ON EQUIVALENT DESCRIPTIONS OF BOUNDARY CONDITIONS FOR FRIEDRICHS SYSTEMS

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Abstract: Recently a new view on the theory of Friedrichs systems has been proposed, rewriting them in terms of Hilbert spaces, and a new way of representing the boundary conditions has been introduced. The admissible boundary conditions are characterised by two intrinsic geometric conditions in the graph space, thus avoiding the traces at the boundary. Another representation of boundary conditions via boundary operators has been discussed as well, which is equivalent to the intrinsic one (with boundary conditions enforced by two geometric requirements) if the sum of two specific subspaces V and $\tilde{V}$ of the graph space is closed. However, the validity of the last condition was left open.

We give a simple criterion (corresponding to the case of one space dimension) which ensures that $V + \tilde{V}$ is closed in the graph space. In the case of one equation in one space dimension we also give a complete classification of admissible boundary conditions.

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0. Introduction

Friedrichs systems form a class of boundary value problems which allow the study of a wide range of differential equations in a unified framework. They were introduced by Kurt Otto Friedrichs [6] in 1958 in an attempt to treat equations of mixed type (such as the Tricomi equation; for some specific applications see [1]). The Friedrichs system is a first order system of partial differential equations (of specific type) together with *admissible* boundary conditions enforced by matrix-valued boundary fields.

In [5] (see also [2,3]) a new view on the theory of Friedrichs systems has been proposed, rewriting them in terms of Hilbert spaces, and a new way of representing the boundary conditions has been introduced. Sufficient conditions for a specific operator to be bijective have been given, and the equivalence of different representations of the boundary condition has been discussed.

The paper is organised as follows. In the first part we restate the main results from [5], with emphasis on different representations of boundary conditions (Theorem 2). In the second section we describe a simple criterion ensuring that two different representations of boundary conditions are equivalent, and in the last section we show by examples that this criterion can be satisfied, at least in the case of one space dimension. There we also give a complete classification of admissible boundary conditions for one equation in one space dimension.

1. A new Hilbert space formalism

Let L be a real Hilbert space (identified to its dual L' by the Riesz representation theorem), and $\mathcal{D} \subseteq L$ a dense subspace. We assume that $T, \tilde{T} : \mathcal{D} \longrightarrow L$ are unbounded linear operators satisfying

$$\text{(T1)} \qquad (\forall\, \varphi, \psi \in \mathcal{D}) \quad \langle\, T\varphi \mid \psi \,\rangle_L = \langle\, \varphi \mid \tilde{T}\psi \,\rangle_L \,,$$

$$\text{(T2)} \qquad (\exists\, c > 0)(\forall\, \varphi \in \mathcal{D}) \quad \|(T + \tilde{T})\varphi\|_L \leq c\|\varphi\|_L \,,$$

$$\text{(T3)} \qquad (\exists\, \mu_0 > 0)(\forall\, \varphi \in \mathcal{D}) \quad \langle\, (T + \tilde{T})\varphi \mid \varphi \,\rangle_L \geq 2\mu_0\|\varphi\|_L^2 \,.$$

If we define $\langle\, \cdot \mid \cdot \,\rangle_T := \langle\, \cdot \mid \cdot \,\rangle_L + \langle\, T\cdot \mid T\cdot \,\rangle_L$, it can easily be seen that $(\mathcal{D}, \langle\, \cdot \mid \cdot \,\rangle_T)$ is an inner product space (the corresponding norm $\|\cdot\|_T$ usually being called *the graph norm*), with completion W_0. Similarly we can define $\langle\cdot\mid\cdot\rangle_{\tilde T}$, and by (T2) the norms $\|\cdot\|_T$ and $\|\cdot\|_{\tilde T}$ are equivalent on $\mathcal{D}$.

As the operators $T, \tilde T : \mathcal{D} \longrightarrow L$ are continuous in the pair $(\|\cdot\|_T, \|\cdot\|_L)$ of norms, they can be extended by density to operators from $\mathcal{L}(W_0; L)$ (the space of bounded linear operators from W_0 to L); note that we keep the same notation, thus $T, \tilde T \in \mathcal{L}(W_0; L)$, and the properties (T1)–(T3) are also valid for those extended operators (for $\varphi, \psi \in W_0$).

