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Heat equation as a Friedrichs system

Journal of Mathematical Analysis and Applications

Abstract

Inspired by recent advances in the theory of (Friedrichs) symmetric positive systems, we apply newly developed results to the heat equation, by showing how the intrinsic theory of Ern, Guermond and Caplain (2007) can be used in order to get a well-posedness result for the Dirichlet initial boundary value problem. We also demonstrate the application of the two-field theory with partial coercivity of Ern and Guermond (2008), originally developed for elliptic problems, and also discuss different possibilities for the construction of appropriate boundary operator.

Keywords: symmetric positive system, initial boundary value problem, second-order parabolic equation

Mathematics subject classification: 35F45, 35K20, 35M32, 47F05

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This work is supported in part by the Croatian MZOS trough projects 037–0372787–2795 and 037–1193086–3226, and DAAD project Centre of Excellence for Applications of Mathematics.

23rd October 2013

1. Introduction

Symmetric positive systems (also known as Friedrichs systems) form a class of boundary value problems which allow the study of a wide range of differential equations in a unified framework. They were introduced by Kurt Otto Friedrichs [F] in 1958 in an attempt to handle transonic flow problems, which are partially hyperbolic and partially elliptic in different parts of domain.

To be specific, in his seminal paper [F] Friedrichs considered a first-order partial differential operator $\mathcal{L} : \mathrm{L}^2(\Omega; \mathbf{R}^r) \longrightarrow \mathcal{D}'(\Omega; \mathbf{R}^r)$ of the form

$$\mathcal{L}\mathsf{u} := \sum_{k=1}^{d} \partial_k(\mathbf{A}_k \mathsf{u}) + \mathbf{C}\mathsf{u}\,,$$

the coefficients being real matrix functions $\mathbf{A}_k \in \mathrm{W}^{1,\infty}(\Omega; \mathrm{M}_r(\mathbf{R}))$, for $k \in 1..d$, while $\mathbf{C} \in \mathrm{L}^\infty(\Omega; \mathrm{M}_r(\mathbf{R}))$, where $\Omega \subseteq \mathbf{R}^d$ is an open bounded set with Lipschitz boundary Γ (we shall denote its closure by $\mathsf{Cl}\,\Omega = \Omega \cup \Gamma$) and $d, r \in \mathbf{N}$. Of course, at that time he assumed more regularity than it is stated here. It is also required that the coefficients satisfy

(F1) $$\text{each } \mathbf{A}_k \text{ is symmetric:} \quad \mathbf{A}_k = \mathbf{A}_k^\top\,,$$

(F2) $$(\exists\, \mu_0 > 0) \quad \mathbf{C} + \mathbf{C}^\top + \sum_{k=1}^{d} \partial_k \mathbf{A}_k \geqslant 2\mu_0 \mathbf{I} \qquad (\text{a.e. on } \Omega)\,,$$

and then such an operator $\mathcal{L}$ is called *the Friedrichs operator* or *the symmetric positive operator*. The corresponding first-order system of partial differential equations $\mathcal{L}\mathsf{u} = \mathsf{f}$, for given $\mathsf{f} \in \mathrm{L}^2(\Omega; \mathbf{R}^r)$, is called *the Friedrichs system* or *the symmetric positive system*. Also note that we have used the divergence form of the differential operator in order to allow coefficients with lower regularity (the difference $(\partial_k \mathbf{A}_k)\mathsf{u}$ can be included in the term $\mathbf{C}\mathsf{u}$).

In order to describe boundary conditions, following Friedrichs [F] we first define a matrix field on the boundary, namely

$$\mathbf{A}_{\boldsymbol{\nu}} := \sum_{k=1}^{d} \nu_k \mathbf{A}_k\,,$$

where $\boldsymbol{\nu} = (\nu_1, \nu_2, \cdots, \nu_d)^\top \in \mathrm{L}^\infty(\Gamma; \mathbf{R}^d)$ is the outward unit normal on Γ. Note that $\mathbf{A}_{\boldsymbol{\nu}}$ is of class L^∞ on Γ. The boundary condition is then prescribed by

$$(\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M})\mathsf{u}_{|_\Gamma} = 0\,,$$

where $\mathbf{M} : \Gamma \longrightarrow \mathrm{M}_r(\mathbf{R})$ is a given matrix field on the boundary, and by varying $\mathbf{M}$ one can enforce different boundary conditions. Friedrichs required the following two conditions (for a.e. $\mathbf{x} \in \Gamma$) to hold:

(FM1) $$(\forall \boldsymbol{\xi} \in \mathbf{R}^r) \quad \mathbf{M}(\mathbf{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geqslant 0\,,$$

(FM2) $$\mathbf{R}^r = \ker\Big(\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x}) - \mathbf{M}(\mathbf{x})\Big) + \ker\Big(\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x}) + \mathbf{M}(\mathbf{x})\Big)\,;$$

and such $\mathbf{M}$ he called *an admissible boundary condition*.

The boundary value problem thus reads: for given $\mathsf{f} \in \mathrm{L}^2(\Omega; \mathbf{R}^r)$ find u such that

(1) $$\begin{cases} \mathcal{L}\mathsf{u} = \mathsf{f} \\ (\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M})\mathsf{u}_{|_\Gamma} = 0 \end{cases}.$$

It is important to emphasise that this setting covers a large number of equations of continuum physics, regardless of their type. Also a different types of (initial) boundary conditions (Dirichlet, Neumann, Robin) can be treated in this way as well. For some specific examples of (initial) boundary value problems that can be treated via theory of Friedrichs systems we refer to [ABV1, BDG, EG, EGC, J].

As assumptions on coefficients are rather weak, the existence of a classical solution (C^1 or $\mathrm{W}^{1,\infty}$) cannot be expected. It can be shown that, in general, the solution belongs only to the graph space of operator $\mathcal{L}$:

$$W = \Big\{ \mathsf{u} \in \mathrm{L}^2(\Omega; \mathbf{R}^r) : \mathcal{L}\mathsf{u} \in \mathrm{L}^2(\Omega; \mathbf{R}^r) \Big\}.$$

For more information about these spaces we refer to [AB1, B, J]. Here we would only like to mention that W is a separable Hilbert space with inner product

$$\langle\, \mathsf{u} \mid \mathsf{v} \,\rangle_{\mathcal{L}} := \langle\, \mathsf{u} \mid \mathsf{v} \,\rangle_{\mathrm{L}^2(\Omega;\mathbf{R}^r)} + \langle\, \mathcal{L}\mathsf{u} \mid \mathcal{L}\mathsf{v} \,\rangle_{\mathrm{L}^2(\Omega;\mathbf{R}^r)}.$$

The corresponding norm is denoted by

$$\|\mathsf{u}\|_{\mathcal{L}} = \sqrt{\|\mathsf{u}\|^2_{\mathrm{L}^2(\Omega;\mathbf{R}^r)} + \|\mathcal{L}\mathsf{u}\|^2_{\mathrm{L}^2(\Omega;\mathbf{R}^r)}},$$

and functions from $\mathrm{C}^\infty_c(\mathsf{Cl}\,\Omega; \mathbf{R}^r)$ (and thus also from $\mathrm{H}^1(\Omega; \mathbf{R}^r)$) are dense in W.

One of the main difficulties in the theory of Friedrichs systems was the interpretation of boundary conditions: it was not a priori clear what would be the meaning of restriction $\mathsf{u}_{|_\Gamma}$ for functions u from the graph space. Later, it was shown that $\mathsf{u}_{|_\Gamma}$ can be interpreted as an element of $\mathrm{H}^{-\frac{1}{2}}(\Gamma; \mathbf{R}^r)$, and the appropriate well-posedness results for the weak formulation of (1), under additional assumptions, were proved [Ra, J].

