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Detecting multifractal stochastic processes under the heavy tails effect

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Abstract
Multifractality of a time series can be analyzed using the partition function method based on empirical moments of the process. In this paper we analyze the method when the underlying process has heavy-tailed increments. A nonlinear estimated scaling function and non-trivial spectrum are usually considered as signs of a multifractal property in the data. We show that a large class of processes can produce these effects and that this behavior can be attributed to heavy tails of the process increments. Examples are provided indicating that multifractal features considered can be reproduced by simple heavy-tailed Lévy process.

Keywords: multifractal stochastic processes, partition function, scaling function, multifractal spectrum, heavy tails

MSC2000: 60G07; 60G51; 62M99;

1 Introduction

The importance of scaling relations in financial data was first stressed in the work of B.B. Mandelbrot. Early references are the seminal papers Mandelbrot (1963) and Mandelbrot (1967); see also Mandelbrot (1997). A first concept of the scaling relation was self-affinity (or self-similarity, see (Mandelbrot 1997, Chapter E6) for explanation of difference). Later the notion “monofractal" has also been used. As a generalization models allowing a richer form of scaling were introduced by Yaglom Yaglom (1966) and later called multifractal in the work of Frisch and Parisi Frisch & Parisi (1985). Multifractals have been introduced as measures to model turbulence. The concept can be easily generalized to stochastic processes, thus extending the notion of self-similar stochastic processes.

In a series of papers, Mandelbrot et al. (1997), Fisher et al. (1997), Calvet et al. (1997) and Calvet & Fisher (2002), the authors develop a theory of multifractal stochastic processes and a new model for financial time series, called the Multifractal model of asset returns (MMAR) (the name Brownian motion in multifractal time (BMMT) is used by Mandelbrot later). BMMT is constructed by compounding a standard (or fractional) Brownian motion with a random time process, which is specified to be multifractal. It incorporates most of the broadly accepted properties of financial data, such as long range dependence, volatility clustering and heavy tails. The multifractal property in this model is built using the notion of multiplicative cascade. Later, many models have been built

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possessing the multiscaling property, see e.g. Bacry et al. (2001a), Bacry & Muzy (2003a), Anh et al. (2008), Anh et al. (2009), Leonenko et al. (2013).

Although multifractal models are very appealing, there is certain controversy over its use. Many authors have reported to find no evidence of multifractal scaling in different data sets and report spurious multiscaling for different model types (see Schertzer & Lovejoy (1983), Chechkin & Gonchar (2000), Lux (2004), Jiang & Zhou (2008b), Sly (2005), Heyde (2009), Heyde & Sly (2008), Zhou (2012)). On the other hand, there is a range of papers confirming the multifractal behavior in various contexts by different methods: Feldmann et al. (1998), Schmitt et al. (1999), Xu & Gençay (2003), Wei & Huang (2005), Jiang & Zhou (2008a), Zunino et al. (2009) and, of course, Fisher et al. (1997) and Calvet & Fisher (2002).

In order to detect the multifractal property of a certain data set, one needs statistical methods. For multifractal stochastic process the multiscaling property is usually defined in terms of the moment scaling. This gives a simple detection method based on estimating the scaling function using a partition function. While for self-similar processes the scaling function is linear, for multifractal it should be nonlinear but always concave. Thus by estimating the scaling function, it is possible to distinguish the scaling nature of the process. Another detection method is based on the estimation of the multifractal spectrum. The spectrum can be obtained as a Legendre transform of the estimated scaling function provided so-called multifractal formalism holds (see Riedi (1999)). However if the scaling function is unreliable, then the same is true for the spectrum.

In this paper we want to stress out that concavity of the estimated scaling function can be attributed to the presence of heavy tails in the data rather than multifractality. We derive an asymptotic form of the estimated scaling function for a large class of processes with stationary, heavy-tailed and weakly dependent increments. Estimation will yield a bilinear scaling function when the tail index is less than 2. This result is known for α-stable Lévy processes (see Schmitt et al. (1999) and Chechkin & Gonchar (2000)). Processes we consider, unlike stable Lévy motion, are not assumed to be self-similar or to satisfy the moment scaling relation (see Equation (3) below). Our results also show that this class of processes will behave as if they obey the moment scaling relation. When the tail index is larger than 2, scaling function will have a shape that is hard to recognize as bilinear. We illustrate through examples that this shape can be mistakenly regarded as evidence of multifractality. Therefore estimated scaling functions can be misleading, especially for financial data which is widely believed to be heavy-tailed (see, e.g., Heyde & Liu (2001)).

Some authors define multifractality in terms of wavelets. This is usually done by basing the definition of the partition function on wavelet decomposition of the process (see e.g. Riedi (1999), Audit et al. (2002)). This leads to different methods for multifractal analysis based on wavelets. However, this type of definition is also sensitive to diverging moments. This has been noted in Gonçalves & Riedi (2005), where a wavelet based estimator of the tail index is proposed.

In the next section we recall some facts related to multifractal processes and statistical methods for analyzing multifractality using the partition function. In Section 3 we establish the asymptotic behavior of the partition function for a certain class of stochastic processes, in particular for processes with stationary independent heavy-tailed increments.

Using this we derive an asymptotic form of the scaling function and multifractal spectrum for these processes. Results show that nonlinearity of the scaling function and a non-trivial spectrum can be caused by the presence of heavy tails. In Section 4, we present examples that indicate that empirical facts considered typical for multifractals can be reproduced by a simple heavy-tailed Lévy process.

2 Multifractal stochastic processes

The best known scaling relation is self-similarity. A stochastic process $\{X(t), t \geq 0\}$ is said to be self-similar if for some $H \geq 0$ and for any $c > 0$

$$\{X(ct)\} \overset{d}{=} \{c^H X(t)\},$$

where equality is in finite dimensional distributions. The exponent H is usually called the Hurst parameter or index and we say $\{X(t)\}$ is H-ss. Brownian motion is known to be self-similar with exponent $H = 1/2$ and an α-stable Lévy process is $1/\alpha$-s.s. Both of these have stationary and independent increments. On the other hand, fractional Brownian motion (FBM) and fractional stable motion can be constructed with arbitrary $0 < H < 1$. They have stationary but dependent increments, exhibiting the long range dependence property. For details see Embrechts & Maejima (2002) and Samorodnitsky & Taqqu (1994).

The definition of a multifractal is motivated by generalizing the scaling rule of self-similar processes in the following manner:

$$\{X(ct)\} \overset{d}{=} \{M(c)X(t)\}, \tag{1}$$

where for every $c > 0$, $M(c)$ is a random variable independent of $\{X(t)\}$, whose distribution does not depend on t. When $M(c)$ is non-random and $M(c) = c^H$ then the definition reduces to H-self-similarity. The scaling factor $M(c)$ should satisfy the following property:

$$M(ab) \overset{d}{=} M_1(a)M_2(b), \tag{2}$$

for every choice of a and b, whereby M_1 and M_2 are independent copies of M. A motivation for this property can be found in Mandelbrot et al. (1997).

However instead of using definition (1), multifractality is usually defined by specifying scaling properties in terms of moments. In Mandelbrot et al. (1997) it is claimed that this approach leads to a more elegant theory and provides direct graphical and testable implications. We will however show that specifying the scaling rule in terms of moments can be misleading and hard to examine in practical situations.

A stochastic process $\{X(t)\}$ is said to be multifractal if it has stationary increments and there exist functions $c(q)$ and $\tau(q)$ such that

$$E|X(t)|^q = c(q)t^{\tau(q)}, \quad \text{for all } t \in [0, T], q \in [q_-, q_+], \tag{3}$$

for some $T > 0$ and $q_-, q_+ \in \mathbb{R}$. This definition is slightly different from the one in Mandelbrot et al. (1997), which specifies the exponent to be of the from $\tau(q) + 1$. Many processes studied as multifractals only obey the definition for t in some small range or for asymptotically small t. The condition of stationary increments can also be relaxed. For example multiplicative cascades obey the definition only for a discrete grid of time points thus having a discrete form of scaling invariance. A class of processes having continuous scale invariance, stationary increments and even satisfying Equation (1) was given in Bacry & Muzy (2003*b*), Muzy & Bacry (2002). These include the so-called Multifractal Random Walk developed earlier in Bacry et al. (2001*b*).

