Time-changed SIRV model for epidemic of SARS-CoV-2 virus

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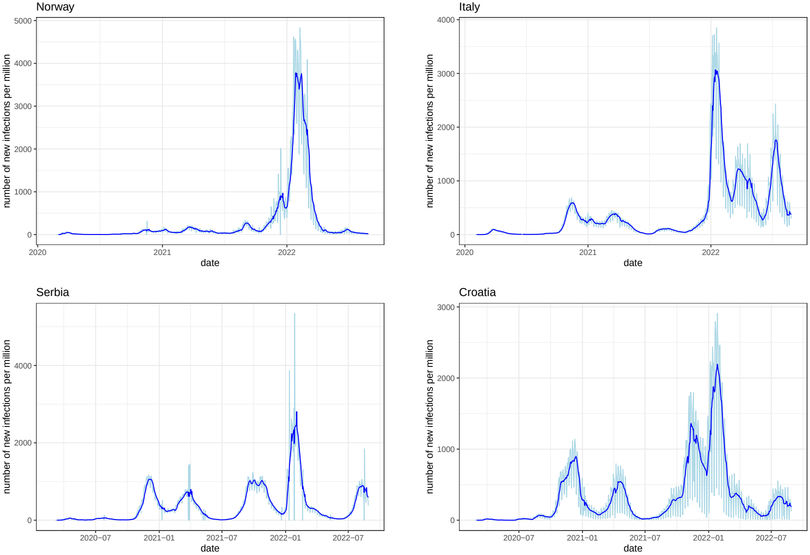
SARS-CoV-2 - daily number of infections



regularly updated data can be found on `https://ourworldindata.org/covid-cases`

Compartmental epidemiological models

- models for spread of the epidemic in population divided into several disjoint **compartments** or classes (e.g. susceptible **S**, infected **I**, recovered **R** and vaccinated **V** individuals)

- **population size** - either constant N or time-varying $(N(t),\, t \geq 0)$

- **deterministic case** - e.g. systems of difference equations; systems of ODEs

- **stochastic case** - e.g. multidimensional Markov chains in discrete or continuous time; systems of SDEs governed by **Brownian motion** or some other **Lévy process**

- models depend of several **parameters** - the most important is the **contact rate**, governing the dynamics of transition from susceptible to infected classes

SARS-CoV-2 - key terms in epidemic dynamics

- **contact rate (β)**

  the expected number of adequate contacts of infectious individual per day; an adequate contact between susceptible and infected individual is one that is sufficient for the transmission of infection

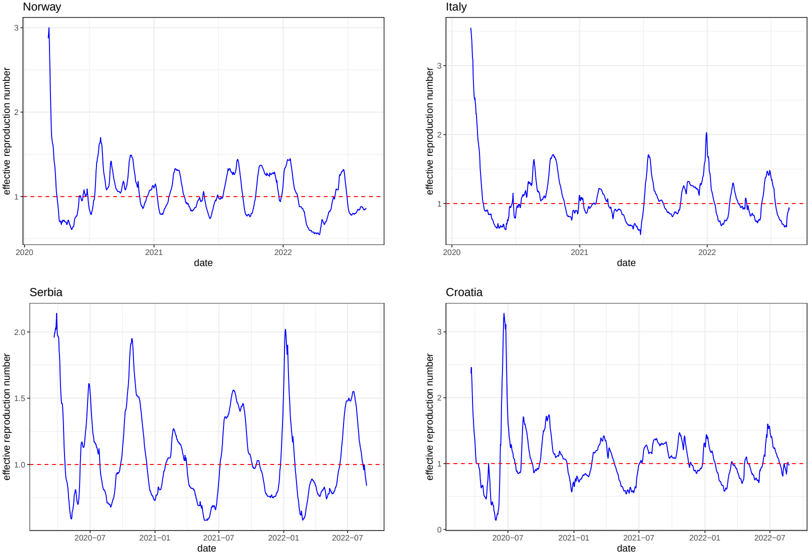
- **basic reproduction number (R_0)**

  the expected number of secondary infections produced by a single infected individual in a disease-free population; $R_0 = f(\beta)$ for a specific function f

- **effective reproduction number (R_e)**

  the expected number of secondary infections produced by a single infected individual in a population made up of both susceptible and non-susceptible hosts; $R_e(t) = R_0 \cdot \frac{S(t)}{N(t)} = f(\beta) \cdot \frac{S(t)}{N(t)}$

SARS-CoV-2 - effective reproduction number



(Arroyo-Marioli et al., 2021)

SIRV model - compartments

the human population is divided into four mutually exclusive compartments:

- **S** - susceptible individuals
- **I** - infected individuals
- **R** - recovered individuals
- **V** - vaccinated individuals
- the total population size at time $t \geq 0$ is $N(t) = S(t) + I(t) + R(t) + V(t)$