Abstract theory of Friedrichs systems

More recently, Ern, Guermond and Caplain [EG1, EGC] suggested another approach to the Friedrichs theory, which completely avoids the question of traces for functions from the graph space. They developed an abstract theory of Friedrichs systems written in terms of operators acting on Hilbert spaces, and gave an intrinsic description of boundary conditions. The trace operator was replaced by the *boundary operator* $D \in \mathcal{L}(W; W')$ defined by

$$_{W'}\langle\, D\mathsf{u}, \mathsf{v} \,\rangle_W := \langle\, \mathcal{L}\mathsf{u} \mid \mathsf{v} \,\rangle_{\mathrm{L}^2(\Omega;\mathbf{R}^r)} - \langle\, \mathsf{u} \mid \tilde{\mathcal{L}}\mathsf{v} \,\rangle_{\mathrm{L}^2(\Omega;\mathbf{R}^r)}, \qquad \mathsf{u}, \mathsf{v} \in W,$$

where $\tilde{\mathcal{L}} : \mathrm{L}^2(\Omega; \mathbf{R}^r) \longrightarrow \mathcal{D}'(\Omega; \mathbf{R}^r)$, the formally adjoint operator to $\mathcal{L}$, was defined by

$$\tilde{\mathcal{L}}\mathsf{v} := -\sum_{k=1}^d \partial_k(\mathbf{A}_k\mathsf{v}) + \Big(\mathbf{C}^\top + \sum_{k=1}^d \partial_k\mathbf{A}_k\Big)\mathsf{v}.$$

In fact, it turned out that operator D had better properties than the trace operator, among which was its symmetry:

$$(\forall\, \mathsf{u}, \mathsf{v} \in W) \qquad {}_{W'}\langle\, D\mathsf{u}, \mathsf{v} \,\rangle_W = {}_{W'}\langle\, D\mathsf{v}, \mathsf{u} \,\rangle_W.$$

Using these properties Ern at al. [EGC] proved a weak well-posedness result in this abstract setting. When applied to the classical partial differential operator $\mathcal{L}$, their main result reads:

Theorem 1. *Let (F1)–(F2) hold, and let subspaces V and $\widetilde{V}$ of W satisfy*

(V1)
$$(\forall\, \mathsf{u} \in V) \qquad {}_{W'}\langle\, D\mathsf{u}, \mathsf{u} \,\rangle_W \geqslant 0,$$
$$(\forall\, \mathsf{v} \in \widetilde{V}) \qquad {}_{W'}\langle\, D\mathsf{v}, \mathsf{v} \,\rangle_W \leqslant 0,$$

(V2)
$$V = D(\widetilde{V})^0, \qquad \widetilde{V} = D(V)^0,$$

where ^{0} stands for the annihilator.

Then the restrictions of operators $\mathcal{L}_{|_V} : V \longrightarrow L$ and $\tilde{\mathcal{L}}_{|_{\widetilde{V}}} : \widetilde{V} \longrightarrow L$ are isomorphisms. ∎

Note that the information about boundary conditions is *hidden* in the structure of subspace V. In the sequel we shall refer to both properties (V1) and (V2) as (V), and similarly we shall use (F), (FM), etc.

Ern et al. [EGC] also investigated different representations of boundary conditions in the abstract setting which correlated with those known in the classical Friedrichs theory. They were also interested in their mutual relationship, which raised a number of open questions. In [AB1, AB2, AB3] we closed the most important question by proving that those abstract conditions were, in fact, all equivalent. The new development was based on the fact that the theory can be expressed in terms of Kreĭn spaces (a particular kind of indefinite inner product spaces). This approach allowed us to simplify a number of earlier proofs as well.

One of these representations, corresponding to the matrix boundary field $\mathbf{M}$ satisfying (FM), uses an operator $M \in \mathcal{L}(W; W')$ with properties

$$\text{(M1)} \qquad (\forall \mathsf{u} \in W) \quad {}_{W'}\langle M\mathsf{u}, \mathsf{u} \rangle_W \geqslant 0\,,$$

and

$$\text{(M2)} \qquad W = \ker(D - M) + \ker(D + M)\,.$$

Such an operator is also called the boundary operator, as $\ker M = \ker D = W_0 := \mathsf{Cl}_W \mathrm{C}^\infty_c(\Omega; \mathbf{R}^r)$.

In [AB2] we proved the equivalence between properties (V) and (M) in the following sense: two subspaces V and $\widetilde{V}$ of W satisfy (V) if and only if there is (not necessary unique) operator $M \in \mathcal{L}(W; W')$ with properties (M), such that $V = \ker(D - M)$ and $\widetilde{V} = \ker(D + M^*)$. Thus, the weak well-posedness result can also be expressed as follows.

Theorem 2. *Let (F) be valid and assume that there exists an operator $M \in \mathcal{L}(W; W')$ satisfying (M). Then the restricted operators*

$$\mathcal{L}_{|_{\ker(D-M)}} : \ker(D - M) \longrightarrow \mathrm{L}^2(\Omega; \mathbf{R}^r) \qquad \textit{and} \qquad \tilde{\mathcal{L}}_{|_{\ker(D+M^*)}} : \ker(D + M^*) \longrightarrow \mathrm{L}^2(\Omega; \mathbf{R}^r)$$

are isomorphisms. ∎

The above simplification of abstract theory paved the way to new investigations of the precise relationship between the classical Friedrichs theory and its abstract counterpart.

The analogy between the properties (M) for operator M and (FM) for matrix boundary condition $\mathbf{M}$ is apparent. A natural question to be investigated is that of the nature of relationship between matrix field $\mathbf{M}$ and boundary operator M. More precisely, to find additional conditions on the matrix field $\mathbf{M}$ satisfying (FM) which will guarantee the existence of a *suitable operator* $M \in \mathcal{L}(W; W')$ with properties (M). Here, a *suitable operator* means that the result of Theorem 2 really presents a *weak well–posedness result* for problem (1) in the following sense: if for given $\mathsf{f} \in \mathrm{L}^2(\Omega; \mathbf{R}^r)$, $\mathsf{u} \in \ker(D - M)$ is such that $\mathcal{L}\mathsf{u} = \mathsf{f}$, where we additionally have $\mathsf{u} \in \mathrm{C}^1(\Omega; \mathbf{R}^r) \cap \mathrm{C}(\mathsf{Cl}\,\Omega; \mathbf{R}^r)$, then u satisfies (1) in the classical sense.

In [AB4, ABV1, ABV2] we found sufficient conditions insuring that matrix field $\mathbf{M}$ defines an appropriate operator M. The connection between $\mathbf{M}$ and M is given by

$$\text{(2)} \qquad (\forall \mathsf{u}, \mathsf{v} \in \mathrm{C}^\infty_c(\mathsf{Cl}\,\Omega; \mathbf{R}^r)) \qquad {}_{W'}\langle M\mathsf{u}, \mathsf{v} \rangle_W = \int_\Gamma \mathbf{M}(\mathbf{x})\mathsf{u}_{|_\Gamma}(\mathbf{x}) \cdot \mathsf{v}_{|_\Gamma}(\mathbf{x}) dS(\mathbf{x})\,,$$

as a result of the known connection [AB1, B, EGC] between operator D and matrix field $\mathbf{A}_{\boldsymbol{\nu}}$:

$$\text{(3)} \qquad (\forall \mathsf{u}, \mathsf{v} \in \mathrm{C}^\infty_c(\mathsf{Cl}\,\Omega; \mathbf{R}^r)) \qquad {}_{W'}\langle D\mathsf{u}, \mathsf{v} \rangle_W = \int_\Gamma \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\mathsf{u}_{|_\Gamma}(\mathbf{x}) \cdot \mathsf{v}_{|_\Gamma}(\mathbf{x}) dS(\mathbf{x})\,.$$

In fact, both above formulæ can easily be extended to $\mathsf{u}, \mathsf{v} \in \mathrm{H}^1(\Omega; \mathbf{R}^r)$, provided that the restriction to Γ is replaced by the trace operator.