The function $\tau(q)$ is called the scaling function, it is easy to show that when $\{X(t)\}$ is H-s.s., then $\tau(q) = Hq$, also $\tau(q)$ is always concave (Mandelbrot et al. (1997)). So multifractal processes can be roughly characterized as those having a nonlinear scaling function. One can also show that a process can be multifractal only over a bounded time horizon. However one immediately sees the drawback of involving moments in the defining property as moments could be infinite. This can be hard to examine in practical situations. As we show later, infinite moments can affect estimation of the scaling function and produce concavity. One has to assume finiteness of the moments involved in order for statements like (3) to have sense. This is a serious drawback, especially for financial data which is widely believed to have only a range of moments finite.

Closely related to scaling function is the multifractal spectrum given by the Legendre transform:

$$d(h) = \inf_q \left(hq - \tau(q) + 1\right), \tag{4}$$

when it is defined. For a H-s.s. process we have $d(h) \neq -\infty$ only for $h = H$, thus the term monofractal is also used for self-similar processes. If the so-called multifractal formalism holds, then $d(h)$ is a Hausdorff dimension of a set of time points having pointwise Hölder exponent equal to h (see e.g. Calvet et al. (1997), Jaffard (1999), Jaffard (2000)). The validity of the multifractal formalism is known to be narrow when the scaling function is specified with a moment scaling relation (Muzy et al. (1993)). Scaling based on wavelet coefficients is also unable to yield a full spectrum of singularities. In Jaffard (2004) formalism based on wavelet leaders has been proposed. In our analysis we will consider only moments of positive order, this can yield at best only an increasing part of the spectrum. The spectrum of singularities can be evaluated for a full range of moments (including negative order moments), see for example Jaffard et al. (2007) and references therein. In this case it is necessary to prove for the model used the multifractal property based on the Hölder exponents, in order to use the powerful results of Jaffard Jaffard (2004). Note that generally this is not an easy task, and in fact for the models based on the products of stationary processes developed in Anh et al. (2008) or Anh et al. (2009) there is no theoretical bases to use such results. However the multifractal scaling of the moments is available for these models, see again Anh et al. (2008) and Anh et al. (2009).

2.1 Statistical methods for detecting multifractal behavior

The main method for detecting multifractal behavior of the data is based on exploiting the fact that the scaling function is linear for self-similar processes. Every departure from linearity can therefore be accredited to multiscaling. So, the main problem is to estimate the scaling function from the data and inspect its shape. We now present the methodology provided in Fisher et al. (1997) but in the next subsection we show that it leads to false conclusions in the presence of heavy-tails.

Consider a process $X(t)$ defined for $t \in [0,T]$ and suppose $X(0) = 0$. Denote by $X(t,s) = X(t+s) - X(t)$ the increment of the process over the interval $[t, t+s]$. Divide the interval $[0,T]$ into N blocks of length Δt and define the partition function (sometimes called structure function):

$$S_q(T, \Delta t) = \frac{1}{N} \sum_{i=1}^{N} \left| X\left((i-1)\Delta t, \Delta t\right)\right|^q . \tag{5}$$

If $\{X(t)\}$ is multifractal, then it has stationary increments and so $ES_q(T,\Delta t) = E|X(0,\Delta t)|^q = c(q)\Delta t^{\tau(q)}$. So,

$$\ln ES_q(T,\Delta t) = \tau(q) \ln \Delta t + \ln c(q). \tag{6}$$

There are other ways to interpret the definition of the partition function. One can see $S_q(T,\Delta t)$ as the empirical counterpart of the left hand side of (3). If we denote $Y_i, i = 1,\dots,T$ to be one step increments $Y_i = X(i) - X(i-1) = X(i-1,1)$, then $S_q(T,\Delta t)$ is the same in distribution as

$$\frac{1}{N} \sum_{i=1}^{N} \left| \sum_{j=1}^{\Delta t} X((i-1)\Delta t + j, 1) \right|^q = \frac{1}{N} \sum_{i=1}^{N} \left| \sum_{j=1}^{\Delta t} Y_{(i-1)\Delta t + j} \right|^q . \tag{7}$$

since $X\left((i-1)\Delta t, \Delta t\right) = \sum_{j=1}^{\Delta t} X((i-1)\Delta t + j, 1)$. Therefore the relation (6) holds for the quantity above also. In what follows, we will not make distinction between the two alternative forms of the partition function.

Multiscaling behavior is inspected through the use of Equation (6). Based on the data sample, $X_i, i = 1,\dots T$, the following methodology (further called FCM) is developed in Fisher et al. (1997):

1. For fixed value of q, one computes the logarithm of the partition function for a range of values Δt and plots it against $\ln \Delta t$. If the scaling exists, the plot should be approximately a linear line.

2. Following Equation (6) the slope of the line can be estimated by linear regression of $\ln ES_q(T,\Delta t)$ on $\ln \Delta t$. The value obtained provides an estimate for the scaling function $\tau(q)$ at point q.

3. Repeating this for a range of q values, one is able to plot the empirical scaling function. If the plot is nonlinear then one can suspect the existence of multifractal scaling.

4. After an estimate $\hat{\tau}$ of the scaling function is obtained, it is possible to calculate the spectrum using Equation (4) with τ replaced by $\hat{\tau}$.

In Fisher et al. (1997) the method is applied to the DEM/USD exchange rate as well as to other financial data. The examples suggest a linear relation of the form (6) holds and the scaling function exhibits nonlinear behavior. We next explain that these effects can be contributed to the presence of heavy tails.

3 Asymptotic behavior of the estimated scaling function

In this section we analyze the behavior of the estimated scaling function when the underlying process has heavy-tailed increments. A part of this analysis under different assumptions has also been made in Heyde (2009).

3.1 Assumptions

We define a class of stochastic processes for which scaling functions will be considered. To do this we first recall some standard terms of probability theory.

The distribution of a random variable Z is said to be heavy-tailed with index $\alpha > 0$ if it has a regularly varying tail with index $-\alpha$. This implies that

$$P(|Z| > x) = \frac{L(x)}{x^\alpha},$$

where $L(t), t > 0$ is a slowly varying function, that is $L(tx)/L(x) \to 1$ as $|x| \to \infty$, for every $t > 0$. In particular, this implies that $E|Z|^q < \infty$ for $q < \alpha$ and $E|Z|^q = \infty$ for $q > \alpha$, which is sometimes also used to define heavy tails. Heavy-tailed distributions have been known to model well the logarithm of asset returns (see Hurst & Platen (1997) and Heyde & Liu (2001)).

The process considered is allowed to have weakly dependent increments. More precisely, for two sub-σ-algebras, $\mathcal{A} \subset \mathcal{F}$ and $\mathcal{B} \subset \mathcal{F}$ on the same complete probability space $(\Omega, \mathcal{F}, P)$ define

$$a(\mathcal{A}, \mathcal{B}) = \sup_{A \in \mathcal{A}, B \in \mathcal{B}} |P(A \cap B) - P(A)P(B)|.$$

Now for a process, Y_t, $t \geq 0$, consider $\mathcal{F}_t = \sigma\{Y_s, s \leq t\}$, $\mathcal{F}^{t+\tau} = \sigma\{Y_s, s \geq t + \tau\}$. We say that $\{Y_t\}$ has a strong mixing property if $a(\tau) = \sup_{t \geq 0} a(\mathcal{F}_t, \mathcal{F}^{t+\tau}) \to 0$ as $\tau \to \infty$. Strong mixing is sometimes also called α-mixing (see Doukhan (1994*a*) for more details). If $a(\tau) = O(e^{-b\tau})$ for some $b > 0$ we say that the strong mixing property has an exponentially decaying rate. We note that results following could probably be proven under some other type of weak dependence.

We will call processes considered to be of type $\mathfrak{L}$.

Definition 1. A stochastic process $\{X(t), t \geq 0\}$ is said to be of type $\mathfrak{L}$, if $Y_t = X(t) - X(t-1), t \in \mathbb{N}$ is a strictly stationary sequence having heavy-tailed marginal distribution with index α, satisfying the strong mixing property with an exponentially decaying rate and such that $EY_t = 0$ when $\alpha > 1$.