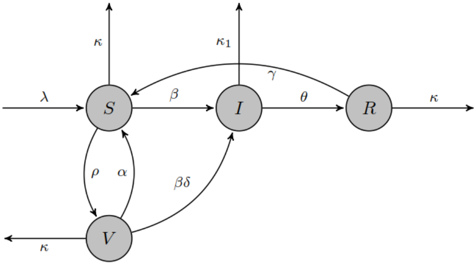


Figure 1: Scheme of SIRV model with temporary immunity

SIRV model - system of ODEs and interpretation of parameters

$$\begin{aligned}
dS(t) &= \left(\left(\lambda - \kappa - \rho - \tfrac{\beta}{N(t)} I(t) \right) S(t) + \alpha V(t) + \gamma R(t) \right) dt \\
dI(t) &= \left(\tfrac{\beta}{N(t)} I(t) \left(S(t) + \delta V(t) \right) - (\kappa_1 + \theta) I(t) \right) dt \\
dR(t) &= \left(\theta I(t) - (\kappa + \gamma) R(t) \right) \; dt \\
dV(t) &= \left(\rho S(t) - (\kappa + \alpha + \tfrac{\delta\beta}{N(t)} I(t)) V(t) \right) dt
\end{aligned}$$

Parameter	Description	Units
λ	birth rate	per day
β	contact rate	per day
ρ	vaccination rate within class S	per day
δ	effectiveness of vaccination	$[0, 1]$
γ	rate of immunity loss in class R	per day
α	rate of immunity loss in class V	per day
θ	recovery rate	per day
κ	natural death rate	per day
κ_1	disease-induced death rate	per day

SIRV model - natural assumptions

- number of organisms which can survive regarding to the resources available in the ecosystem is limited - **carrying capacity of the ecosystem** (K)

- from the perspective of modeling, for spread of the epidemic it is reasonable to consider **positive** and **bounded** process, i.e. for every $t \geq 0$:

  - $(S(t), I(t), R(t), V(t)) \in \mathbb{R}_+^4$

  - processes $S(t), I(t), R(t), V(t)$ have a lower and an upper bound

$$0 < \underline{S} < S(t) < \overline{S} < K$$
$$0 < \underline{I} < I(t) < \overline{I} < K$$
$$0 < \underline{R} < R(t) < \overline{R} < K$$
$$0 < \underline{V} < V(t) < \overline{V} < K$$

Motivation for modeling contact rate β

- **stochasticity** in epidemic models usually comes from the **modeling of the contact rate** β
- **usual approaches** for modeling β:
  - $\beta \longrightarrow \beta(t)$, where $\beta(t)$ is some appropriately chosen time-dependent (e.g. a piecewise constant) function, e.g. (Pardoux, 2021)
  - $\beta\,dt \longrightarrow \beta + dB_t$, where $(B_t,\, t \geq 0)$ is standard Brownian motion, e.g. (Đorđević et al., 2021a, 2021b)
  - $\beta\,dt \longrightarrow \beta(t)$, where $(\beta(t),\, t \geq 0)$ follows the Ornstein-Uhlenbeck process
  $$d\beta(t) = -\theta(\beta(t) - b)\,dt + \sigma\,dB_t,$$
  $\theta, b, \sigma > 0$, e.g. (Allen, 2017)

Model for contact rate in SIRV model

- based on the **time-changed Lévy noise** introduced in

Di Nunno, G., & Sjursen, S. (2014). BSDEs driven by time-changed Lévy noises and optimal control. *Stochastic Processes and their Applications*, 124(4), 1679-1709.

- chosen model for contact rate is time-dependent function with added noise driven by the **random measure** μ:

$$\beta\, dt \mapsto \beta(t)dt + \int_{\mathbb{R}} \sigma_t(z)\mu(dt, dz),$$

where μ is the mixture of a **conditional Brownian measure** B on $[0, T] \times \{0\}$ and a **centered doubly stochastic Poisson measure** $\widetilde{H}$ on $[0, T] \times \mathbb{R}_0$, $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$, and therefore

$$\beta\, dt \mapsto \beta(t)dt + \sigma_t(0)dB_t + \int_{\mathbb{R}_0} \sigma_t(z)\widetilde{H}(dt, dz)$$

Driving random measure

- $(\Omega, \mathcal{F}, \mathbb{P})$ - a complete probability space
- $X = [0,T] \times \mathbb{R} = ([0,T] \times \{0\}) \cup ([0,T] \times \mathbb{R}_0)$, $T > 0$
- $\mathcal{B}_X$ - Borel σ-algebra on X
- $\Delta \subset X$ - an element Δ in $\mathcal{B}_X$
- $\lambda := \left(\lambda^B, \lambda^H\right)$ - a two dimensional stochastic process such that each component λ^l, $l = B, H$ satisfies
  (i) $\lambda_t^l \geq 0$ $\mathbb{P}$-a.s. for all $t \in [0,T]$
  (ii) $\lim_{h \to 0} \mathbb{P}\left(\left|\lambda_{t+h}^l - \lambda_t^l\right| \geq \varepsilon\right) = 0$ for all $\varepsilon > 0$ and almost all $t \in [0,T]$
  (iii) $\mathbb{E}\left[\int_0^T \lambda_t^l \, dt\right] < \infty$

  $\mathcal{L}$ - space of all processes $\lambda := \left(\lambda^B, \lambda^H\right)$ satisfying (i)-(iii)

Driving random measure

- **random measure** Λ on X:

$$\Lambda(\Delta) := \int_0^T \mathbf{1}_{\{(t,0)\in\Delta\}}(t)\lambda_t^B \, dt + \int_0^T \int_{\mathbb{R}_0} \mathbf{1}_\Delta(t,z)\nu(dz)\lambda_t^H \, dt,$$

  where ν is a deterministic, σ-finite measure on the Borel sets of $\mathbb{R}_0$ satisfying

$$\int_{\mathbb{R}_0} z^2 \nu(dz) < \infty$$

- $\Lambda^B(\Delta)$ - restriction of Λ to $[0,T]\times\{0\}$
  $\Lambda^H(\Delta)$ - restriction of Λ to $[0,T]\times\mathbb{R}_0$

$$\Lambda(\Delta) = \Lambda^B(\Delta\cap[0,T]\times\{0\}) + \Lambda^H(\Delta\cap[0,T]\times\mathbb{R}_0)$$