In [ABV1] the following theorem was proved and applied to some hyperbolic and elliptic equations.

Theorem 3. *Let* $\mathbf{P} : \mathsf{Cl}\,\Omega \longrightarrow \mathrm{M}_r(\mathbf{R})$ *be a Lipschitz matrix function satisfying:*

(P1) $(\exists\, \mathbf{R} \in \mathrm{W}^{1,\infty}(\Omega; \mathrm{M}_r(\mathbf{R})))(\forall\, k \in 1..d) \qquad \mathbf{A}_k\mathbf{P} = \mathbf{R}\mathbf{A}_k,$

(P2) for almost every $\mathbf{x} \in \Gamma$ *the matrix* $\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})(\mathbf{I} - 2\mathbf{P}(\mathbf{x}))$ *is positive semidefinite, and*

(P3) for almost every $\mathbf{x} \in \Gamma$ *it holds that* $\ker\Big(\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\mathbf{P}(\mathbf{x})\Big) + \ker\Big((\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})(\mathbf{I} - \mathbf{P}(\mathbf{x}))\Big) = \mathbf{R}^r.$

Then formula (2), for $\mathbf{M}(\mathbf{x}) := \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})(\mathbf{I}-2\mathbf{P}(\mathbf{x}))$ *on* Γ*, defines a bounded operator* $M \in \mathcal{L}(W; W')$ *satisfying (M).* ∎

In this paper we first try to apply this theorem to the heat equation, and pinpoint some obstacles that prevents us in getting good results. At the same time we shall use the *intrinsic result* of Theorem 1 in order to get appropriate well-possednes results for the initial–boundary value problem for heat equation.

Better understanding of the theory for the heat equation written as a Friedrichs system will hopefully bring us closer to the satisfactory theory for mixed-type equations. As a more immediate goal, we hope to be able to get a homogenisation result. The unification of equations of different type (elliptic/parabolic/hyperbolic) within the framework of Friedrichs systems has already shown practical benefits in their numerical treatment. Namely, the convergence analysis can be done in a more unified way, while the numerical code can also be shared. A number of recent results on discontinuous Galerkin methods for Friedrichs systems can be found in [BDG, DE, EG1, EG2, EG3, J].

The paper is organised as follows: In the second section we write the heat equation as a Friedrichs system and show that the method described in [ABV1], i. e. Theorem 3 cannot be applied here. In the third section we describe the graph space W and the boundary operator D. In order to get a well-posedness result it is important to represent D by boundary integrals, which is done in Theorem 4. In the following section we achieve well-posedness result for the Dirichlet boundary condition: first we try to *guess* a good boundary operator M, and then turn our attention to intrinsic conditions of Theorem 1, which enables us to get a well-posedness result. At the end of this section we show that the two-field theory for Friedrichs systems with partial coercivity [EG2, EG3], initially developed for elliptic equations, can be applied to the parabolic equation as well. In the fifth section we try to clarify whether our starting boundary operator M is continuous and satisfies (M). We reduce this to the question of specific decomposition of the graph space (Corollary 2), which remains open. Finally, we close the paper with some concluding remarks, and an Appendix containing basic information regarding the evolution spaces, which were used in the text.

2. Heat equation as a Friedrichs system

Let $\Omega \subseteq \mathbf{R}^d$ be an open and bounded set with the Lipschitz boundary Γ, $T > 0$ and $\Omega_T := \Omega \times \langle 0, T\rangle$. The parts of boundary $\partial\Omega_T$ we denote as follows: $\Gamma_0 := \Omega \times \{0\}$, $\Gamma_T := \Omega \times \{T\}$ and $\Gamma_d := \Gamma \times \langle 0, T\rangle$. We consider the heat equation in the following form

$$\partial_t u - \mathsf{div}_{\mathbf{x}}(\mathbf{A}\nabla_{\mathbf{x}} u) + \mathbf{b}\cdot\nabla_{\mathbf{x}} u + cu = f \qquad \text{in } \Omega_T\,,$$

where $f \in \mathrm{L}^2(\Omega_T)$, $c \in \mathrm{L}^\infty(\Omega_T)$, $\mathbf{b} \in \mathrm{L}^\infty(\Omega_T; \mathbf{R}^d)$ and $\mathbf{A} \in \mathrm{L}^\infty(\Omega_T; \mathrm{M}_d(\mathbf{R}))$ (note that the coefficients depend both on $\mathbf{x}$ and t). Suppose that there exist constants $\beta \geqslant \alpha > 0$ such that $\mathbf{A}(\mathbf{x}, t)$ is a symmetric matrix with eigenvalues between α and β, almost everywhere on Ω_T.

Similarly as it was the case for the elliptic equation [ABV1], this equation can be rewritten as a Friedrichs system in the following way: consider a new unknown vector function taking values in $\mathbf{R}^{d+1}$

$$\mathsf{u} = \begin{bmatrix} -\mathbf{A}\nabla u \\ u \end{bmatrix}\,,$$

matrices $\mathbf{A}_k = \mathbf{e}_k \otimes \mathbf{e}_{d+1} + \mathbf{e}_{d+1} \otimes \mathbf{e}_k \in \mathrm{M}_{d+1}(\mathbf{R})$, for $k \in 1..d$ (here by $\mathbf{e}_1, \dots \mathbf{e}_{d+1}$ we have denoted the standard basis for $\mathbf{R}^{d+1}$) and block matrix function

$$\mathbf{C} = \begin{bmatrix} \mathbf{A}^{-1} & \mathbf{0} \\ -(\mathbf{A}^{-1}\mathbf{b})^\top & c \end{bmatrix} \in \mathrm{L}^\infty((\Omega_T; \mathrm{M}_{d+1}(\mathbf{R}))) ,$$

all coinciding with those used for the elliptic equation, while $\mathbf{A}_{d+1} = \mathbf{e}_{d+1} \otimes \mathbf{e}_{d+1} \in \mathrm{M}_{d+1}(\mathbf{R})$. Note that our domain is now Ω_T, and that first d variables are space variables of the original equation, while $(d+1)$-st variable is the time variable. We choose this order of variables in order to have our matrices $\mathbf{A}_k$ and $\mathbf{C}$ in a form as close as possible to that in the elliptic case.

The positivity condition $\mathbf{C} + \mathbf{C}^\top \geqslant 2\mu_0 \mathbf{I}$ is fulfilled if and only if the Schur complement $c - \frac{1}{4}\mathbf{A}^{-1}\mathbf{b}\cdot\mathbf{b}$ is uniformly positive, i.e. if there exists a constant $\gamma > 0$ such that $c - \frac{1}{4}\mathbf{A}^{-1}\mathbf{b}\cdot\mathbf{b} \geqslant \gamma$ on Ω_T (see [ABV1] for details).