This class includes many examples like all Lévy processes with $X(1)$ heavy-tailed, this includes for example α-stable Lévy processes with $0 < \alpha < 2$. A richer modeling ability is provided by a Student Lévy process, which allows for arbitrary tail index parameter. Student's t-distribution $T(\nu, \delta, \mu)$ is given by the probability density function

$$\text{student}[\nu, \delta, \mu](x) = \frac{\Gamma(\frac{\nu+1}{2})}{\delta\sqrt{\pi}\Gamma(\frac{\nu}{2})} \left(1 + \left(\frac{x-\mu}{\delta}\right)^2\right)^{-\frac{\nu+1}{2}}, \qquad x \in \mathbb{R}, \tag{8}$$

(the so-called symmetric scaled Student's t-distribution), where $\delta > 0$ is the scaling parameter, ν the tail parameter (usually called degrees of freedom) and $\mu \in \mathbb{R}$ is the location parameter. This distribution is heavy-tailed with tail index ν. Since the t-distribution is infinitely divisible, a Lévy process such that $X(1) \stackrel{d}{=} T(\nu, \delta, \mu)$ surely exists, see Heyde & Leonenko (2005) for details.

The class $\mathfrak{L}$ also includes cumulative sums of stationary processes like Ornstein-Uhlenbeck (OU) type processes or diffusions with heavy-tailed marginal distributions. Recall that a stochastic process $\{Y(t), t \geq 0\}$ is said to be of OU-type if it satisfies a stochastic differential equation (SDE) of the form

$$\mathrm{d}Y(t) = -\lambda Y(t)\mathrm{d}t + \mathrm{d}L(\lambda t), \quad t \geq 0, \tag{9}$$

where $\{L(t), t \geq 0\}$ is the background driving Lévy process (BDLP) and $\lambda > 0$. We consider strictly stationary solutions of SDE (9). For every self-decomposable distribution $\mathfrak{D}$ there exists a strictly stationary stochastic process $\{Y(t), t \geq 0\}$, which has a marginal distribution $\mathfrak{D}$ and is referred to as OU-type process. OU-type processes can be shown to posses the strong mixing property with an exponentially decaying rate (see Masuda (2004)). Since the Student t-distribution is self-decomposable there exists Student OU-type process with heavy-tailed marginal distributions (see Heyde & Leonenko (2005) and references therein).

Different diffusion-type models can be defined as solutions of particular SDEs. Under weak regularity conditions, a diffusion process with a given marginal distribution can be described by a suitably chosen SDE (see Bibby et al. (2005)). For example a Student diffusion is defined using the SDE:

$$\mathrm{d}Y(t) = -\theta\left(Y(t) - \mu\right)\mathrm{d}t + \sqrt{\frac{2\theta\delta^2}{\nu - 1}\left(1 + \left(\frac{Y(t) - \mu}{\delta}\right)^2\right)}\mathrm{d}B(t), \quad t \geq 0, \tag{10}$$

where $\nu > 1,\ \delta > 0, \mu \in \mathbb{R},\ \theta > 0$, and $\{B(t), t \geq 0\}$ is a standard Brownian motion. The SDE (10) admits a unique ergodic Markovian weak solution $\{Y(t), t \geq 0\}$ which is a diffusion process with the invariant symmetric scaled Student's t-distribution with probability density function (8). The diffusion process which solves the SDE (10) is called the Student diffusion. If $Y(0) \stackrel{d}{=} T(\nu, \delta, \mu)$, the Student diffusion is strictly stationary. According to Leonenko & Šuvak (2010) the Student diffusion is a strong mixing process with an exponentially decaying rate. For more examples of heavy-tailed diffusions see Avram et al. (2013).

If $\{Y(t), t \geq 0\}$ is a strictly stationary OU-type process or diffusion with heavy-tailed marginals having the strong mixing property with an exponentially decaying rate, then the process

$$X(t) = \sum_{i=0}^{\lfloor t \rfloor} Y(i), \quad t \geq 0$$

will be of type $\mathfrak{L}$. This provides a variety of examples with dependent increments.

Remark 1. For what follows, we will assume that $X_1, \dots, X_T$ is a sample observed at discrete equally spaced time instants from a stochastic process $\{X(t), t \geq 0\}$. Notice that it is not a restriction to impose conditions on one step increments. Suppose that the process $\{X_t\}$ is sampled at regularly spaced time instants $\delta, 2\delta, \dots, n\delta$ and $\delta \neq 1$, then we know the values of the process $\{\tilde{X}_t\} = \{X_{\delta t}\}$ at times $1, 2, \dots n$. But then $\tau_{\tilde{X}}(q) = \tau_X(q)$, since

$$E|\tilde{X}(t)|^q = E|X(\delta t)|^q = \left(c(q)\delta^{\tau(q)}\right) t^{\tau(q)}.$$

We can therefore assume that the process is sampled at time instants $1, 2, \dots n$.

3.2 Asymptotic behavior of the estimated scaling function

Define $Y_t = X(t) - X(t-1) = X(t-1, 1), t \in \mathbb{N}$ to be one-step increments of a process $\{X(t)\}$. In order to establish asymptotic properties of the estimated scaling function, we analyze a special type of limiting behavior of the partition function. Using representation (7) and dividing $0, 1 \dots, T$ into blocks of size Δt, the partition function calculated from the sample will take the form

$$S_q(T, \Delta t) = \frac{1}{\lfloor T/\Delta t \rfloor} \sum_{i=1}^{\lfloor T/\Delta t \rfloor} \left| \sum_{j=1}^{\lfloor \Delta t \rfloor} Y_{(i-1)\lfloor \Delta t \rfloor + j} \right|^q . \tag{11}$$

Instead of keeping Δt fixed we take it to be of the form $\Delta t = T^s$ for some $s \in (0, 1)$, which allows the blocks to grow as the sample size increases. It is clear that $S_q(T, T^s)$ will diverge since $s > 0$. As part of the analysis, we will be interested in the rate of divergence of this statistic, i.e., we consider the limiting behavior of $\ln S_q(T, T^s) / \ln T$. The proof of the following theorem is given in Section 6.

Theorem 1. *If $\{X(t)\}$ is of type $\mathfrak{L}$, then for $q > 0$ and every $s \in (0, 1)$*

$$\frac{\ln S_q(T, T^s)}{\ln T} \xrightarrow{P} R_\alpha(q, s) := \begin{cases} \frac{sq}{\alpha}, & \text{if } q \leq \alpha \text{ and } \alpha \leq 2, \\ s + \frac{q}{\alpha} - 1, & \text{if } q > \alpha \text{ and } \alpha \leq 2, \\ \frac{sq}{2}, & \text{if } q \leq \alpha \text{ and } \alpha > 2, \\ \max\left\{s + \frac{q}{\alpha} - 1, \frac{sq}{2}\right\}, & \text{if } q > \alpha \text{ and } \alpha > 2, \end{cases} \tag{12}$$

as $T \to \infty$, where $\xrightarrow{P}$ stands for convergence in probability.

The theorem establishes the rate of growth for the partition function in the context considered. To understand the implications of the theorem denote

$$\varepsilon_T = \frac{S_q(T, \log_T \Delta t)}{n^{R_\alpha(q, \log_T \Delta t)}},$$

taking the logarithm and rewriting yields

$$\ln S_q(T, \log_T \Delta t) = R_\alpha(q, \log_T \Delta t) \ln T + \ln \varepsilon_T. \tag{13}$$

Notice that, when $\alpha \leq 2$, $R_\alpha(q,s)$ is linear in s, i.e. it can be written in form $R_\alpha(q,s) = a(q)s + b(q)$ for some functions $a(q)$ and $b(q)$. This also holds if $\alpha > 2$ and $q \leq \alpha$. This means we can rewrite (13) as

$$\ln S_q(T, \log_T \Delta t) = a(q) \ln \Delta t + b(q) \ln T + \ln \varepsilon_T. \tag{14}$$

It follows that the partition function is approximately linear in $\ln \Delta t$, i.e. that the relation of type (6) holds up to some random variable. So for processes of type $\mathfrak{L}$ step (1) of FCM methodology will always be satisfied. If $\alpha > 2$ and $q > \alpha$, $R_\alpha(q,s)$ is not linear in s due to the maximum term in (12). It is actually bilinear with the breakpoint depending on the values of q and α. However if q is not much greater than α, $s \mapsto R_\alpha(q,s)$ is very close to the linear function. We can therefore write $R_\alpha(q,s) \approx a(q)s + b(q)$ and the relation of type (14) would again hold approximately.