- $\mathcal{F}^\Lambda$ - σ-algebra generated by values of Λ

Driving random measure

Definition (Di Nunno & Sjursen, 2014)

B **is a signed random measure** on Borel sets of $[0,T]\times\{0\}$ satisfying:

(i) $\mathbb{P}\left(B(\Delta)\le x \mid \mathcal{F}^{\Lambda}\right)=\mathbb{P}\left(B(\Delta)\le x \mid \Lambda^{B}(\Delta)\right)=\Phi\left(\frac{x}{\sqrt{\Lambda^{B}(\Delta)}}\right), x\in\mathbb{R}, \Delta\subseteq [0,T]\times\{0\}$

(ii) $B(\Delta_1)$ and $B(\Delta_2)$ are conditionally independent given $\mathcal{F}^{\Lambda}$ whenever Δ_1 and Δ_2 are disjoint

H **is a signed random measure** on Borel sets of $[0,T]\times\mathbb{R}_0$ satisfying:

(iii) $\mathbb{P}\left(H(\Delta)=k \mid \mathcal{F}^{\Lambda}\right)=\mathbb{P}\left(H(\Delta)=k \mid \Lambda^{H}(\Delta)\right)=\frac{\Lambda^{H}(\Delta)^k}{k!}e^{-\Lambda^{H}(\Delta)}, k\in\mathbb{N}, \Delta\subseteq[0,T]\times\mathbb{R}_0$

(iv) $H(\Delta_1)$ and $H(\Delta_2)$ are conditionally independent given $\mathcal{F}^{\Lambda}$ whenever Δ_1 and Δ_2 are disjoint

(v) B and H are conditionally independent given $\mathcal{F}^{\Lambda}$.

- (i) - conditional on Λ, B is a **Gaussian random measure**
- (iii)- conditional on Λ, H is a **Poisson random measure**

Driving random measure

Definition (Di Nunno & Sjursen, 2014)

The random measure μ on the Borel subsets of X is defined by

$$\mu(\Delta) := B(\Delta \cap [0,T] \times \{0\}) + \widetilde{H}\left(\Delta \cap [0,T] \times \mathbb{R}_0\right), \quad \Delta \subseteq X$$

where $\widetilde{H} := H - \Lambda^H$ is a measure given by

$$\widetilde{H}(\Delta) := H(\Delta) - \Lambda^H(\Delta), \quad \Delta \subset [0,T] \times \mathbb{R}_0.$$

Driving random measure

- properties of μ:
  - $\mathbb{E}\left[B(\Delta) \mid \mathcal{F}^{\Lambda}\right] = 0$ & $\mathbb{E}\left[H(\Delta) \mid \mathcal{F}^{\Lambda}\right] = \Lambda^{H}(\Delta) \implies \mathbb{E}\left[\mu(\Delta) \mid \mathcal{F}^{\Lambda}\right] = 0$
  - $\mathbb{E}\left[B(\Delta)^2 \mid \mathcal{F}^{\Lambda}\right] = \Lambda^{B}(\Delta)$ & $\mathbb{E}\left[\widetilde{H}(\Delta)^2 \mid \mathcal{F}^{\Lambda}\right] = \Lambda^{H}(\Delta) \implies \mathbb{E}\left[\mu(\Delta)^2 \mid \mathcal{F}^{\Lambda}\right] = \Lambda(\Delta)$
  - conditionally on $\mathcal{F}^{\Lambda}$, for disjoint Δ_1 and Δ_2 $\mu(\Delta_1)$ and $\mu(\Delta_2)$ are orthogonal

- μ is a martingale with respect to the following filtrations:
  - $\mathbb{F}^{\mu} = \{\mathcal{F}^{\mu}_t,\, t \in [0,T]\}$ is the filtration generated by $\mu(\Delta)$, $\Delta \subseteq [0,t] \times \mathbb{R}$
  - $\mathbb{G} = \{\mathcal{G}_t,\, t \in [0,T]\}$, $\mathcal{G}_t = \mathcal{F}^{\mu}_t \vee \mathcal{F}^{\Lambda}$

Driving random measure

- random measures B and H are related to a specific form of **time-change** for Brownian motion and pure jump Lévy process:

$$B_t := B([0,t] \times \{0\}), \quad \Lambda_t^B := \int_0^t \lambda_s^B \, ds, \quad t \in [0,T]$$

$$\eta_t := \int_0^t \int_{\mathbb{R}_0} z \tilde{H}(ds, dz), \quad \widehat{\Lambda}_t^H := \int_0^t \lambda_s^H \, ds, \quad t \in [0,T]$$

Theorem (Serfozo, 1972)