Let us now apply Theorem 3 (with $d+1$ in place of d and Ω_T in place of Ω): in order to fulfil condition (P1) we need to determine all possible $\mathbf{P} \in \mathrm{W}^{1,\infty}(\Omega_T; \mathrm{M}_r(\mathbf{R}))$ such that

$$(\exists\, \mathbf{R} \in \mathrm{W}^{1,\infty}(\Omega_T; \mathrm{M}_r(\mathbf{R})))(\forall\, k \in 1..d+1) \qquad \mathbf{A}_k\mathbf{P} = \mathbf{R}\mathbf{A}_k .$$

After testing this condition for first d matrices $\mathbf{A}_k$, we obtain

$$\mathbf{P} = \begin{bmatrix} a\mathbf{I} & \boldsymbol{\eta} \\ \mathbf{0}^\top & b \end{bmatrix} \quad \text{and} \quad \mathbf{R} = \begin{bmatrix} b\mathbf{I} & \mathbf{0} \\ \boldsymbol{\eta}^\top & a \end{bmatrix} ,$$

where a, b and $\boldsymbol{\eta}$ are Lipschitz functions on Ω_T. Since $\mathbf{A}_{d+1}$ has to satisfy (P1) as well, we easily get the requirement $b = a$, and thus

$$\mathbf{P} = \begin{bmatrix} a\mathbf{I} & \boldsymbol{\eta} \\ \mathbf{0}^\top & a \end{bmatrix} \quad \text{and} \quad \mathbf{R} = \begin{bmatrix} a\mathbf{I} & \mathbf{0} \\ \boldsymbol{\eta}^\top & a \end{bmatrix} .$$

Note that the outward unit normal $\boldsymbol{\nu} = (\boldsymbol{\nu}_d, \nu_t)^\top$ on boundary $\partial\Omega_T$ is now a $(d+1)$-dimensional vector function on $\partial\Omega_T$, whose first d components, corresponding to the space directions, we denote by $\boldsymbol{\nu}_d$, while its last component, corresponding to the time direction, we denote by ν_t. Therefore, for matrix functions $\mathbf{A}_{\boldsymbol{\nu}}$ and $\mathbf{M}$ we have

$$\mathbf{A}_{\boldsymbol{\nu}} = \begin{bmatrix} \mathbf{0} & \boldsymbol{\nu}_d \\ \boldsymbol{\nu}_d^\top & \nu_t \end{bmatrix} \qquad \text{and} \qquad \mathbf{M} = \mathbf{A}_{\boldsymbol{\nu}}(I - 2\mathbf{P}) = \begin{bmatrix} \mathbf{0} & (1-2a)\boldsymbol{\nu}_d \\ (1-2a)\boldsymbol{\nu}_d^\top & (1-2a)\nu_t - 2\boldsymbol{\nu}_d\cdot\boldsymbol{\eta} \end{bmatrix} .$$

Condition (P2) now reads

$$(\forall\, \boldsymbol{\xi} \in \mathbf{R}^{d+1}) \quad 2(1-2a)\xi_{d+1}\boldsymbol{\nu}\cdot\boldsymbol{\xi} - (1-2a)\nu_t\xi_{d+1}^2 - 2\xi_{d+1}^2\boldsymbol{\nu}_d\cdot\boldsymbol{\eta} \geqslant 0 ,$$

which leads us to consider the following two cases:

a) $a = \frac{1}{2}$ and $\boldsymbol{\nu}_d\cdot\boldsymbol{\eta} \leqslant 0$;

b) $a \neq \frac{1}{2}$, $\boldsymbol{\nu}_d = 0$, and $(1-2a)\nu_t \geqslant 0$.

It order to check (P3), we first write it in a form more suitable for calculations.

Lemma 1. *The condition (P3) of Theorem 3 is equivalent to equality* $\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\mathbf{P}(\mathbf{x})(\mathbf{I}-\mathbf{P}(\mathbf{x})) = \mathbf{0}$ *valid for almost every* $\mathbf{x}$ *from the boundary of domain.*

Dem. Let (P3) hold, and for an arbitrary $\boldsymbol{\xi} \in \mathbf{R}^{d+1}$ let $\boldsymbol{\xi}_1 \in \ker(\mathbf{A}_{\boldsymbol{\nu}}\mathbf{P})$ be such that $\boldsymbol{\xi} - \boldsymbol{\xi}_1 \in \ker(\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{I}-\mathbf{P}))$ (we suppress explicitly writing $\mathbf{x}$ in this proof). Then

$$0 = \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{I}-\mathbf{P})(\boldsymbol{\xi}-\boldsymbol{\xi}_1) = \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{I}-\mathbf{P})\boldsymbol{\xi} - \mathbf{A}_{\boldsymbol{\nu}}\boldsymbol{\xi}_1 = \mathbf{A}_{\boldsymbol{\nu}}\Big((\mathbf{I}-\mathbf{P})\boldsymbol{\xi} - \boldsymbol{\xi}_1\Big) ,$$

which implies $(\mathbf{I}-\mathbf{P})\boldsymbol{\xi} - \boldsymbol{\xi}_1 \in \ker \mathbf{A}_{\boldsymbol{\nu}}$. Since $\ker \mathbf{A}_{\boldsymbol{\nu}} \subseteq \ker(\mathbf{A}_{\boldsymbol{\nu}}\mathbf{P})$, it follows that

$$0 = \mathbf{A}_{\boldsymbol{\nu}}\mathbf{P}\Big((\mathbf{I}-\mathbf{P})\boldsymbol{\xi} - \boldsymbol{\xi}_1\Big) = \mathbf{A}_{\boldsymbol{\nu}}\mathbf{P}(\mathbf{I}-\mathbf{P})\boldsymbol{\xi} ,$$

and therefore $\mathbf{A}_{\boldsymbol{\nu}}\mathbf{P}(\mathbf{I}-\mathbf{P}) = \mathbf{0}$, by the arbitrariness of $\boldsymbol{\xi}$.

In order to prove the converse statement, for $\boldsymbol{\xi} \in \mathbf{R}^{d+1}$ define $\boldsymbol{\xi}_1 := (\mathbf{I}-\mathbf{P})\boldsymbol{\xi}$, so that $\boldsymbol{\xi} - \boldsymbol{\xi}_1 = \mathbf{P}\boldsymbol{\xi}$. Now it easily follows

$$\mathbf{A}_{\boldsymbol{\nu}}\mathbf{P}\boldsymbol{\xi}_1 = \mathbf{A}_{\boldsymbol{\nu}}\mathbf{P}(\mathbf{I}-\mathbf{P})\boldsymbol{\xi} = 0 ,$$
$$\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{I}-\mathbf{P})(\boldsymbol{\xi}-\boldsymbol{\xi}_1) = \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{I}-\mathbf{P})\mathbf{P}\boldsymbol{\xi} = 0 ,$$

which proves the statement.

Q.E.D.

For the heat equation, the condition $\mathbf{A}_{\boldsymbol{\nu}}\mathbf{P}(\mathbf{I}-\mathbf{P}) = \mathbf{0}$ becomes

$$a(1-a)\boldsymbol{\nu}_d = 0 \qquad \text{and} \qquad (1-2a)\boldsymbol{\nu}_d \cdot \boldsymbol{\eta} + a(1-a)\nu_t = 0\,.$$

With these conditions the above case (a) leads to $\boldsymbol{\nu}_d = 0$ and $\nu_t = 0$, which contradicts $|\boldsymbol{\nu}| = 1$. Since the case (b) already contains the constraint $\boldsymbol{\nu}_d = 0$ it means that on the part Γ_d of boundary Γ conditions (P2) and (P3) of Theorem 3 cannot be achieved, and thus we cannot apply Theorem 3 to our Friedrichs systems. However, we should keep in mind that Theorem 3 gives only sufficient conditions for well-posedness, so in the sequel we shall try to obtain the well-posedness result for the heat equation in a different way.