As follows from the preceding discussion it makes sense to consider the slope of the linear regression of $\ln S_q(T, \log_T \Delta t)$ on $\ln \Delta t$. This is step (2) of the FCM methodology that leads to the empirical scaling function. Using the well known formula for the slope of the linear regression line, we can define the empirical scaling function for $q > 0$

$$\hat{\tau}_{N,T}(q) = \frac{\sum_{i=1}^N \ln \Delta t_i \ln S_q(n, \Delta t_i) - \frac{1}{N} \sum_{i=1}^N \ln \Delta t_i \sum_{j=1}^N \ln S_q(n, \Delta t_i)}{\sum_{i=1}^N (\ln \Delta t_i)^2 - \frac{1}{N} \left(\sum_{i=1}^N \ln \Delta t_i \right)^2}, \tag{15}$$

where $1 \leq \Delta t_i \leq T$ for $i = 1, \dots, N$. The next theorem derives the asymptotic form of the scaling function obtained by estimation from Equation (15).

Theorem 2. *Suppose $\{X(t)\}$ is of type $\mathfrak{L}$ and suppose Δt_i is of the form $T^{\frac{i}{N}}$ for $i = 1, \dots, N$. Then, for every $q > 0$,*

$$\lim_{N \to \infty} \operatorname*{plim}_{T \to \infty} \hat{\tau}_{N,T}(q) = \tau_\infty(q),$$

where plim *stands for limit in probability and*

$$\tau_\infty(q) = \begin{cases} \frac{q}{\alpha}, & \text{if } 0 < q \leq \alpha \ \& \ \alpha \leq 2, \\ 1, & \text{if } q > \alpha \ \& \ \alpha \leq 2, \\ \frac{q}{2}, & \text{if } 0 < q \leq \alpha \ \& \ \alpha > 2, \\ \frac{q}{2} + \frac{2(\alpha - q)^2 (2\alpha + 4q - 3\alpha q)}{\alpha^3 (2-q)^2}, & \text{if } q > \alpha \ \& \ \alpha > 2. \end{cases} \tag{16}$$

Proof. This theorem is similar in spirit to Theorem 2 in Grahovac et al. (2013) but we repeat it here for completeness. Fix a $q > 0$ and denote $y_T(s) = \ln S_q(T, T^s)/\ln T$. By dividing denominator and numerator of the right hand side of (15) by $(\ln T)^2$ we get

$$\hat{\tau}_{N,T}(q) = \frac{\sum_{i=1}^N \frac{i}{N} \frac{\ln S_q(n, \Delta t_i)}{\ln T} - \frac{1}{N} \sum_{i=1}^N \frac{i}{N} \sum_{j=1}^N \frac{\ln S_q(n, \Delta t_i)}{\ln T}}{\sum_{i=1}^N \left(\frac{i}{N} \right)^2 - \frac{1}{N} \left(\sum_{i=1}^N \frac{i}{N} \right)^2}. \tag{17}$$

We first show that

$$\operatorname*{plim}_{T \to \infty} \hat{\tau}_{N,T}(q) = \frac{\sum_{i=1}^N \frac{i}{N} R_\alpha(q, \frac{i}{N}) - \frac{1}{N} \sum_{i=1}^N \frac{i}{N} \sum_{j=1}^N R_\alpha(q, \frac{i}{N})}{\sum_{i=1}^N \left(\frac{i}{N} \right)^2 - \frac{1}{N} \left(\sum_{i=1}^N \frac{i}{N} \right)^2}. \tag{18}$$

Let $\varepsilon > 0$ and $\delta > 0$. By Theorem 1 for each $i = 1, \dots, N$ there exists a T_i such that

$$P\left(\left|y_T\left(\frac{i}{N}\right) - R_\alpha(q, \frac{i}{N})\right| > \frac{\varepsilon}{N}\right) < \frac{\delta}{N},$$

for $T \geq T_i$. Take $T_{max} = \max_{i=1,\dots,N} T_i$. Then for all $T \geq T_{max}$,

$$\begin{aligned} P\left(\sum_{i=1}^{N} \frac{i}{N}\left|y_T\left(\frac{i}{N}\right) - R_\alpha(q, \frac{i}{N})\right| > \varepsilon\right) &\leq P\left(\sum_{i=1}^{N}\left|y_T\left(\frac{i}{N}\right) - R_\alpha(q, \frac{i}{N})\right| > \varepsilon\right) \\ &\leq (N)P\left(\left|y_T\left(\frac{i}{N}\right) - R_\alpha(q, \frac{i}{N})\right| > \frac{\varepsilon}{N}\right) < \delta. \end{aligned}$$

This proves the convergence for two terms depending on T and the claim now follows by the continuous mapping theorem. By dividing the denominator and numerator of the fraction in the limit (18) by $1/N$, one can see all the sums involved as Riemann sums based on an equidistant partition. The functions $s \mapsto sR_\alpha(q,s)$, $s \mapsto R_\alpha(q,s)$, $s \mapsto s$ and $s \mapsto s^2$ are all bounded continuous on $[0,1]$, so all sums converge to integrals when the partition is refined, i.e. when $N \to \infty$. Thus

$$\begin{aligned} \lim_{N\to\infty} \operatorname*{plim}_{n\to\infty} \hat{\tau}_{N,n}(q) &= \frac{\int_0^1 sR_\alpha(q,s)ds - \int_0^1 sds \int_0^1 R_\alpha(q,s)ds}{\int_0^1 s^2 ds - \left(\int_0^1 sds\right)^2} \\ &= 12\int_0^1 sR_\alpha(q,s)ds - 6\int_0^1 R_\alpha(q,s)ds. \end{aligned}$$

Solving the integrals using expression for $R_\alpha(q,s)$, one gets $\tau_\infty(q)$ as in (16). $\square$

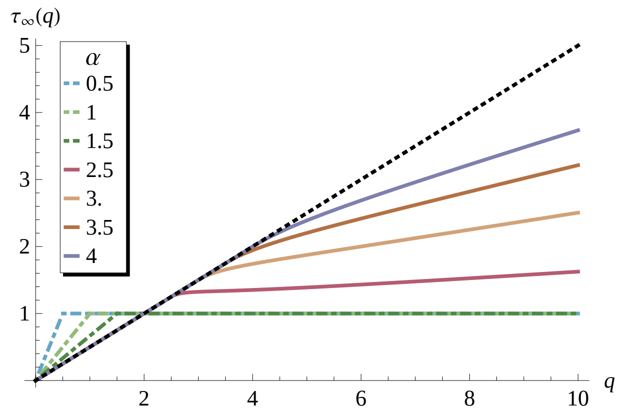
Theorem 2 establishes the behavior of the estimated scaling function in a special asymptotic regime. The assumptions impose natural conditions. Indeed taking Δt_i to be of the form $T^{\frac{i}{N}}$ ensures as the sample size grows ($T \to \infty$), the number of points included in the regression based on (6) grows. Letting $N \to \infty$ has the same effect ensuring that more and more points are used in regression.

To conclude if the underlying process has stationary, heavy-tailed, zero mean, weakly dependent increments then the scaling function estimated from the data will behave approximately as $\tau_\infty(q)$ in (16). When $\alpha \leq 2$, $\tau_\infty(q)$ is bilinear with slope $1/\alpha > 1/2$ for $q \leq \alpha$ and a horizontal line for $q > \alpha$. Therefore when $\alpha \leq 2$ there will be a sharp slope change around the point where q is equal to the tail index in the graph. When $\alpha > 2$ the plot of $\tau_\infty(q)$ is concave and appears approximately bilinear with slope $1/2$ for $q < \alpha$ and with a slowly decreasing slope for $q > \alpha$. When α is large, i.e., $\alpha \to \infty$, it follows from (16) that $\tau_\infty(q) = q/2$. This case corresponds to the data coming from a distribution with all moments finite, e.g. an independent normally distributed sample. This line will be referred to as the baseline.

The shape of the scaling function is determined by the value of the tail index α. Theoretical plots of scaling functions for a range of values of α are shown in Figure 1.

Thus under the assumptions considered, the difference between the linear and non-linear estimated scaling function can be described by the presence of heavy tails in the data. It is therefore extremely dangerous to conclude multifractality by examining the

Figure 1: Plots of scaling functions $\tau_\infty(q)$ for a range of α values



estimated scaling function under heavy tails. Moreover this effect is so regular that one can make inference about the unknown tail index of the underlying distribution by analyzing the scaling function behavior. This was done in Grahovac et al. (2013), giving a reasonably good estimator of the tail parameter.

