013) *Correlation structure of fractional Pearson diffusions*, Comput. Math. Appl., **66**(5), 737–745

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