Let us denote by $\tilde T^* \in \mathcal{L}(L; W_0')$ the adjoint operator (in the sense of Banach spaces) of $\tilde T : W_0 \longrightarrow L$:

$$(\forall\, u \in L)(\forall\, v \in W_0) \quad {}_{W_0'}\langle\, \tilde T^* u, v\,\rangle_{W_0} = \langle\, u \mid \tilde T v\,\rangle_L \,.$$

Having in mind the Gelfand triple (W_0 is equipped with norm $\|\cdot\|_T$, and L with $\|\cdot\|_L$)

$$W_0 \hookrightarrow L \equiv L' \hookrightarrow W_0' \,,$$

it is immediate that $T = \tilde T^*_{|W_0}$. In a similar way one can show that $\tilde T = T^*_{|W_0}$, so we shall abuse the notation again and denote the operators $\tilde T^*, T^* \in \mathcal{L}(L; W_0')$ by T and $\tilde T$, respectively. Having this in mind, we have $T, \tilde T \in \mathcal{L}(L; W_0')$, and (T1) becomes

$$(\forall\, u \in L)(\forall\, \varphi \in W_0) \qquad \begin{array}{l} {}_{W_0'}\langle\, Tu, \varphi\,\rangle_{W_0} = \langle\, u \mid \tilde T \varphi\,\rangle_L \\ {}_{W_0'}\langle\, \tilde T u, \varphi\,\rangle_{W_0} = \langle\, u \mid T \varphi\,\rangle_L \end{array} ,$$

while (T2)–(T3) are valid for $\varphi \in L$ (see [3,5] for details).

Next we define *the graph space*

$$W := \{u \in L : Tu \in L\} = \{u \in L : \tilde T u \in L\}\,,$$

and one can easily prove that $(W, \langle\cdot\mid\cdot\rangle_T)$ is a Hilbert space.

In order to write down sufficient conditions on $V \leq W$ which ensure that the restriction $T_{|V} : V \longrightarrow L$ is an isomorphism, we define *a boundary operator* $D \in \mathcal{L}(W; W')$ by

$${}_{W'}\langle\, Du, v\,\rangle_W := \langle\, Tu \mid v\,\rangle_L - \langle\, u \mid \tilde T v\,\rangle_L \,, \qquad u, v \in W \,.$$

Let V and $\tilde V$ be subspaces of W satisfying

$$\text{(V1)} \qquad \begin{array}{ll} (\forall\, u \in V) & {}_{W'}\langle\, Du, u\,\rangle_W \geq 0\,, \\ (\forall\, v \in \tilde V) & {}_{W'}\langle\, Dv, v\,\rangle_W \leq 0\,, \end{array}$$

(V2) $$V = D(\tilde{V})^0\,, \qquad \tilde{V} = D(V)^0\,,$$

where ^{0} stands for the annihilator. One can easily check that V and $\tilde{V}$ are closed, and $\mathrm{Ker}\, D = W_0 \subseteq V \cap \tilde{V}$.

The following well–posedness result is proved in [5,4].

Theorem 1. *Under assumptions (T1)–(T3) and (V1)–(V2), the restrictions of operators* $T_{|_V} : V \longrightarrow L$ *and* $\tilde{T}_{|_{\tilde{V}}} : \tilde{V} \longrightarrow L$ *are isomorphisms.*

In the next theorem we describe an alternative way to enforce the boundary conditions, which is related to Friedrichs' original idea—*the admissible boundary conditions*, and relate it to (V1)–(V2).

Theorem 2. *If an operator* $M \in \mathcal{L}(W; W')$ *satisfies:*

(M1) $$(\forall\, u \in W) \quad {}_{W'}\langle\, Mu, u \,\rangle_W \geq 0\,,$$

(M2) $$W = \mathrm{Ker}\,(D - M) + \mathrm{Ker}\,(D + M)\,,$$

then the subspaces $V = \mathrm{Ker}\,(D - M)$ *and* $\tilde{V} = \mathrm{Ker}\,(D + M^*)$ *satisfy (V1)–(V2).*

Vice versa, if two subspaces V *and* $\tilde{V}$ *of* W *satisfy (V1)–(V2), and if additionally* $V + \tilde{V}$ *is closed, then there is an operator* $M \in \mathcal{L}(W; W')$ *satisfying (M1)–(M2), and* $V = \mathrm{Ker}\,(D - M)$.

Remark. The above two theorems imply that the conditions (M1)–(M2) are sufficient for operator $T_{|_{\mathrm{Ker}\,(D-M)}} : \mathrm{Ker}\,(D - M) \longrightarrow L$ to be an isomorphism.