Let $W = (W_t, t \in [0,T])$ be a Brownian motion and $N = (N_t, t \in [0,T])$ be a centered pure jump Lévy process with Lévy measure ν. Assume that both W and N are independent of Λ. Then B satisfies (i) and (ii) if and only if, for any $t \geq 0$

$$B_t \overset{d}{=} W_{\Lambda_t^B},$$

and η satisfies (iii) and (iv) if and only if for any $t \geq 0$

$$\eta_t \overset{d}{=} N_{\hat{\Lambda}_t^H}.$$

Building stochastic SIRV model

- contact rate model:

$$\begin{aligned}\beta\, dt &\mapsto \beta(t)dt + \sigma_t(0)dB_t + \int_{\mathbb{R}_0} \sigma_t(z)\widetilde{H}(dt,dz)\\ \beta\, dt &\mapsto \beta(t)dt + \int\int_{\mathbb{R}} \sigma_t(z)\mu(dt,dz),\end{aligned}$$

- SIRV system of ODEs:

$$\begin{aligned}dS(t) &= \left(\left(\lambda-\kappa-\rho-\tfrac{\beta}{N(t)}I(t)\right)S(t)+\alpha V(t)+\gamma R(t)\right)\, dt\\ dI(t) &= \left(\tfrac{\beta}{N(t)}I(t)\left(S(t)+\delta V(t)\right)-(\kappa_1+\theta)I(t)\right)\, dt\\ dR(t) &= \left(\theta I(t)-(\kappa+\gamma)R(t)\right)\;dt\\ dV(t) &= \left(\rho S(t)-(\kappa+\alpha+\tfrac{\delta\beta}{N(t)}I(t))V(t)\right)\, dt\end{aligned}$$

SIRV model driven by random measure μ

$$
\begin{aligned}
dS(t) &= \left((\lambda - \rho - \kappa)S(t) - \frac{\beta(t)}{N(t)} S(t)I(t) + \alpha V(t) + \gamma R(t) \right) dt \\
&\quad - \int_{\mathbb{R}} \sigma_t(z) \frac{S(t)}{N(t)} I(t)\, \mu(dt, dz) \\
dI(t) &= \left(\frac{\beta(t)}{N(t)} \left(S(t) + \delta V(t)\right) - (\kappa_1 + \theta) \right) I(t)\, dt \\
&\quad + \int_{\mathbb{R}} \sigma_t(z) [S(t) + \delta V(t)] \frac{I(t)}{N(t)}\, \mu(dt, dz) \qquad (1) \\
dR(t) &= \left(\theta I(t) - (\kappa + \gamma) R(t)\right) dt \\
dV(t) &= \left(\rho S(t) - (\kappa + \alpha) V(t) - \delta \frac{\beta(t)}{N(t)} V(t) I(t) \right) dt \\
&\quad - \int_{\mathbb{R}} \sigma_t(z) \delta \frac{V(t)}{N(t)} I(t)\, \mu(dt, dz)
\end{aligned}
$$

SIRV model - analysis of the solution

Theorem

The following statements hold:

1. *Since the* ***capacity of the population*** *is bounded by a positive constant* K*, it follows that*

$$\limsup_{t\to\infty} N(t) = K_1 = \begin{cases} K, & \lambda > \kappa \\ N(0), & \lambda = \kappa \\ 0, & \lambda < \kappa. \end{cases}$$

2. *For any initial value* $(S(0), I(0), R(0), V(0)) \in \langle 0, K]^4$ *there exist a* ***unique global solution*** $((S(t), I(t), R(t), V(t)), t \geq 0)$ *of the SDE system* (1) *that almost surely* ***remains*** *in* $\langle 0, K]^4$.

SIRV model - outline of the proof (1)

- by solving the **differential equation for** $N(t)$ and by applying the L'Hospital rule, for $\lambda > \kappa$ it follows:

$$N(t) \leq e^{-\kappa t} \left(N(0) + \int_0^t \lambda S(s) e^{\kappa s} ds \right)$$

$$\downarrow$$

$$\limsup_{t \to \infty} N(t) \leq \limsup_{t \to \infty} \frac{\lambda S(t) e^{\kappa t}}{\kappa e^{\kappa t}} \leq \frac{\lambda}{\kappa} \limsup_{t \to \infty} S(t) \leq K$$

- furthermore, by summing all four equations from system (1), under natural assumption $\kappa_1 \geq \kappa$, it follows:

$$dN(t) = (\lambda S(t) - \kappa N(t) - (\kappa_1 - \kappa) I(t))\, dt$$

$$\downarrow (\kappa_1 \geq \kappa)$$

$$dN(t) \leq (\lambda S(t) - \kappa N(t))\, dt \leq (\lambda - \kappa) N(t) dt$$

$$\downarrow$$

$$N(t) \leq N(0) e^{t(\lambda - \kappa)}$$

SIRV model - outline of the proof (2)

- **the existence and uniqueness** of solution of system (1) for any initial value $(S(0), I(0), R(0), V(0)) \in \mathbb{R}_+^4$ on $[0, \tau_0]$, where τ_0 is the explosion time, follows from (Jacod, 1971)

- in order to prove that the solution of system (1) is **global**, it needs to be proven that $\tau_0 = \infty$ $\mathbb{P}$-a.s.