3. Graph space and boundary operator

For the heat equation written as a Friedrichs system, the operator $\mathcal{L}$ is given by (note that we use $\mathsf{u} = (\mathsf{u}_d, u_{d+1})^\top$ below)

$$\mathcal{L}\begin{bmatrix} \mathsf{u}_d \\ u_{d+1} \end{bmatrix} = \begin{bmatrix} \nabla_{\mathbf{x}} u_{d+1} + \mathbf{A}^{-1}\mathsf{u}_d \\ \partial_t u_{d+1} + \mathsf{div}_{\mathbf{x}}\mathsf{u}_d + c u_{d+1} - \mathbf{A}^{-1}\mathsf{b}\cdot \mathsf{u}_d \end{bmatrix}, \tag{4}$$

while the corresponding graph space is

$$\begin{aligned} W &= \Big\{\mathsf{u} \in \mathrm{L}^2(\Omega_T; \mathbf{R}^{d+1}) : \nabla_{\mathbf{x}} u_{d+1} \in \mathrm{L}^2(\Omega_T;\mathbf{R}^d) \quad \& \quad \partial_t u_{d+1} + \mathsf{div}_{\mathbf{x}}\mathsf{u}_d \in \mathrm{L}^2(\Omega_T)\Big\} \\ &= \Big\{\mathsf{u} \in \mathrm{L}^2_{\mathrm{div}}(\Omega_T) : \nabla_{\mathbf{x}} u_{d+1} \in \mathrm{L}^2(\Omega_T;\mathbf{R}^d)\Big\} \\ &= \Big\{\mathsf{u} \in \mathrm{L}^2_{\mathrm{div}}(\Omega_T) : u_{d+1} \in \mathrm{L}^2(0,T;\mathrm{H}^1(\Omega))\Big\}\,, \end{aligned}$$

where

$$\mathrm{L}^2_{\mathrm{div}}(\Omega_T) = \Big\{\mathsf{u} \in \mathrm{L}^2(\Omega_T;\mathbf{R}^{d+1}) : \mathsf{div}\,\mathsf{u} = \partial_t u_{d+1} + \mathsf{div}_{\mathbf{x}}\mathsf{u}_d \in \mathrm{L}^2(\Omega_T)\Big\}\,.$$

Here we identified spaces $\mathrm{L}^2(\Omega_T)$ and $\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))$ as they are isometrically isomorphic (see Appendix for some basic properties of L^p spaces with values in Banach spaces which will be frequently used in the sequel). One can easily see that norm $\|\cdot\|_{\mathcal{L}}$ on W is equivalent to ($\sim$ stands for equivalence between norms)

$$\begin{aligned} \|\mathsf{u}\|_{\mathcal{L}} &\sim \sqrt{\|\mathsf{u}\|^2_{\mathrm{L}^2(\Omega_T;\mathbf{R}^{d+1})} + \|\nabla_{\mathbf{x}} u_{d+1}\|^2_{\mathrm{L}^2(\Omega_T;\mathbf{R}^d)} + \|\mathsf{div}\,\mathsf{u}\|^2_{\mathrm{L}^2(\Omega_T)}} \\ &\sim \sqrt{\|\mathsf{u}\|^2_{\mathrm{L}^2_{\mathrm{div}}(\Omega_T)} + \|\nabla_{\mathbf{x}} u_{d+1}\|^2_{\mathrm{L}^2(\Omega_T;\mathbf{R}^d)}} \\ &\sim \sqrt{\|\mathsf{u}_d\|^2_{\mathrm{L}^2(\Omega_T;\mathbf{R}^d)} + \|u_{d+1}\|^2_{\mathrm{L}^2(0,T;\mathrm{H}^1(\Omega))} + \|\mathsf{div}\,\mathsf{u}\|^2_{\mathrm{L}^2(\Omega_T)}}\,. \end{aligned} \tag{5}$$

Let us now introduce

$$W(0,T) = \Big\{u \in \mathrm{L}^2(0,T;\mathrm{H}^1(\Omega)) : \partial_t u \in \mathrm{L}^2(0,T;\mathrm{H}^{-1}(\Omega))\Big\}\,,$$

which is a Banach space when equipped by norm

$$\|\mathsf{u}\|_{W(0,T)} = \sqrt{\|u\|^2_{\mathrm{L}^2(0,T;\mathrm{H}^1(\Omega))} + \|\partial_t u\|^2_{\mathrm{L}^2(0,T;\mathrm{H}^{-1}(\Omega))}}\,.$$

The following lemma, together with the imbedding of $W(0,T)$ to $\mathrm{C}([0,T];\mathrm{L}^2(\Omega))$ (see Appendix), will be of great importance for obtaining well-posedness results.

Lemma 2. *The projection* $\mathsf{u} = (\mathsf{u}_d, u_{d+1})^\top \mapsto u_{d+1}$ *is a continuous linear operator from* W *to* $W(0,T)$.

Dem. Let $\mathsf{u} \in W$. Then $\partial_t u_{d+1} + \mathsf{div}_{\mathbf{x}} \mathsf{u}_d \in \mathrm{L}^2(\Omega_T) \simeq \mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))$. Using the continuity of imbedding of $\mathrm{L}^2(\Omega)$ in $\mathrm{H}^{-1}(\Omega)$ and Theorem 10 (Appendix), we get $\partial_t u_{d+1} + \mathsf{div}_{\mathbf{x}} \mathsf{u}_d \in \mathrm{L}^2(0,T;\mathrm{H}^{-1}(\Omega))$. Analogously, using the continuity of $\mathsf{div}_{\mathbf{x}} : \mathrm{L}^2(\Omega) \longrightarrow \mathrm{H}^{-1}(\Omega)$, we get $\mathsf{div}_{\mathbf{x}} \mathsf{u}_d \in \mathrm{L}^2(0,T;\mathrm{H}^{-1}(\Omega))$, and thus $\partial_t u_{d+1} \in \mathrm{L}^2(0,T;\mathrm{H}^{-1}(\Omega))$. Finally, $u_{d+1} \in W(0,T)$, and as all operators involved in the above calculations where continuous, we get the continuity of projection as well.

Q.E.D.

One can easily see that boundary operator $D \in \mathcal{L}(W;W')$ takes the form

$$
{}_{W'}\langle\, D\mathsf{u}, \mathsf{v}\,\rangle_W = \int_0^T \int_\Omega (\nabla_{\mathbf{x}} u_{d+1} \cdot \mathsf{v}_d + v_{d+1} \mathsf{div}\, \mathsf{u} + \nabla_{\mathbf{x}} v_{d+1} \cdot \mathsf{u}_d + u_{d+1} \mathsf{div}\, \mathsf{v})\, d\mathbf{x}\, dt\,,
$$

for any $\mathsf{u} = (\mathsf{u}_d, u_{d+1})^\top, \mathsf{v} = (\mathsf{v}_d, v_{d+1})^\top \in W$. The low regularity of functions in W does not allow us to apply the divergence theorem directly. For that purpose, it is enough to strengthen the condition $\mathsf{div}\, \mathsf{u} = \partial_t u_{d+1} + \mathsf{div}_{\mathbf{x}} \mathsf{u}_d \in \mathrm{L}^2(\Omega_T)$ by assuming that both summands belong to $\mathrm{L}^2(\Omega_T)$. This motivates the definition of

$$
\widetilde{W} := \{\mathsf{u} \in W : \partial_t u_{d+1}, \mathsf{div}_{\mathbf{x}} \mathsf{u}_d \in \mathrm{L}^2(\Omega_T)\}\,,
$$

which is a dense subspace of W. Notice that for $\mathsf{u}, \mathsf{v} \in \widetilde{W}$ we have

$$
\begin{aligned}
&\nabla_{\mathbf{x}} u_{d+1} \cdot \mathsf{v}_d + v_{d+1} \mathsf{div}\, \mathsf{u} + \nabla_{\mathbf{x}} v_{d+1} \cdot \mathsf{u}_d + u_{d+1} \mathsf{div}\, \mathsf{v} \\
&\quad = \nabla_{\mathbf{x}} u_{d+1} \cdot \mathsf{v}_d + u_{d+1} \mathsf{div}_{\mathbf{x}} \mathsf{v}_d + \nabla_{\mathbf{x}} v_{d+1} \cdot \mathsf{u}_d + v_{d+1} \mathsf{div}_{\mathbf{x}} \mathsf{u}_d + u_{d+1} \partial_t v_{d+1} + v_{d+1} \partial_t u_{d+1}\,.
\end{aligned}
$$