In the classical Friedrichs theory the analogous conditions to (M1)–(M2) and (V1)–(V2) are equivalent. Theorem 2 is proved in [5] in an attempt to get a similar result in the abstract setting. However, the question whether (V1)–(V2) imply that $V + \tilde{V}$ is closed is left open. In the sequel we discuss some situations where $V + \tilde{V}$ is closed.

2. The case of finite codimension

The quotient space $\hat{W} := W/W_0$ is isomorphic to the orthogonal complement $W_0^\perp$ of W_0 in W, the isomorphism being given by $\hat{x} \mapsto Qx$, $x \in W$; where $Q : W \longrightarrow W_0^\perp$ is the orthogonal projector, and

$\hat{x} := x + W_0$. Therefore, $\hat{W}$ is a Hilbert space. As W_0 is closed in W, we have the following result (see [7, p. 140] or [8, p. 393]).

Lemma 1. *A subspace V of W, such that $W_0 \subseteq V$, is closed if and only if $\hat{V} := \{\hat{v} : v \in V\}$ is closed in $\hat{W}$.*

We can easily provide one simple instance where $V + \tilde{V}$ is closed.

Theorem 3. *If $\operatorname{codim} W_0 (= \dim W/W_0)$ is finite, then for every two subspaces V and $\tilde{V}$ of W such that $W_0 \subseteq V + \tilde{V}$, we have that $V + \tilde{V}$ is closed in W.*

Proof. As the quotient space $\hat{W}$ is of finite dimension, its subspaces are closed; in particular, $\widehat{(V + \tilde{V})}$ is closed in $\hat{W}$. Combined with $W_0 \subseteq V + \tilde{V}$, Lemma 1 implies that $V + \tilde{V}$ is closed in W.

Lemma 2. *If $I = \langle a, b\rangle \subseteq \mathbf{R}$ is a finite interval, then the codimension of the Sobolev space $\mathrm{H}^1_0(I)$ in $\mathrm{H}^1(I)$ is 2.*

Proof. Let $e, f \in \mathrm{H}^1(I)$ be such that $e(a) = 1$, $e(b) = 0$, $f(a) = 0$, and $f(b) = 1$. We shall show that $B := \{\hat{e}, \hat{f}\}$ is a basis for $\widehat{\mathrm{H}^1(I)} = \mathrm{H}^1(I)/\mathrm{H}^1_0(I)$. If B were linearly dependent, then from $\hat{f} - \alpha\hat{e} = \hat{0} = \mathrm{H}^1_0(I)$ it would follow $f - \alpha e \in \mathrm{H}^1_0(I)$, which is a contradiction with $(f - \alpha e)(b) = 1$. B spans $\widehat{\mathrm{H}^1(I)}$; as for given $g \in \mathrm{H}^1(I)$ we have

$$g - g(a)e - g(b)f \in \mathrm{H}^1_0(I)\,,$$

which implies $\hat{g} = g(a)\hat{e} + g(b)\hat{f}$.

3. Examples

We are going to show that assumptions of Theorem 3 can be satisfied, at least in one space dimension. In Example 1 we also give a complete classification of subspaces V and $\tilde{V}$ satisfying (V1)–(V2).

Example 1. Let $I = \langle a, b\rangle \subseteq \mathbf{R}$, $L = \mathrm{L}^2(I)$, $\mathcal{D} = \mathrm{C}^\infty_c(I)$, and let the operators $T, \tilde{T} : \mathcal{D} \longrightarrow L$ be given by formulæ

$$\begin{aligned} Tu &= u' + \gamma u\,, \\ \tilde{T}u &= -u' + \gamma u\,, \end{aligned}$$

where $\gamma > 0$ is a constant. Then T and $\tilde{T}$ satisfy (T1)–(T3), while $W = \mathrm{H}^1(I) \hookrightarrow \mathrm{C}(I)$ and $W_0 = \mathrm{H}^1_0(I)$. We also have

$${}_{W'}\langle\, Du, v\,\rangle_W = u(b)v(b) - u(a)v(a)\,.$$

From Theorem 3 and Lemma 2 it follows that in this example $V+\tilde{V}$ is closed, whenever V and $\tilde{V}$ satisfy (V1)–(V2). We shall give a complete classification of pairs $(V,\tilde{V})$ satisfying (V1)–(V2) (in one dimensional case).