3.3 Estimation of the spectrum

As mentioned before, using Equation (4) the spectrum can be estimated as the Legendre transform of the estimated scaling function, i.e.

$$\hat{d}_{N,T}(h) = \inf_q \left(hq - \hat{\tau}_{N,T}(q) + 1\right). \tag{19}$$

Adding 1 on the right hand side depends whether the exponent in (3) is defined as τ or $\tau - 1$. Since $\tau(q)$ is defined only for $q > 0$ it only makes sense to consider the infimum over positive q. This means considering only moments of positive orders and can yield only the left part of the spectrum. One way to assess the spectrum numerically is to interpolate $\hat{\tau}_{N,T}$ based on some estimated points and then proceed with numerical minimization. Since $\hat{\tau}_{N,T}(q)$ can be estimated at any point q, interpolation can be made arbitrary precise.

The Legendre transform has a following geometric meaning. Consider $d(h) = \inf_q (hq - \tau(q))$ and suppose τ is concave. Given q_0 we can find the tangent at q_0 on τ, call it $s(q) = aq + b$, such that $\tau(q) \leq s(q)$ with equality at q_0. If τ is differentiable this tangent will be unique. Then $aq - \tau(q) \geq aq - s(q) = -b$ with equality at q_0 and so

$$d(a) = aq_0 - \tau(q_0) = -b.$$

If we suppose τ is differentiable at q_0, then s is unique and

$$d(\tau'(q_0)) = q_0\tau'(q_0) - \tau(q_0).$$

One can show that d is concave (see e.g. (Riedi et al. 1999, Appendix A)). Thus the line $l_{q_0}(h) = q_0 h - \tau(q_0)$ is a tangent of d at point a. This gives an idea of how to estimate the spectrum graphically, as simply plotting l_{q_0} for a range of q_0 values will yield an envelope for d. We will use this method in examples in the next section.

Taking in mind the asymptotic behavior of the scaling function, we can expect the estimated spectrum for processes of type $\mathcal{L}$ to behave as

$$d_\infty(h) = \inf_{q>0} \left(hq - \tau_\infty(q) + 1 \right). \tag{20}$$

When $\alpha \leq 2$, we can explicitly calculate

$$\underline{d}_\infty(h) = \min\left\{ \inf_{0<q\leq\alpha} \left(hq - \frac{q}{\alpha} + 1 \right),\ \inf_{q\geq\alpha} (hq) \right\} \tag{21}$$

$$= \begin{cases} -\infty, & \text{if } h < 0, \\ \alpha h, & \text{if } 0 \leq h \leq \frac{1}{\alpha} \\ 1, & \text{if } h > \frac{1}{\alpha}. \end{cases} \tag{22}$$

If the infimum is taken over all q, the part of the spectrum for $h > \frac{1}{\alpha}$ would depend on $\tau_\infty(q)$ for negative q. This part refers to right (decreasing) part of the spectrum and requires negative order moments to be estimated, thus cannot be considered reliable.

When $\alpha > 2$ we have

$$\overline{d}_\infty(h) = \min\left\{ \inf_{0<q\leq\alpha} \left(hq - \frac{q}{2} + 1 \right),\ \inf_{q\geq\alpha} \left(hq - \frac{q}{2} - \frac{2(\alpha-q)^2(2\alpha+4q-3\alpha q)}{\alpha^3(2-q)^2} \right) \right\}.$$

Values $h > 1/2$ yield the right part of the spectrum and $\overline{d}_\infty(h) = 1$. On the other hand if

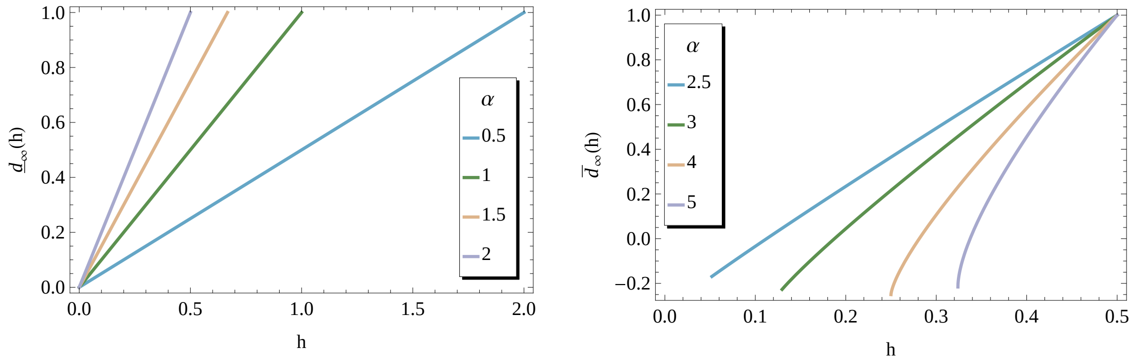
$$h < \lim_{q\to\infty} \frac{\tau_\infty(q)}{q} = \frac{(\alpha-2)^2(\alpha+4)}{2\alpha^3},$$

then $\overline{d}_\infty(h) = -\infty$ is attained when $q \to \infty$. Thus the left part of the spectrum is finite for

$$h \in \left[\frac{(\alpha-2)^2(\alpha+4)}{2\alpha^3}, \frac{1}{2} \right].$$

On this interval the spectrum is nonlinear and approximately parabolic but the explicit formula is complicated. Figure 2 shows the shape of the spectrum one would expect when estimation is done using scaling function. We conclude that even processes that posses no scaling property, in the presence of heavy-tails can yield a non-trivial estimated spectrum.

Figure 2: Spectrum estimated from τ_∞



(a) Case $\alpha \leq 2$

(b) Case $\alpha > 2$

4 Simulations and examples

In this section we provide examples showing that nonlinearity of the estimated scaling functions can be reconstructed just by using a process with heavy tailed increments. For this purpose we set $\{X(t)\}$ to be a Student Lévy process, i.e. a stochastic process with stationary independent increments such that $X(0) = 0$ and $X(1)$ has Student's t-distribution.

It is important to stress that we do not advocate using Student Lévy process as a model in any of the examples. Besides independence of increments is an unrealistic property for financial data. Our goal is simply to show that nonlinear scaling functions can be reproduced using heavy-tailed models. However in Heyde & Leonenko (2005) and Leonenko et al. (2011) the authors provide examples of Student processes with different dependence structures which could be more appropriate for financial data.

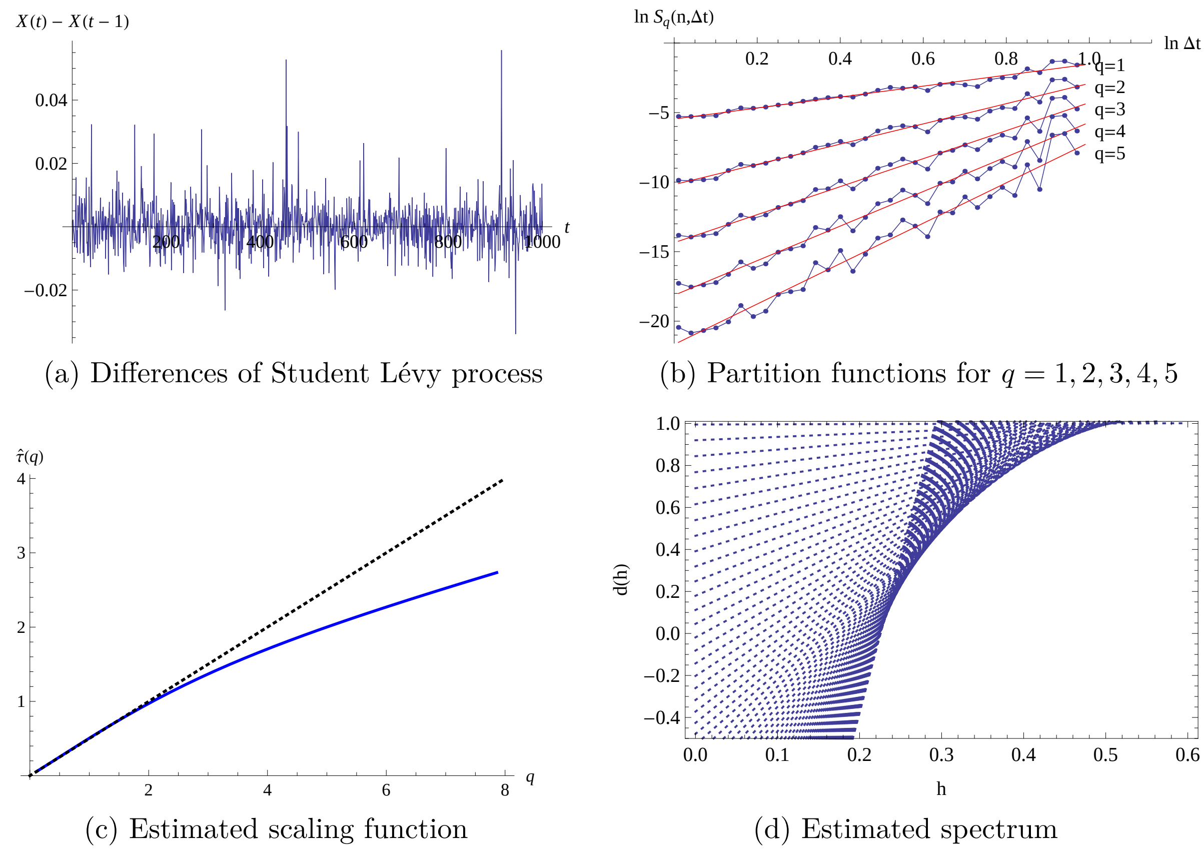
4.1 Mandelbrot, Fisher and Calvet example