Since $\mathsf{v} \in \widetilde{W}$, v_d belongs to $\mathrm{L}^2(0,T;\mathrm{L}^2_{\mathrm{div}}(\Omega))$ so (by Theorem 10) its normal trace $\boldsymbol{\nu}_d \cdot \mathsf{v}_d$ on the boundary Γ_d belongs to $\mathrm{L}^2(0,T;\mathrm{H}^{-\frac{1}{2}}(\Gamma))$. Similarly, as $\mathsf{u} \in W$, we have that $u_{d+1} \in \mathrm{L}^2(0,T;\mathrm{H}^1(\Omega))$, so its trace on Γ belongs to $\mathrm{L}^2(0,T;\mathrm{H}^{\frac{1}{2}}(\Gamma))$. Using Green's first formula, almost everywhere on $\langle 0,T\rangle$ we have

$$
\int_\Omega (\nabla_{\mathbf{x}} u_{d+1} \cdot \mathsf{v}_d + u_{d+1} \mathsf{div}_{\mathbf{x}} \mathsf{v}_d)\, d\mathbf{x} = {}_{\mathrm{H}^{-\frac{1}{2}}(\Gamma)}\langle\, \boldsymbol{\nu}_d \cdot \mathsf{v}_d, u_{d+1}\,\rangle_{\mathrm{H}^{\frac{1}{2}}(\Gamma)}\,.
$$

By the generalised Hölder inequality for evolution spaces (see Appendix), both sides of the equality belong to $\mathrm{L}^1(\langle 0,T\rangle)$. Analogously,

$$
\int_\Omega (\nabla_{\mathbf{x}} v_{d+1} \cdot \mathsf{u}_d + v_{d+1} \mathsf{div}_{\mathbf{x}} \mathsf{u}_d)\, d\mathbf{x} = {}_{\mathrm{H}^{-\frac{1}{2}}(\Gamma)}\langle\, \boldsymbol{\nu}_d \cdot \mathsf{u}_d, v_{d+1}\,\rangle_{\mathrm{H}^{\frac{1}{2}}(\Gamma)} \in \mathrm{L}^1(\langle 0,T\rangle)\,.
$$

Finally, for $\mathsf{u}, \mathsf{v} \in \widetilde{W}$, we have $u_{d+1}\partial_t v_{d+1} + v_{d+1}\partial_t u_{d+1} = \partial_t(u_{d+1}v_{d+1}) \in \mathrm{L}^1(\Omega_T)$, so Green's first formula gives

$$
\int_0^T \int_\Omega (u_{d+1}\partial_t v_{d+1} + v_{d+1}\partial_t u_{d+1})\, d\mathbf{x}\, dt = \int_\Omega u_{d+1}(\cdot,T) v_{d+1}(\cdot,T)\, d\mathbf{x} - \int_\Omega u_{d+1}(\cdot,0) v_{d+1}(\cdot,0)\, d\mathbf{x}\,.
$$

These calculations lead to the following expression for D:

$$
\begin{aligned}
{}_{W'}\langle\, D\mathsf{u}, \mathsf{v}\,\rangle_W &= \int_0^T \Big({}_{\mathrm{H}^{-\frac{1}{2}}(\Gamma)}\langle\, \boldsymbol{\nu}_d \cdot \mathsf{v}_d, u_{d+1}\,\rangle_{\mathrm{H}^{\frac{1}{2}}(\Gamma)} + {}_{\mathrm{H}^{-\frac{1}{2}}(\Gamma)}\langle\, \boldsymbol{\nu}_d \cdot \mathsf{u}_d, v_{d+1}\,\rangle_{\mathrm{H}^{\frac{1}{2}}(\Gamma)} \Big)\, dt \\
&\quad + \int_\Omega u_{d+1}(\cdot,T) v_{d+1}(\cdot,T)\, d\mathbf{x} - \int_\Omega u_{d+1}(\cdot,0) v_{d+1}(\cdot,0)\, d\mathbf{x}\,, \quad \mathsf{u}, \mathsf{v} \in \widetilde{W}\,.
\end{aligned} \tag{6}
$$

If we additionally assume that $\mathsf{v}_d \in \mathrm{L}^2(0,T;\mathrm{H}^1(\Omega;\mathbf{R}^d))$, the normal trace $\boldsymbol{\nu}_d \cdot \mathsf{v}_d$ belongs to $\mathrm{L}^2(0,T;\mathrm{L}^2(\Gamma))$, so the above dual product can be replaced by an integral:

$$
{}_{\mathrm{H}^{-\frac12}(\Gamma)}\langle\, \boldsymbol{\nu}_d \cdot \mathsf{v}_d, u_{d+1}\,\rangle_{\mathrm{H}^{\frac12}(\Gamma)} = \int_\Gamma u_{d+1}\, \boldsymbol{\nu}_d \cdot \mathsf{v}_d \, dS_{\mathbf{x}} \,.
$$

Particularly, as $\mathrm{H}^1(\Omega_T;\mathbf{R}^{d+1})$ is dense in W [AB1], the linear operator $D \in \mathcal{L}(W;W')$ is uniquely determined by

$$
\begin{aligned}
{}_{W'}\langle\, D\mathsf{u},\mathsf{v}\,\rangle_W = &\int_0^T\!\!\int_\Gamma (u_{d+1}\,\boldsymbol{\nu}_d\cdot\mathsf{v}_d + v_{d+1}\,\boldsymbol{\nu}_d\cdot\mathsf{u}_d)\, dS_{\mathbf{x}}\,dt + \int_\Omega u_{d+1}(\cdot,T)v_{d+1}(\cdot,T)\,d\mathbf{x} \\
&- \int_\Omega u_{d+1}(\cdot,0)v_{d+1}(\cdot,0)\,d\mathbf{x}\,, \quad \mathsf{u},\mathsf{v}\in \mathrm{H}^1(\Omega_T;\mathbf{R}^{d+1})\,.
\end{aligned} \tag{7}
$$

Theorem 4. *Let* $\mathsf{u} \in W$, *and* $\mathsf{u}^n, \mathsf{v} \in \mathrm{H}^1(\Omega_T;\mathbf{R}^{d+1})$ *be such that the sequence* (u^n) *converges to* u *in* W. *Then*

$$
\begin{aligned}
{}_{W'}\langle\, D\mathsf{u},\mathsf{v}\,\rangle_W = &\lim_n \int_0^T\!\!\int_\Gamma v_{d+1}\,\boldsymbol{\nu}_d\cdot\mathsf{u}_d^n\, dS_{\mathbf{x}}\,dt + \int_0^T\!\!\int_\Gamma u_{d+1}\,\boldsymbol{\nu}_d\cdot\mathsf{v}_d\, dS_{\mathbf{x}}\,dt \\
&+ \int_\Omega u_{d+1}(\cdot,T)v_{d+1}(\cdot,T)\,d\mathbf{x} - \int_\Omega u_{d+1}(\cdot,0)v_{d+1}(\cdot,0)\,d\mathbf{x}\,.
\end{aligned} \tag{8}
$$

Dem. Using formula (7) and the continuity of $D: W \longrightarrow W'$ we get

$$
\begin{aligned}
{}_{W'}\langle\, D\mathsf{u},\mathsf{v}\,\rangle_W &= \lim_n {}_{W'}\langle\, D\mathsf{u}^n,\mathsf{v}\,\rangle_W \\
&= \lim_n \Big(\int_0^T\!\!\int_\Gamma v_{d+1}\,\boldsymbol{\nu}_d\cdot\mathsf{u}_d^n\, dS_{\mathbf{x}}\,dt + \int_0^T\!\!\int_\Gamma u_{d+1}^n\,\boldsymbol{\nu}_d\cdot\mathsf{v}_d\, dS_{\mathbf{x}}\,dt \\
&\qquad + \int_\Omega u_{d+1}^n(\cdot,T)v_{d+1}(\cdot,T)\,d\mathbf{x} - \int_\Omega u_{d+1}^n(\cdot,0)v_{d+1}(\cdot,0)\,d\mathbf{x} \Big)\,,
\end{aligned} \tag{9}
$$

and from (5_3) it follows that convergence $\mathsf{u}^n \longrightarrow \mathsf{u}$ in W is equivalent to the following three convergences

$$
\begin{aligned}
u_{d+1}^n &\longrightarrow u_{d+1} \ \text{ in } \mathrm{L}^2(0,T;\mathrm{H}^1(\Omega))\,, \\
\mathsf{u}_d^n &\longrightarrow \mathsf{u}_d \ \text{ in } \mathrm{L}^2(0,T;\mathrm{L}^2(\Omega;\mathbf{R}^d))\,, \\
\operatorname{div}\mathsf{u}^n &\longrightarrow \operatorname{div}\mathsf{u} \ \text{ in } \mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))\,.
\end{aligned}
$$