- for each $k > k_0$ define the stopping time

$$\tau_k = \inf \Big\{ t \in [0, \tau_0\rangle : \min \{S(t), I(t), R(t), V(t)\} \leq \frac{1}{k} \text{ or } \max \{S(t), I(t), R(t), V(t)\} \geq k \Big\},$$

where $k_0 > 0$ is a constant large enough such that $S(0), I(0), R(0), V(0)$ belong to the interval $[1/k_0, k_0]$ and $\inf \emptyset = \infty$

SIRV model - outline of the proof (2)

- note that τ_k increases as $k \to \infty$ and denote $\lim_{k\to\infty} \tau_k = \tau_\infty$

- if $\tau_\infty = \infty$ $\mathbb{P}$-a.s., then $\tau_0 = \infty$ $\mathbb{P}$-a.s., which means that $(S(t), I(t), R(t), V(t))$ $\mathbb{P}$-a.s. remains in $[0, K]^4$ for all $t > 0$

- the proof that $\tau_\infty = \infty$ $\mathbb{P}$-a.s. follows by **assuming** that there exist a pair of constants $T \geq 0$ and $\varepsilon \in (0, 1)$ such that $\mathbb{P}(\tau_\infty \leq T) \geq \varepsilon$, which leads to **contradiction**

- **technical details** of the proof after assumption $\mathbb{P}(\tau_\infty \leq T) \geq \varepsilon$:
  - define a twice continuously differentiable function

$$Y(S, I, R, V) = (S - 1 - \log(S)) + (I - 1 - \log(I)) + (R - 1 - \log(R)) + (V - 1 - \log(V)),$$

  where the dependence of S, I, R and V on t is omitted

SIRV model - outline of the proof (2)

- by applying the multidimensional **Itô's formula** for semimartingales (Protter, 2005) to Y, it follows that for every $t \geq 0$

$$dY(S,I,R,V) \leq \sum_{X=S,I,R,V} \left(\left(1 - \frac{1}{X(t)}\right) dX(t) + \frac{1}{2X^2(t)} \left(dX(t)\right)^2 \right) + C[\mu,\mu]_t$$

where the **quadratic variation of** μ comes from the "jump part" of the application of Itô's formula:

$$\sum_{0 \leq s \leq t} \left(X(s) - X(s-) - (\log X(s) - \log X(s-)) - \left(1 - \frac{1}{X(s-)}\right) \Delta X_s \right) \leq$$

$$\leq \widetilde{C}_i[X,X]_t \leq C_i[\mu,\mu]_t < \infty,$$

and where

$$C = C_1 + C_2 + C_3 + C_4$$

SIRV model - outline of the proof (2)

- under some technical assumptions

$$\begin{cases} S(t)+I(t)+R(t)+V(t)=N(t)\le K_1 \\ \frac{1}{N(t)} \le \max\left\{\frac{1}{S(t)}, \frac{1}{I(t)} \frac{1}{R(t)}, \frac{1}{V(t)}\right\} \le \widetilde{K}_1 \\ E\left[\int_0^T \sigma_t^2(0)\lambda_s^B ds + \int_0^T \int_{\mathbb{R}_0} \sigma_t^2(z)\nu(dz)\lambda_s^H ds\right] \le K_2, \end{cases} \tag{2}$$

due to **positivity of** (S, I, R, V) **process** and **non-negativity of its parameters**, it follows that

$$\mathbb{E}\left[Y\left(S(\tau_k\wedge T), I(\tau_k\wedge T), R(\tau_k\wedge T), Y(\tau_k\wedge T)\right)\right] \le$$

$$\mathbb{E}\left[Y\left(S(0), I(0), R(0), V(0)\right)\right] + \widetilde{N}(T),$$

where $\widetilde{N}(T)$ is finite quantity depending on T and

$$\mathbb{E}\left[Y\left(S(0), I(0), R(0), V(0)\right)\right] + \widetilde{N}(T) \ge \varepsilon \min\left\{k-1-\log(k), \frac{1}{k}-1+\log(k)\right\}$$

- by letting $k\to\infty$ it follows that

$$\mathbb{E}\left[Y\left(S(0), I(0), R(0), V(0)\right)\right] + \widetilde{N}(T) \ge \infty,$$

which gives a contradiction, i.e. $\tau_\infty = \infty$ $\mathbb{P}$-a.s.

SIRV model - outline of the proof (2)

- the set

$$\Gamma^{\star} = \{(S(t), I(t), R(t), V(t)) : S(t), I(t), R(t), V(t) > 0 \ \& \ N(t) \leq K\}$$

is a **positively invariant set** of the system (1) for every $t > 0$, i.e. if the system starts from $\Gamma^{\star}$, almost surely it never leaves $\Gamma^{\star}$

■

SIRV model - extinction

Theorem

If

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t \left(\left(\lambda_s^B \sigma_s(0) \right)^2 \right)^{-1} ds < \frac{2(\kappa_1 + \theta)}{K^2} \quad \mathbb{P} - a.s.,$$

and

$$\limsup_{t\to\infty} \frac{\Lambda_t}{t} < \infty \quad \mathbb{P} - a.s.,$$

than for any initial value $(S(0), I(0), R(0), V(0)) \in \Gamma^\star$ *it follows that*

$$I(t) \to 0 \quad \mathbb{P} - a.s. \ \ as\ t \to \infty,$$
$$R(t) \to 0 \quad \mathbb{P} - a.s. \ \ as\ t \to \infty,$$

while

$$\limsup_{t\to\infty} (S(t) + V(t)) = K_1 \quad \mathbb{P} - a.s.$$

Extinction - outline of the proof

- according to the boundedness of the process for contact rate and the boundaries (2), by applying the Itô's formula for semimartingales to the function $\ln(I(t))$ and dividing everything by t, it follows:

$$\frac{\ln(I(t))}{t} \leq \frac{\ln(I(0))}{t} + \int_0^t \left(\frac{K^2}{2\sigma_s^2(0)(\lambda_s^B)^2} - (\kappa_1 + \theta) \right) ds + k\frac{M_1(t)}{t},$$