In Calvet & Fisher (2002) (see also Fisher et al. (1997) and Calvet & Fisher (2008)), the authors provide an example with DM/USD exchange rate data with a plot of the estimated scaling function (see Figure 6. in Calvet & Fisher (2002)). Concavity is ascribed to multifractality. Considering Theorem 2 and comparing the plot with Figure 1 one can conclude that the data exhibits heavy-tail characteristics and a rough estimate of the tail index is around 4. This is consistent with other research suggesting risky asset returns are usually heavy-tailed with tail index between 3 and 5 (see Hurst & Platen (1997) and Heyde & Liu (2001)).

We try to reproduce the same figures as in Calvet & Fisher (2002) by simulating the data taking $\{X(t)\}$ to be Student Lévy process. Figure 3a shows the one step increments of a sample path of a Student Lévy process with $X(1) \stackrel{d}{=} T(4, 0.1, 0)$. The process was generated with 1000 observations. Linear behavior of the partition function in the sense of relation (6) is confirmed by Figure 3b. The plot was made in the log-log scale for a range of moments q. Adjusted R^2 values for the linear fit are approximately 0.97 for $q = 1, \dots, 5$, which confirms the linear relation. Similar analysis was done in Calvet & Fisher (2002), Figure 5, and we want to stress the similarity of two plots. Figure 3c shows the estimated scaling function together with the baseline. It is clear that concavity has nothing to do with scaling properties, one can notice the resemblance with Figure 6 in Calvet & Fisher (2002). Finally, we estimate the spectrum by plotting tangents forming an envelope of the spectrum. The shape of the spectrum is almost identical to the one presented in Figure 7 in Calvet & Fisher (2002).

It has been observed that in many data sets the partition function scales linearly in the sense of (6) but for small and large lags there is a breakdown of the linear relation. This has been studied especially for environmental and earth variables; see Neuman et al. (2013), Guadagnini et al. (2012) and the references therein. In Neuman et al. (2013) the authors provide a model based on subordinating a truncated fractional Gaussian random field which has the ability of reproducing this type of behavior. For Student Lévy process, one can see from Figure 3b a certain break in linearity for larger lags. However we find no evidence that the reason for this anything other than undersampling caused by a small number of blocks for larger lags. For larger q this can also be explained by Theorem 1, by noting that (14) holds only approximately when $q > \alpha$.

Figure 3: Student Lévy process



(a) Differences of Student Lévy process

(b) Partition functions for $q = 1, 2, 3, 4, 5$

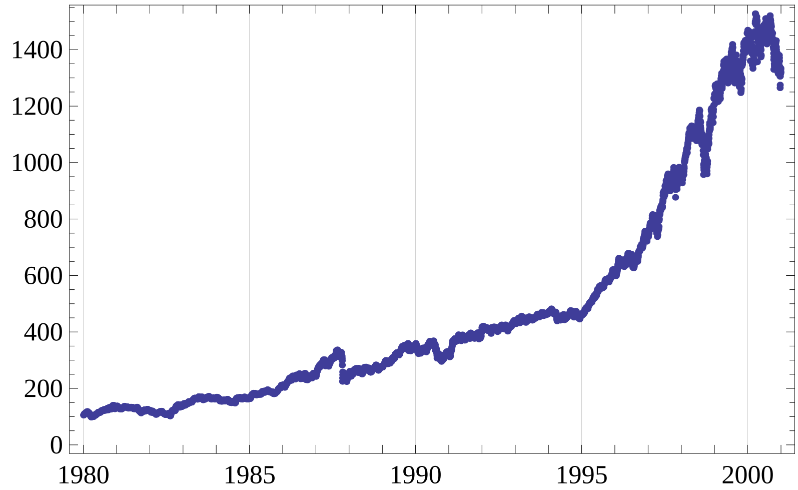
(c) Estimated scaling function

(d) Estimated spectrum

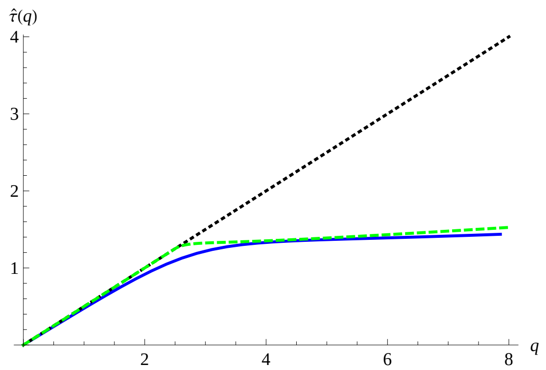
4.2 S&P 500 index

We provide another example to illustrate that quantities related to multifractality can be simply reproduced with the heavy-tailed Lévy process. The data consists of 5307 daily closing values of the S&P500 stock market index collected in the period from January 1, 1980 until December 31, 2000 and is shown on Figure 4a. For the analysis we consider log-differences of this series and subtract the mean. For estimating the scaling function at every point q the time points chosen are $1, 2, 3, 4, 5, 7, 15, 30, 60, 90, 180$. The scaling function estimated from the data is shown on Figure 4b together with the baseline and plot of τ_∞ for $\alpha = 2.5$ (dashed). One can see the resemblance which indicates the data may be heavy-tailed with the tail index around 2.5. We additionally generate a sample path of the same length for a Student Lévy process, with $X(1) \overset{d}{=} T(2.5, 0.0072, 0)$, where the second parameter is estimated from the data by the method of moments. Figure 4c shows the estimated scaling function for the generated process. The similarity of the two scaling functions is also naturally reflected in the estimated multifractal spectrum shown on Figures 4d and 4e.