Note that (V1)–(V2) take the following form:

$$\text{(V1)}\qquad \begin{aligned} &(\forall\, u\in V) \quad u^2(b)-u^2(a)\geq 0\,,\\ &(\forall\, v\in \tilde{V}) \quad v^2(b)-v^2(a)\leq 0\,, \end{aligned}$$

$$\text{(V2)}\qquad \begin{aligned} &\tilde{V}=\{v\in \mathrm{H}^1(I):(\forall\, u\in V)\quad u(b)v(b)-u(a)v(a)=0\}\,,\\ &V=\{u\in \mathrm{H}^1(I):(\forall\, v\in \tilde{V})\quad u(b)v(b)-u(a)v(a)=0\}\,. \end{aligned}$$

Also note that $V\neq \mathrm{H}^1_0(I)$, otherwise (V2) would imply $\tilde{V}=\mathrm{H}^1(I)$, which contradicts (V1). Analogously we conclude that $\tilde{V}\neq \mathrm{H}^1_0(I)$. One can easily check that in each of the following cases:

$$\text{(a)}\qquad \begin{aligned} &V=\tilde{V}=\{u\in \mathrm{H}^1(I):u(a)=u(b)\}\quad \text{or}\\ &V=\tilde{V}=\{u\in \mathrm{H}^1(I):u(a)=-u(b)\}\,. \end{aligned}$$

$$\text{(b)}\qquad \begin{aligned} &V=\{u\in \mathrm{H}^1(I):u(a)=0\}\,,\\ &\tilde{V}=\{u\in \mathrm{H}^1(I):u(b)=0\}\,. \end{aligned}$$

$$\text{(c)}\qquad \begin{aligned} &V=\{u\in \mathrm{H}^1(I):u(b)=\alpha u(a)\}\,,\\ &\tilde{V}=\{u\in \mathrm{H}^1(I):u(b)=\frac{1}{\alpha}u(a)\}\,, \end{aligned}$$

where α is a real constant such that $|\alpha|>1$, the spaces V and $\tilde{V}$ satisfy (V1)–(V2). For (a) we clearly have $V+\tilde{V}=V=\tilde{V}$, while for (b) and (c) one can easily see that $V+\tilde{V}=\mathrm{H}^1(I)$.

Let us show that any pair $(V,\tilde{V})$ satisfying (V1)–(V2) necessarily has one of the three above forms. We distinguish two cases:
I. Let $V=\tilde{V}$. Then (V1) implies $u^2(b)=u^2(a)$, for each $u\in V$. As $\mathrm{H}^1_0(I)\subset V$, there is $\bar{v}\in V\setminus \mathrm{H}^1_0(I)$ such that

$$\bar{v}(b)=\bar{v}(a)\neq 0 \qquad \text{or} \qquad \bar{v}(b)=-\bar{v}(a)\neq 0\,.$$

By (V2) we have

$$\begin{aligned} &(\forall\, u\in V)\quad u(b)\bar{v}(b)=u(a)\bar{v}(a) \qquad \text{or}\\ &(\forall\, u\in V)\quad u(b)\bar{v}(b)=-u(a)\bar{v}(a)\,, \end{aligned}$$

which implies

$$(\forall\, u \in V) \quad u(b) = u(a) \qquad \text{or} \qquad (\forall\, u \in V) \quad u(b) = -u(a)\,.$$

Therefore, if we denote

$$U^{\pm} := \{u \in \mathrm{H}^1(I) : u(a) = \pm u(b)\}\,,$$

we have $V \subseteq U^+$ or $V \subseteq U^-$. Let us assume that $V \subseteq U^+$. Then (V2) implies $D(U^+)^0 \subseteq D(V)^0 = V$. As from (a) we know that $D(U^+)^0 = U^+$, it follows $U^+ = V$. In the same way one can analyse the case $V \subseteq U^-$. Therefore, if $V = \tilde{V}$ then the case (a) has occurred.

II. If $V \neq \tilde{V}$, then we distinguish two subcases:

II.1. Let there exist $\bar{u} \in V$ such that $\bar{u}(a) \neq 0 \neq \bar{u}(b)$. If we denote $\alpha := \frac{\bar{u}(b)}{\bar{u}(a)} \neq 0$ ((V1) implies $|\alpha| > 1$), then from (V2) it follows

$$(\forall\, v \in \tilde{V}) \quad v(b) = \frac{\bar{u}(a)}{\bar{u}(b)} v(a) = \frac{1}{\alpha} v(a)\,.$$

As $\mathrm{H}^1_0(I) \subset \tilde{V}$, there exists $\bar{v} \in \tilde{V}$ such that $\bar{v}(b) = \frac{1}{\alpha}\bar{v}(a) \neq 0$. After applying (V2) we get

$$(\forall\, u \in V) \quad u(b) = \frac{\bar{v}(a)}{\bar{v}(b)} v(a) = \alpha u(a)\,.$$