  where k is a generic constant and

$$M_1(t) := \int_0^t \int_{\mathbb{R}} \sigma_s(z)\mu(ds, dz), \quad \langle M_1, M_1 \rangle_t = \Lambda_t$$

  is a martingale vanishing at 0

- as $\limsup_{t\to\infty} \frac{\langle M_1, M_1 \rangle_t}{t} < \infty$ $\mathbb{P}$-a.s., according to SLLN from (Mao, 2007) it follows that

$$\lim_{t\to\infty} \frac{M_1(t)}{t} = 0 \quad \mathbb{P} - a.s.$$

Extinction - outline of the proof

- then it follows that

$$\limsup_{t\to\infty} \frac{\ln(I(t))}{t} \le \limsup_{t\to\infty} \frac{1}{t}\int_0^t \frac{1}{(\sigma_s(0)\lambda_s^B)^2} ds - \frac{2(\kappa_1+\theta)}{K^2} < 0 \quad \mathbb{P}-a.s.$$

  and therefore, due to positivity of $I(t)$,

$$\lim_{t\to\infty} I(t) = 0 \quad \mathbb{P}-a.s.$$

- by solving the ODE for recovered class explicitly, we obtain that

$$R(t) = e^{-(\kappa+\gamma)t}\left(R(0) + \int_0^t \theta I(s) e^{(\kappa+\gamma)s}\, ds\right)$$

  and by applying the L'Hospital rule it follows that

$$\lim_{t\to\infty} R(t) = 0 \quad \mathbb{P}-\text{a.s.}$$

- at last, it follows that

$$\limsup_{t\to\infty} (S(t)+V(t)) = K_1 \quad \mathbb{P}-\text{a.s.}$$

■

Persistence in mean - definition

- the virus **remains persistent** in population if there is at least one infected individual communicating with susceptible subpopulation
- mathematical concept of persistence - **persistence in mean**
- the system (1) is said to be persistent in mean if

$$\liminf_{t\to\infty}[I(t)] = \liminf_{t\to\infty} \frac{1}{t}\int_0^t I(s)\,ds > 0, \quad \mathbb{P}-\text{a.s.}$$

SIRV model - persistence in mean

Theorem

If

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t \sigma_s^2(0)(\lambda_s^B)^2 ds \leq \widetilde{\beta}\,\frac{\lambda+\rho(\delta-1)}{\kappa}\,\frac{2K_1^2\underline{S}}{\underline{S}-\delta\underline{V}},$$

$$\widetilde{\beta} \leq \liminf_{t\to\infty}\frac{\beta(t)}{N(t)}, \quad \underline{S} \leq S(t), \quad \underline{V} \leq V(t), \quad \forall t \geq 0,$$

and

$$\limsup_{t\to\infty}\frac{\Lambda_t}{t} < \infty \quad \mathbb{P}-a.s.,$$

than for any initial value $(S(0), I(0), R(0), V(0)) \in \Gamma^\star$ *it follows that*

$$\liminf_{t\to\infty}[I(t)] > 0 \quad \mathbb{P}-a.s.$$

Persistence in mean - outline of the proof

- by applying Itô formula to $\ln(I(t))$ ant dividing the result by t it follows that

$$\liminf_{t\to\infty}\frac{\ln I(t)}{t}\geq\frac{\widetilde{\beta}}{\kappa}\liminf_{t\to\infty}\left((\lambda+\rho(\delta-1))\underline{S}-F(t)-\left(\frac{\theta\gamma}{\kappa+\gamma}-\kappa_1\right)[I(t)]\right.$$
$$\left.+\frac{1}{t}\int\limits_0^t\int\limits_{\mathbb{R}}\sigma_s(z)\frac{V(s)}{N(s)}\delta(1-\delta)I(s)\mu(ds,dz)\right)$$
$$-\frac{\underline{S}-\delta\underline{V}}{2K_1^2}\liminf_{t\to\infty}\frac{1}{t}\int_0^t\sigma_s^2(0)(\lambda_s^B)^2ds$$

where

$$F(t):=\frac{S(t)-S(0)}{t}+\frac{I(t)-I(0)}{t}+\delta\frac{V(t)-V(0)}{t}+\frac{\gamma}{\kappa+\gamma}\frac{R(t)-R(0)}{t}$$

and $[I(t)]$ comes from the definition of $F(t)$ after substituting the integral forms for $S(t)$, $I(t)$, $V(t)$ and $R(t)$:

Persistence in mean - outline of the proof

$$[S(t)+\delta V(t)] \geq \frac{1}{\kappa}\Bigg((\lambda+\rho(\delta-1))\underline{S} - K(t) - \left(\frac{\theta\gamma}{\kappa+\gamma}-\kappa_1\right)[I(t)] + \frac{1}{t}\int_0^t\int_{\mathbb{R}} \sigma_s(z)\frac{V(s)}{N(s)}\delta(1-\delta)I(s)\mu(ds,dz)\Bigg)$$