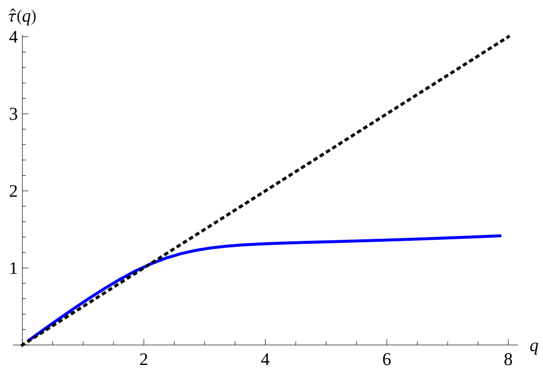
Figure 4: S&P 500 index



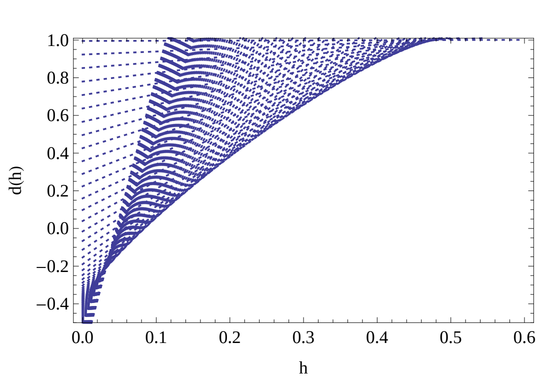
(a) S&P 500 index time series



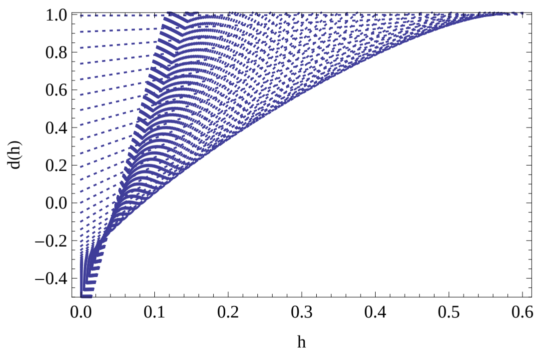
(b) Estimated scaling function of S&P 500 index with τ_∞ for $\alpha = 2.5$



(c) Estimated scaling function of generated Student Lévy process



(d) Estimated spectrum of S&P 500 index



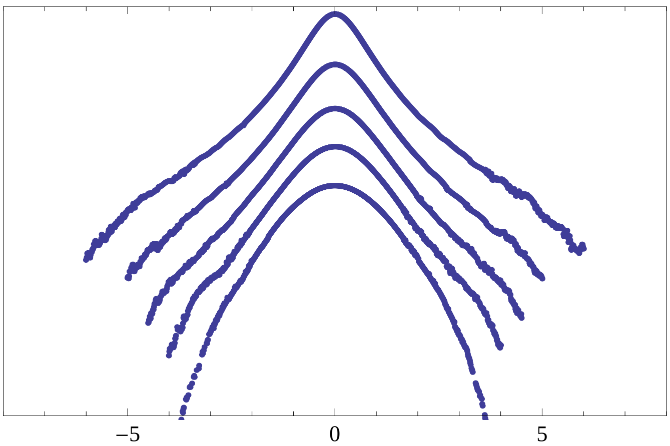
(e) Estimated spectrum of generated Student Lévy process

4.3 A further empirical feature ascribed to multifractality

The nonlinear scaling function estimated from the data is not the only empirical feature that is considered to be typical for multifractals. Here we briefly discuss one other such manifestation. In many applications it has been observed that increments at small lags of time series exhibit heavy tails, while for larger lags the distribution of increments tends to Gaussian (see, e.g., Bacry et al. (2008), Neuman et al. (2013)). Simulations show that multifractal models like the multifractal random walk (Bacry et al. (2001*b*), Muzy & Bacry (2002)) can produce such behavior (see also Kiyono et al. (2007)). This is explained as a consequence of stochastic self-similarity defined in Equation (1). We present a small simulation showing that this phenomenon can also be achieved with Student Lévy process.

We have generated 500 sample paths of length 10000 of a Student Lévy process with $X(1) \overset{d}{=} T(2.5, 1, 0)$. For each sample path increments at lags $1, 2, 4, 8, 30$ have been formed and the pdf has been estimated. All pdfs have been standardized to have unit variance and Figure 5 shows the mean estimates for different lags. The vertical position of each plot is irrelevant as it is arbitrarily chosen and we emphasize the shape of the plots indicating tail behavior. It is clear that tails become lighter as the lag increases. Figure 5 can be compared with a similar one for a multifractal random walk process, see Figure 4 in Bacry et al. (2008).

Figure 5: Standardized log-pdf estimates for increments of Student Lévy process at lags $1, 2, 4, 8, 30$



5 Summary and discussion

We provided a rigorous proof that estimating the scaling function using the partition function can lead to nonlinear estimates under the presence of heavy tails. These results shed a new light on many data sets that have been claimed to be multifractal by using the partition function method. This is particulary important for financial data which can produce nonlinear scaling functions due to its heavy-tailed properties. Scaling functions can be estimated correctly but only when the range of finite moments is known. This makes the multifractal definition based on the moment scaling problematic to use in many practical situations and calls for clarification of the interplay between multifractality and heavy-tailed properties.

Results proved in the paper are concerned with processes with short range dependence properties. However we expect that infinite moments produce similar behavior for the scaling function in the case of long range dependence, with the possible involvement of dependence parameter. We leave this analysis to future research which could apply to strongly correlated and heavy tailed data sets (see e.g. Safonov et al. (2010)).

6 Proof of Theorem 1

For the sake of completeness we provide a short version of the proof of Theorem 1. Full proof can be found in Grahovac et al. (2013). We split the proof into two parts depending on whether $q < \alpha$ or $q > \alpha$. The case $q = \alpha$ follows from this due to monotonicity of $S_q(T, T^s)$ in q. For simplicity we assume here that Y_i are symmetric around 0. Notation follows the one from Section 3.

(a) Suppose first that $q < \alpha$. For $\varepsilon > 0$, we first show the upper bound on the limit:

$$P\left(\frac{\ln S_q(T,T^s)}{\ln T} > \frac{sq}{\beta(\alpha)} + \varepsilon\right) = P\left(S_q(T,T^s) > T^{\frac{sq}{\beta(\alpha)}+\varepsilon}\right)$$
$$\leq P\left(\frac{1}{\lfloor T^{s-1}\rfloor}\sum_{i=1}^{\lfloor T^{1-s}\rfloor}\left|\sum_{j=1}^{\lfloor T^s\rfloor} Y_{T^s(i-1)+j}\right|^q > T^{\frac{sq}{\beta(\alpha)}+\varepsilon}\right) \leq \frac{E\left|\sum_{j=1}^{\lfloor T^s\rfloor} Y_j\right|^q}{T^{\frac{sq}{\beta(\alpha)}+\varepsilon}},$$

where we write $\beta(\alpha) = \alpha$ or 2 corresponding to $\alpha \leq 2$ or $\alpha > 2$. To show that this tends to zero, we first consider the case $\alpha > 2$. It follows from the Rosenthal's inequality for strong mixing sequences ((Doukhan 1994*b*, Section 1.4.1)) and Jensen's inequality that

$$E\left|\sum_{j=1}^{\lfloor T^s\rfloor} Y_j\right|^q \leq C_1 T^{\frac{sq}{2}}.$$

In case $\alpha \leq 2$ we choose γ small enough to make $q < \alpha - \gamma < \alpha$ and get

$$E\left|\sum_{j=1}^{\lfloor T^s\rfloor} Y_j\right|^q \leq \left(E\left|\sum_{j=1}^{\lfloor T^s\rfloor} Y_j\right|^{\alpha-\gamma}\right)^{\frac{q}{\alpha-\gamma}} \leq C_2 T^{\frac{sq}{\alpha-\gamma}}.$$

For the lower bound in the case $\alpha > 2$ denote

$$\sigma^2 = \lim_{T\to\infty}\frac{E\left(\sum_{j=1}^{T} Y_j\right)^2}{n}, \quad \rho_T = P\left(\left|\sum_{j=1}^{\lfloor T^s\rfloor} Y_{T^s(i-1)+j}\right| > T^{\frac{s}{2}}\sigma\right).$$

Since the sequence Y_j is stationary and strong mixing with an exponential decaying rate and since $E|Y_j|^{2+\zeta} < \infty$ for $\zeta > 0$ sufficiently small, the Central Limit Theorem holds (see (Hall & Heyde 1980, Corollary 5.1.)) and σ^2 exists. Since $P(|\mathcal{N}(0,1)|>1)>1/4$, it follows that for T large enough $\rho_T > 1/4$. Recall that if $\mathcal{MB}(T,p)$ is the sum of T stationary mixing indicator variables with expectation p then ergodic theorem implies $\mathcal{MB}(T,p)/T \to p$, a.s.