Therefore, if we denote

$$\begin{aligned} V_\alpha &:= \{u \in \mathrm{H}^1(I) : u(b) = \alpha u(a)\}\,, \\ \tilde{V}_\alpha &:= \{u \in \mathrm{H}^1(I) : u(b) = \frac{1}{\alpha} u(a)\} = V_{\frac{1}{\alpha}}\,, \end{aligned}$$

for $\alpha \neq 0$, we have $V \subseteq V_\alpha$ and $\tilde{V} \subseteq \tilde{V}_\alpha$. From (c) we know that $V_\alpha = D(\tilde{V}_\alpha)^0$ and $\tilde{V}_\alpha = D(V_\alpha)^0$. Now from (V2) it easily follows that $\tilde{V}_\alpha = D(V_\alpha)^0 \subseteq D(V)^0 = \tilde{V}$, and $V_\alpha = D(\tilde{V}_\alpha)^0 \subseteq D(\tilde{V})^0 = V$, which implies $V = V_\alpha$ and $\tilde{V} = \tilde{V}_\alpha$.

II.2. Assume that for every $u \in V$ we have $u(a) = 0$ or $u(b) = 0$. Taking into account $V \neq \mathrm{H}^1_0(I)$, (V1) implies that there exists $\bar{u} \in V$ such that $\bar{u}(a) = 0$ and $\bar{u}(b) \neq 0$. Using (V2) we get

$$(\forall\, v \in \tilde{V}) \quad v(b) = 0\,,$$

which implies the existence of $\bar{v} \in \tilde{V}$ such that $\bar{v}(a) \neq 0$. After using (V2) again, we get

$$(\forall\, u \in V) \quad u(a) = 0\,.$$

Therefore we have $V \subseteq U_a$ and $\tilde{V} \subseteq U_b$, where

$$\begin{aligned} U_a &= \{u \in \mathrm{H}^1(I) : u(a) = 0\}\,, \\ U_b &= \{u \in \mathrm{H}^1(I) : u(b) = 0\}\,. \end{aligned}$$

Similarly as above, using (b) we easily get $V = U_a$ and $\tilde{V} = U_b$.

Therefore, we have given a complete classification of pairs $(V, \tilde{V})$ satisfying (V1)–(V2) in this example.

Example 2. Let $I = \langle a, b\rangle \subseteq \mathbf{R}$, and assume that we are given matrix functions $\mathbf{A} = \mathbf{A}^\top \in \mathrm{W}^{1,\infty}(I; \mathrm{M}_r(\mathbf{R}))$ and $\mathbf{C} \in \mathrm{L}^\infty(I; \mathrm{M}_r(\mathbf{R}))$, such that

$$\mathbf{C} + \mathbf{C}^\top - \mathbf{A}' \geq \mu_0 \mathbf{I}\,,$$

for some positive constant μ_0. It can easily be seen that the operators T and $\tilde{T}$, defined by

$$\begin{aligned} Tu :=& \mathbf{A}u' + \mathbf{C}u\,, \\ \tilde{T}u :=& -\mathbf{A}^\top u' + (\mathbf{C}^\top - \mathbf{A}^\top)u \end{aligned}$$

satisfy (T1)–(T3) (now we have $L = \mathrm{L}^2(I; \mathbf{R}^r)$ and $\mathcal{D} = \mathrm{C}_c^\infty(I; \mathbf{R}^r)$). Assume additionally that $\mathbf{A}(x)$ is regular (a.e. $x \in I$), and $\mathbf{A}^{-1} \in \mathrm{L}^\infty(I; \mathrm{M}_r(\mathbf{R}))$. Then from

$$u \in \mathrm{L}^2(I; \mathbf{R}^r) \quad \text{and} \quad Tu \in \mathrm{L}^2(I; \mathbf{R}^r)$$

we have

$$u' = \mathbf{A}^{-1}(Tu - \mathbf{C}u) \in \mathrm{L}^2(I; \mathbf{R}^r)\,,$$

implying $W = \mathrm{H}^1(I; \mathbf{R}^r)$, and therefore $W_0 = \mathrm{H}_0^1(I; \mathbf{R}^r)$. It follows that

$$W/W_0 \cong \Big(\mathrm{H}^1(I)/\mathrm{H}_0^1(I)\Big)^r\,,$$

which implies

$$\dim W/W_0 = 2^r < \infty\,.$$

Now, from Theorem 3 we deduce that $V + \tilde{V}$ is closed whenever V and $\tilde{V}$ satisfy (V1)–(V2).

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