- from some natural properties of model parameters it follows that

$$\liminf_{t\to\infty}[I(t)] \geq \frac{(\kappa+\gamma)(\lambda+\rho(\delta-1))}{\theta\gamma-\kappa_1(\kappa+\gamma)}\underline{S} - \frac{\kappa(\kappa+\gamma)}{\widetilde{\beta}(\theta\gamma-\kappa_1(\kappa+\gamma))}\frac{\underline{S}-\delta\underline{V}}{2K_1^2}\liminf_{t\to\infty}\frac{1}{t}\int_0^t \sigma_s^2(0)(\lambda_s^B)^2\,ds$$

which is positive if

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t \sigma_s^2(0)(\lambda_s^B)^2 ds \leq \widetilde{\beta}\,\frac{\lambda+\rho(\delta-1)}{\kappa}\,\frac{2K_1^2\underline{S}}{\underline{S}-\delta\underline{V}} \quad \mathbb{P}-a.s.$$

■

Extinction and persistence - remarks

- the condition
$$\limsup_{t\to\infty} \frac{\Lambda_t}{t} < \infty \quad \mathbb{P} - a.s.$$
can be interpreted as the **"long term" comparability** of the time-change process and the real time

- it can be replaced by a stronger assumption of **ergodicity** of the integrands in the absolutely continuous time-change processes Λ^B and $\widehat{\Lambda}^H$ (Serfozo, 1972)

- this condition is always fulfilled when the time-change process is **slowing down the real time**, i.e. when $\Lambda(t) \leq t$ for all $t \geq 0$

Simulation study - contact rate model

- natural assumptions for contact rate model
  - **non-negativity**
  - **mean-reverting property**
  - **presence of jumps and clustering**

- an example of model for contact rate - **time-changed CIR jump diffusion**

- **SDE** for the CIR jump diffusion **(without time-change)**:

$$db(t) = -\theta\,(b(t) - \beta)\,dt + \sigma\sqrt{b(t)}\,dB_t + kZ_t$$

where $(Z_t,\, t \geq 0)$ is the compound Poisson process, k is the intensity of the jumps, σ is the volatility coefficient, β is the long-term level of the process and θ is the speed of reversion to β

Simulation study - time-change model

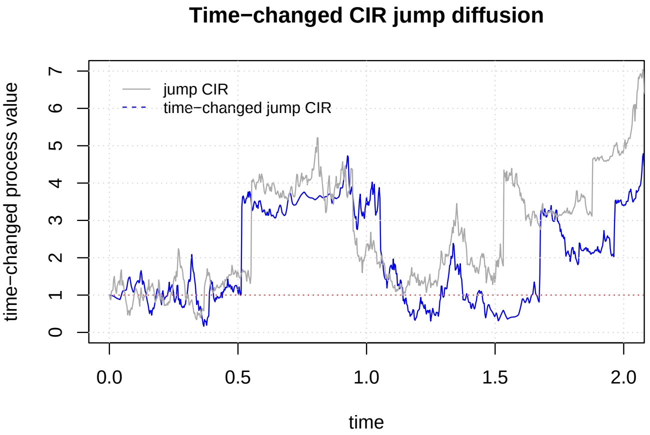
- choice of the **absolutely-continuous time-change processes** in Brownian and CPP part of the CIR jump diffusion - integrated process $(\lambda_t,\, t \geq 0)$:

  1. integrated **periodic function**
     $\lambda_t = a \sin(kt)$
  2. integrated **compound Poisson process (CPP) with drift**
     $$\lambda_t = dt + \sum_{k=0}^{N_t} X_k$$
  3. integrated **inverse-Gaussian subordinator** with Lévy measure
     $\pi(dx) = \frac{\delta}{\sqrt{2\pi x^3}} e^{-\alpha^2/2\, dx},\ x, \alpha, \delta > 0$
  4. integrated **Ornstein-Uhlenbeck process**
     $d\lambda_t = -\theta(\lambda_t - \mu) + \sigma dB_t$

- **algorithm for building the time-changed process** from simulated time-change process and simulated base process is given in (Magdziarz et al., 2007)

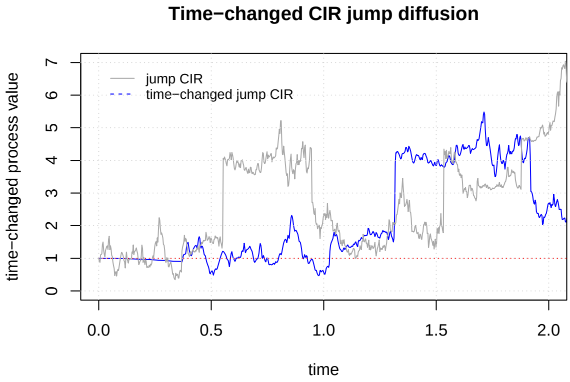
Contact rate - CIR jump diffusion time-changed by integrated periodic function

- $\lambda_t^B = \lambda_t^H = a\sin(kt),\ a = 1.5,\ k = 4$



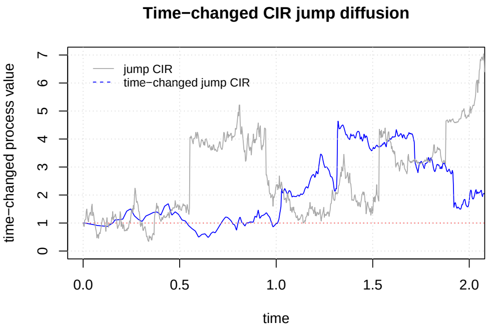
Contact rate - CIR jump diffusion time-changed by integrated CPP with drift