$$
\begin{aligned}
P\left(\frac{\ln S_q(T,T^s)}{\ln T} < \frac{sq}{2}-\varepsilon\right) &\leq P\left(\sum_{i=1}^{\lfloor T^{1-s}\rfloor}\left|\sum_{j=1}^{\lfloor T^s\rfloor} Y_{T^s(i-1)+j}\right|^q < T^{\frac{sq}{2}-\varepsilon+1-s}\right)\\
&\leq P\left(\sum_{i=1}^{\lfloor T^{1-s}\rfloor}\mathbf{1}\left(\left|\sum_{j=1}^{\lfloor T^s\rfloor} Y_{T^s(i-1)+j}\right| > T^{\frac{s}{2}}\sigma\right) < \frac{T^{1-s-\epsilon}}{\sigma^q}\right)\\
&\leq P\left(\mathcal{MB}(\lfloor T^{1-s}\rfloor, 1/4) < \frac{T^{1-s-\epsilon}}{\sigma^q}\right) \to 0.
\end{aligned}
$$

hence $\operatorname{plim}_{T\to\infty} \ln S_q(T,T^s)/\ln T \geq sq/2$. For the case $\alpha \leq 2$, we use the fact that $\max_{j=1,\ldots,\lfloor T^s\rfloor}|Y_j|/T^{s/\alpha}$ behaves as in the independent case and converges in distribution to some positive random variable (see e.g. Embrechts et al. (1997)). This means we can choose some constant $m>0$ such that for large enough T

$$
P\left(\frac{\max_{j=1,\ldots,\lfloor T^s\rfloor}|Y_j|}{T^{\frac{s}{\alpha}}} > 2m\right) > \frac{1}{4}.
$$

Denote by $|Y_l| = \max_{j=1,\ldots,\lfloor T^s\rfloor}|Y_j|$. Then it follows that

$$
P\left(\left|\sum_{j=1}^{\lfloor T^s\rfloor} Y_j\right| > mT^{\frac{s}{\alpha}}\right) \geq P\left(|Y_l| > 2mT^{\frac{s}{\alpha}}\right) + P\left(\left|\sum_{j=1, j\neq l}^{\lfloor T^s\rfloor} Y_j\right| < mT^{\frac{s}{\alpha}}\right) > \frac{1}{4}.
$$

The argument now follows as in the case $\alpha > 2$.

(b) Suppose $q>\alpha$. For the upper bound of the limit in probability let $\varepsilon>0$. Let $\delta>0$ and define

$$
Z_{j,T} = Y_j\mathbf{1}\left(|Y_j| \leq T^{\frac{1}{\alpha}+\delta}\right), \quad j=1,\ldots,T.
$$

For fixed T, $Z_{j,T}, j=1,\ldots,T$ is a stationary sequence with zero mean (due to symmetry) and finite moments of every order. Moreover the mixing properties of $Z_{j,T}$ are inherited from those of the sequence Y_j. By using Karamata's theorem (Embrechts et al. (1997)), for arbitrary $r>\alpha$ it follows

$$
\begin{aligned}
E|Z_{j,T}|^r &= \int_0^\infty P(|Z_{j,T}|^r > x)dx = \int_0^{T^{r(\frac{1}{\alpha}+\delta)}} P(|Y_j|^r > x)dx\\
&= \int_0^{T^{r(\frac{1}{\alpha}+\delta)}} L(x^{\frac{1}{r}})x^{-\frac{\alpha}{r}}dx \leq CT^{r(\frac{1}{\alpha}+\delta)(-\frac{\alpha}{r}+1)} = CT^{\frac{r}{\alpha}-1+\delta(r-\alpha)}
\end{aligned}
$$

Using this and Rosenthal's inequality for mixing sequences one can show the following bounds

$$E\left|\sum_{j=1}^{\lfloor T^s\rfloor} Z_{T^s(i-1)+j,T}\right|^q \leq C_1 T^{s+\frac{q}{\alpha}-1+\delta q}, \qquad \text{if } \alpha \leq 2,$$

$$E\left|\sum_{j=1}^{\lfloor T^s\rfloor} Z_{T^s(i-1)+j,T}\right|^q \leq C_2 T^{\max\left\{s+\frac{q}{\alpha}-1+\delta q, \frac{sq}{2}\right\}}, \qquad \text{if } \alpha > 2.$$

Next, notice that

$$P\left(\max_{i=1,\dots,T}|Y_i| > T^{\frac{1}{\alpha}+\delta}\right) \leq \sum_{i=1}^{T} P\left(|Y_i| > T^{\frac{1}{\alpha}+\delta}\right) \leq T\frac{L(T^{\frac{1}{\alpha}+\delta})}{(T^{\frac{1}{\alpha}+\delta})^{\alpha}} \leq C_3\frac{L(T^{\frac{1}{\alpha}+\delta})}{T^{\alpha\delta}}.$$

If $\alpha \leq 2$ it now follows from the Markov's inequality

$$\begin{aligned}
&P\left(\frac{\ln S_q(T,T^s)}{\ln T} > s+\frac{q}{\alpha}-1+\delta q+\varepsilon\right) \\
&\leq P\left(\frac{1}{\lfloor T^{1-s}\rfloor}\sum_{i=1}^{\lfloor T^{1-s}\rfloor}\left|\sum_{j=1}^{\lfloor T^s\rfloor} Z_{\lfloor T^s\rfloor(i-1)+j}\right|^q > T^{s+\frac{q}{\alpha}-1+\delta q+\varepsilon}\right) + P\left(\max_{i=1,\dots,T}|Y_i| > T^{\frac{1}{\alpha}+\delta}\right) \\
&\leq \frac{C_1 T^{s+\frac{q}{\alpha}-1+\delta q}}{T^{s+\frac{q}{\alpha}-1+\delta q+\varepsilon}} + C_3\frac{L(T^{\frac{1}{\alpha}+\delta})}{T^{\alpha\delta}} \to 0.
\end{aligned}$$

The case $\alpha > 2$ follows by the same arguments and this proves the upper bound on the probability limit.

The lower bound for the case $\alpha > 2$ follows exactly as in part (a) if $s+\frac{q}{\alpha}-1 \leq \frac{sq}{2}$. If $s+\frac{q}{\alpha}-1 > \frac{sq}{2}$ we can assume that $\varepsilon < \frac{1}{\alpha}-\frac{s}{2}$. Let $l \in \mathbb{N}$ be such that $|Y_l| = \max_{j=1,\dots,T}|Y_j|$. Then, for some $k \in \{1,2,\dots,\lfloor T^{1-s}\rfloor\}$ we have $l \in \mathcal{J} := \{\lfloor T^s\rfloor(k-1)+1,\dots,\lfloor T^s\rfloor k\}$. Assumption $\alpha > 2$ ensures that $E|Y_1|^{2+\zeta} < \infty$ for some $\zeta > 0$. Applying Markov's inequality and Rosenthal's inequality yields

$$\begin{aligned}
P\left(\left|\sum_{j\in\mathcal{J},j\neq l} Y_j\right| > T^{\frac{1}{\alpha}-\epsilon}\right) &\leq \frac{E\left(\sum_{j\in\mathcal{J},j\neq l} Y_j\right)^2}{T^{\frac{2}{\alpha}-2\varepsilon}} \\
&\leq \frac{K_1 T^s}{T^{\frac{2}{\alpha}-2\varepsilon}} = K_2 T^{s-\frac{2}{\alpha}+2\varepsilon} \to 0, \quad \text{as } T\to\infty.
\end{aligned}$$

Since $\max_{j=1,\dots,T}|Y_j|/T^{1/\alpha}$ converges in distribution to some positive random variable we have that

$$\begin{aligned}
&P\left(\frac{\ln S_q(T,T^s)}{\ln T} < s+\frac{q}{\alpha}-1-\varepsilon\right) \leq P\left(\sum_{i=1}^{\lfloor T^{1-s}\rfloor}\left|\sum_{j=1}^{\lfloor T^s\rfloor} Y_{T^s(i-1)+j}\right|^q < T^{\frac{q}{\alpha}-q\epsilon}\right) \\
&\leq P\left(\left|\sum_{j\in\mathcal{J}} Y_j\right|^q < T^{\frac{q}{\alpha}-q\varepsilon}\right) = P\left(\left|\sum_{j\in\mathcal{J}} Y_j\right| < T^{\frac{1}{\alpha}-\varepsilon}\right) \\
&\leq P\left(|Y_l| < 2T^{\frac{1}{\alpha}-\varepsilon}\right) + P\left(\left|\sum_{j\in\mathcal{J},j\neq l} Y_j\right| > T^{\frac{1}{\alpha}-\varepsilon}\right) \to 0,
\end{aligned}$$

as $T\to\infty$. Similar arguments apply in the case $\alpha \leq 2$ and this completes the proof.

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