As the first step we calculate the limit of the last two integrals in (9). By Lemma 2, $u_{d+1}^n \longrightarrow u_{d+1}$ in $W(0,T)$, and, by the imbedding (Appendix, Theorem 12), also in $\mathrm{C}([0,T];\mathrm{L}^2(\Omega))$. In particular, $u_{d+1}^n(\cdot,0) \longrightarrow u_{d+1}(\cdot,0)$ and $u_{d+1}^n(\cdot,T) \longrightarrow u_{d+1}(\cdot,T)$ in $\mathrm{L}^2(\Omega)$, and therefore

$$
\begin{aligned}
\int_\Omega u_{d+1}^n(\cdot,T)v_{d+1}(\cdot,T)\,d\mathbf{x} &\longrightarrow \int_\Omega u_{d+1}(\cdot,T)v_{d+1}(\cdot,T)\,d\mathbf{x}\,, \\
\int_\Omega u_{d+1}^n(\cdot,0)v_{d+1}(\cdot,0)\,d\mathbf{x} &\longrightarrow \int_\Omega u_{d+1}(\cdot,0)v_{d+1}(\cdot,0)\,d\mathbf{x}\,.
\end{aligned}
$$

The convergence $u_{d+1}^n \longrightarrow u_{d+1}$ in $\mathrm{L}^2(0,T;\mathrm{H}^1(\Omega))$ implies the convergence of traces on Γ, by Theorem 10: $u_{d+1}^n \longrightarrow u_{d+1}$ in $\mathrm{L}^2(0,T;\mathrm{H}^{\frac12}(\Gamma))$. Since the normal trace of v_d on Γ_d belongs to $\mathrm{L}^2(\Gamma_d)$, it follows

$$
\lim_n \int_0^T\!\!\int_\Gamma u_{d+1}^n\,\boldsymbol{\nu}_d\cdot\mathsf{v}_d\, dS_{\mathbf{x}}\,dt = \int_0^T\!\!\int_\Gamma u_{d+1}\,\boldsymbol{\nu}_d\cdot\mathsf{v}_d\, dS_{\mathbf{x}}\,dt\,,
$$

which proves the theorem.

Q.E.D.

4. Dirichlet boundary conditions

Let us now investigate the possibility of representing the initial boundary value problem for the heat equation

$$(10)\qquad \begin{cases} \partial_t u - \mathsf{div}_{\mathbf{x}}(\mathbf{A}\nabla_{\mathbf{x}} u) + \mathbf{b}\cdot\nabla_{\mathbf{x}} u + cu = f \text{ in } \Omega_T \\ \qquad\qquad\qquad\qquad\qquad\qquad u = 0 \text{ on } \Gamma_0\cup\Gamma_d \end{cases}$$

as a boundary value problem for the Friedrichs system.

Remark. If one wants to treat nonhomogeneous initial and boundary conditions, their homogenisation needs to be done first; this can be done for sufficiently regular initial and boundary conditions [W, Section 28]. ∎

We shall use the following approach: first we *guess* a matrix field $\mathbf{M}$ satisfying (FM), and then try to prove that this matrix field defines an operator $M\in\mathcal{L}(W;W')$ satisfying (M) via formula (2).

If we choose

$$\mathbf{M} = \begin{bmatrix} \mathbf{0} & -\boldsymbol{\nu}_d \\ \boldsymbol{\nu}_d^\top & a\nu_t \end{bmatrix},$$

where a is a scalar function defined on $\partial\Omega_T$, such that $a_{|_{\Gamma_0}} = -1$ and $a_{|_{\Gamma_T}} = 1$, then

$$\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M} = \begin{bmatrix} \mathbf{0} & 2\boldsymbol{\nu}_d \\ \mathsf{0}^\top & (1-a)\nu_t \end{bmatrix},$$

which imposes the Dirichlet boundary condition for the heat equation (with homogeneous initial condition)

$$u_{|_{\Gamma_0\cup\Gamma_d}} = 0\,.$$

The nonnegativity condition (FM1) then reduces to

$$(\forall\boldsymbol{\xi}\in\mathbf{R}^{d+1})\quad a\nu_t\xi_{d+1}^2 \geqslant 0\,,$$

which is satisfied with the above a. One can easily check that we have $\ker(\mathbf{A}_{\boldsymbol{\nu}}+\mathbf{M}) = \mathbf{R}^{d+1}$ on Γ_0 , $\ker(\mathbf{A}_{\boldsymbol{\nu}}-\mathbf{M}) = \mathbf{R}^{d+1}$ on Γ_T , while on Γ_d we have $\ker(\mathbf{A}_{\boldsymbol{\nu}}-\mathbf{M}) = \mathbf{R}^d\times\{0\}$ and $\ker(\mathbf{A}_{\boldsymbol{\nu}}+\mathbf{M})$ is described by $\boldsymbol{\nu}_d\cdot\boldsymbol{\xi}_d = 0$, for $\boldsymbol{\xi} = (\boldsymbol{\xi}_d, \xi_{d+1})^\top\in\mathbf{R}^{d+1}$. It follows that (FM2) holds as well.

The corresponding operator M given by (2) then takes the form

$$(11)\qquad \begin{aligned} {}_{W'}\langle\, M\mathsf{u},\mathsf{v}\,\rangle_W = &\int_0^T\!\!\int_\Gamma (-u_{d+1}\,\boldsymbol{\nu}_d\cdot\mathsf{v}_d + v_{d+1}\,\boldsymbol{\nu}_d\cdot\mathsf{u}_d)\,dS_{\mathbf{x}}\,dt \\ &+\int_\Omega u_{d+1}(\cdot,T)v_{d+1}(\cdot,T)\,d\mathbf{x} + \int_\Omega u_{d+1}(\cdot,0)v_{d+1}(\cdot,0)\,d\mathbf{x}\,, \end{aligned}$$

for any $\mathsf{u},\mathsf{v}\in\mathrm{H}^1(\Omega_T;\mathbf{R}^{d+1})$. However, we do not know whether the above formula defines a continuous operator $M: W\longrightarrow W'$. If this is the case, then

$$(12)\qquad {}_{W'}\langle\,(D-M)\mathsf{u},\mathsf{v}\,\rangle_W = 2\int_0^T\!\!\int_\Gamma u_{d+1}\,\boldsymbol{\nu}_d\cdot\mathsf{v}_d\,dS_{\mathbf{x}}\,dt - 2\int_\Omega u_{d+1}(\cdot,0)v_{d+1}(\cdot,0)\,d\mathbf{x}\,,$$

and, proceeding as in the proof of Theorem 4, one can easily see that this formula would be valid for an arbitrary $\mathsf{u}\in W$. The following lemma suggests that $\ker(D-M)$ should consist of those functions in the graph space whose last components satisfy $u_{d+1}{}_{|_{\Gamma_0\cup\Gamma_d}} = 0$.