- $\lambda_t^B = \lambda_t^H = dt + \sum_{k=0}^{N_t} X_k$ **CPP with drift**, where $d = 0.05$, $X_t \sim \mathcal{U}(-1, 0.6)$, $(N_t,\, t \geq 0)$ Poisson process with intensity $\lambda = 2$



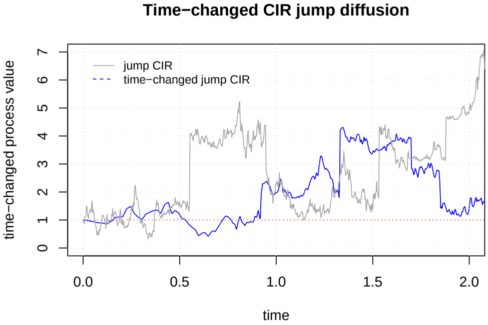
Contact rate - CIR jump diffusion time-changed by integrated IG subordinator and CPP with drift

- λ_t^B **IG(α, δ) subordinator**, $\alpha = 1$, $\delta = 5$
- $\lambda_t^H = dt + \sum_{k=0}^{N_t} X_k$ **CPP with drift**, $d = 0.05$, $X_t \sim \mathcal{U}(-1, 0.6)$, $(N_t,\ t \geq 0)$ Poisson process with intensity $\lambda = 2$



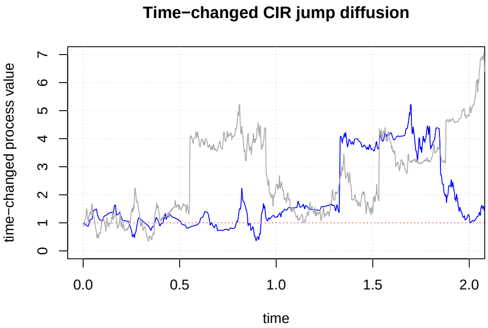
Contact rate - CIR jump diffusion time-changed by integrated IG subordinator and OU process

- λ_t^B **IG(α, δ) subordinator**, $\alpha = 1$, $\delta = 5$
- $d\lambda_t^H = -\theta(\lambda_t^H - \mu) + \sigma dB_t$ **Ornstein-Uhlenbeck process**, $\theta = 5$, $\mu = 0$, $\sigma = 3$



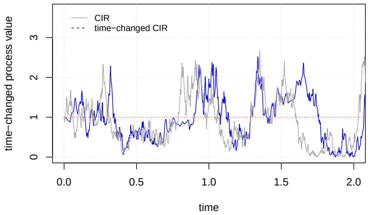
Contact rate - CIR jump diffusion time-changed by integrated OU process

- $\lambda_t^B = \lambda_t^H$
- $d\lambda_t^H = -\theta(\lambda_t^H - \mu) + \sigma dB_t$ **Ornstein-Uhlenbeck process**, $\theta = 5$, $\mu = 0$, $\sigma = 3$

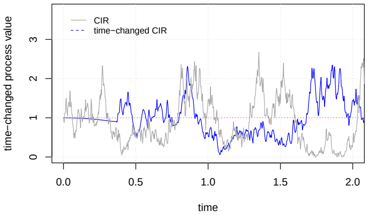


Contact rate
time-changed CIR diffusion without jumps

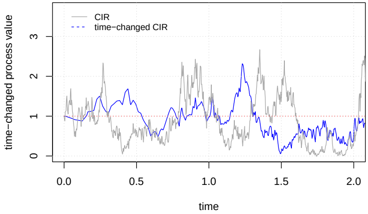
Time−changed CIR diffusion (sin)



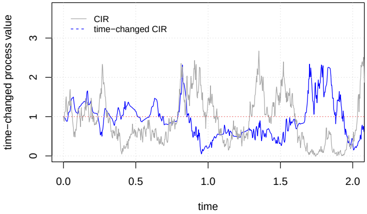
Time−changed CIR diffusion (CPP)



Time−changed CIR diffusion (IGsub)



Time−changed CIR diffusion (OU)



Contact rate - remarks and questions

- if 0 is the **absorbing barrier** of the process describing the dynamics of contact rate, the **extinction** appears after the first hitting time to 0
- if 0 is **reflecting barrier** and the process is **mean-reverting**, then the epidemic model is always in the **persistence regime**?
- what about **extinction**?
- **recovering**
    - contact rate process?
    - time-change process?

Recovering contact rate

- **contact rate is not directly observable**, it is "hidden" within the observable epidemiological data (number of susceptible, infected, vaccinated and recovered individuals)

- **model-based recovery** - depends on the model and its parameters (Mummert, 2012), (Pollicot et al., 2012)

- the simplest model for $\beta(t)$, according to (Pollicot et al., 2012) is

$$\beta(t) = \frac{I(t+1)}{I(t)S(t)}$$

- in (Pollicot et al., 2012) the recovery algorithm for $\beta(t)$ in SIR model with permanent immunity is based on the **inverse problem** for the SIR system

- for SIRV model with **non-permanent immunity** the inverse problem yields the **implicit result for** $\beta(t)$ - numerical procedures?

Recovering the time-change process

- if the day-by-day values of the contact rate are recoverd and the model without time-change is proposed, what **would be the right choice of the time-change processes**?

- (Winkel, 2001)
  for a given Lévy process $(Y(t),\, t \geq 0)$ and an independent time-change process $(\tau(t),\, t \geq 0)$, the case when both processes are **completely determined** by time-changed process $(X(\tau(t)),\, t \geq 0)$ are identified

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