Lemma 3. *If formula (11) defines an operator $M \in \mathcal{L}(W;W')$ satisfying (M), then*

$$\ker(D-M) = \Big\{ \mathsf{u} \in W : u_{d+1} \in \mathrm{L}^2(0,T;\mathrm{H}^1_0(\Omega)), \quad u_{d+1}(\cdot,0) = 0 \text{ a.e. } on\ \Omega \Big\}.$$

Dem. By using the density argument, one proves that $(D-M)\mathsf{u} = \mathsf{0}$ if and only if ${}_{W'}\langle\, (D-M)\mathsf{u}, \mathsf{v}\,\rangle_W = 0$, for every $\mathsf{v} \in \mathrm{H}^1(\Omega_T;\mathbf{R}^{d+1})$. From (12) it trivially follows that the right hand side of the above identity is included in $\ker(D-M)$.

In order to prove the converse inclusion, let us take $\mathsf{u} \in \ker(D-M)$ and $\mathsf{v} = (\mathsf{0}, v_{d+1})^\top \in \mathrm{H}^1(\Omega_T;\mathbf{R}^{d+1})$ such that $v_{d+1}(\mathbf{x},t) = g(t)h(\mathbf{x})$ with $g \in \mathrm{C}^\infty([0,T])$, $g(0) = 1$ and $h \in \mathrm{H}^1(\Omega)$. Then by (12)

$$0 = {}_{W'}\langle\, (D-M)\mathsf{u}, \mathsf{v}\,\rangle_W = -2\int_\Omega u_{d+1}(\cdot,0)v_{d+1}(\cdot,0)\,d\mathbf{x} = -2\int_\Omega u_{d+1}(\cdot,0)h\,d\mathbf{x}\,,$$

and therefore $u_{d+1}(\cdot,0)$ vanishes by the arbitrariness of h.

If we now take an arbitrary $\varphi \in \mathrm{L}^2(\Gamma_d)$ and a sequence (v^n_d) in $\mathrm{H}^1(\Omega_T;\mathbf{R}^d)$ such that the corresponding sequence of traces on Γ_d converges to $\varphi\boldsymbol{\nu}_d$ in $\mathrm{L}^2(\Gamma_d;\mathbf{R}^d)$, then for $\mathsf{v}^n = (\mathsf{v}^n_d, 0)^\top \in \mathrm{H}^1(\Omega_T;\mathbf{R}^{d+1})$ we have

$$\begin{aligned} 0 = {}_{W'}\langle\, (D-M)\mathsf{u}, \mathsf{v}^n\,\rangle_W &= \lim_n {}_{W'}\langle\, (D-M)\mathsf{u}, \mathsf{v}^n\,\rangle_W \\ &= 2\lim_n \int_0^T\!\!\int_\Gamma u_{d+1}\,\boldsymbol{\nu}_d\cdot \mathsf{v}^n_d\,dS_{\mathbf{x}}\,dt \;= 2\int_0^T\!\!\int_\Gamma u_{d+1}\,\varphi\,dS_{\mathbf{x}}\,dt\,. \end{aligned}$$

As φ is arbitrary, it follows that $u_{d+1} = 0$ in $\mathrm{L}^2(\Gamma_d)$, and thus $u_{d+1} \in \mathrm{L}^2(0,T;\mathrm{H}^1_0(\Omega))$.

Q.E.D.

The result of the above lemma is in agreement with the Dirichlet boundary condition which we want to impose; however, we are still not able to prove that M is continuous. Therefore, we change our approach, and try to use the intrinsic conditions of Theorem 1.

We start with the subspace V of W, imposing the required initial and boundary conditions

$$V = \Big\{ \mathsf{u} \in W : u_{d+1} \in \mathrm{L}^2(0,T;\mathrm{H}^1_0(\Omega)), \quad u_{d+1}(\cdot,0) = 0 \text{ a.e. on } \Omega \Big\},$$

and

$$\widetilde{V} = \Big\{ \mathsf{v} \in W : v_{d+1} \in \mathrm{L}^2(0,T;\mathrm{H}^1_0(\Omega)), \quad v_{d+1}(\cdot,T) = 0 \text{ a.e. on } \Omega \Big\}.$$

For the application of Theorem 1, one should calculate annihilators of $D(V)$ and $D(\widetilde{V})$, where the annihilator G^0 of set $G \subseteq W'$ is defined by

$$G^0 = \Big\{ \mathsf{v} \in W : (\forall\, g \in G) \quad {}_{W'}\langle\, g, \mathsf{v}\,\rangle_W = 0 \Big\}.$$

Let us first prove one technical result. The density of space $\mathrm{H}^1(\Omega_T;\mathbf{R}^{d+1})$ in the graph space W of any first order partial differential operator holds in general, but for our application we have to additionally consider the homogeneous Dirichlet boundary condition for the last component.

Lemma 4. *The space*

$$S_b = \Big\{ \mathsf{u} \in \mathrm{H}^1(\Omega_T;\mathbf{R}^{d+1}) : u_{d+1} = 0 \ on\ \Gamma_d \Big\}$$

is dense in

$$V_b = \Big\{ \mathsf{u} \in W : u_{d+1} \in \mathrm{L}^2(0,T;\mathrm{H}^1_0(\Omega)) \Big\},$$

in the norm of the graph space W.

Dem. Using the partition of unity, for given $0 < \varepsilon < \frac{T}{2}$ one can introduce functions $\varphi_1, \varphi_2 \in \mathrm{C}^\infty([0,T])$ such that $\varphi_1 + \varphi_2 = 1$ with $\varphi_1 = 0$ on $[0,\varepsilon]$ and $\varphi_2 = 0$ on $[T-\varepsilon, T]$. A given function $\mathsf{u} \in V_b$ can be represented by the sum $\mathsf{u} = \varphi_1 \mathsf{u} + \varphi_2 \mathsf{u}$ of functions in W, vanishing in a neighbourhood of $t = 0$ and $t = T$, respectively. Therefore, for the proof of the lemma it suffices to approximate these two terms separately by sequences in S_b. We shall consider only the former, as the latter can be treated analogously.

Let us take a function $\mathsf{u} \in W$ which vanishes in a neighbourhood of $t = 0$. For $n \in \mathbf{N}$ by τ_n we shall denote the translation operator in time variable: $\tau_n \mathsf{u}(\mathbf{x}, t) := \mathsf{u}(\mathbf{x}, t - \frac{1}{n})$. It is a classical result that, as $n \longrightarrow \infty$, we have L^2 convergence on $\mathbf{R}^{d+1}$ of sequences $\tau_n \mathsf{u}$, $\tau_n \mathsf{div}\, \mathsf{u}$ and $\tau_n \nabla_{\mathbf{x}} u_{d+1}$ towards u, $\mathsf{div}\, \mathsf{u}$ and $\nabla_{\mathbf{x}} u_{d+1}$, respectively. More precisely, here we consider the extension of the original functions to whole $\mathbf{R}^{d+1}$ by zero (as L^2 functions).

Let the mollifying sequence in time variable be denoted by $\rho_m(t) = m\rho(mt)$, where ρ is a nonnegative infinitely differentiable function with support in $\langle -1, 1 \rangle$ and integral equal to 1. Now we follow the Cantor diagonal procedure: for given n we choose $m(n) \geqslant n$ sufficiently large such that (recall that the convolution is taken only in the time variable)

$$
\begin{aligned}
\|\rho_{m(n)} * \tau_n \mathsf{u} - \tau_n \mathsf{u}\|_{\mathrm{L}^2(\mathbf{R}^{d+1})} &< \frac{1}{n}\,, \\
\|\rho_{m(n)} * \tau_n \mathsf{div}\, \mathsf{u} - \tau_n \mathsf{div}\, \mathsf{u}\|_{\mathrm{L}^2(\mathbf{R}^{d+1})} &< \frac{1}{n}\,, \\
\|\rho_{m(n)} * \tau_n \nabla_{\mathbf{x}} \mathsf{u} - \tau_n \nabla_{\mathbf{x}} \mathsf{u}\|_{\mathrm{L}^2(\mathbf{R}^{d+1})} &< \frac{1}{n}\,.
\end{aligned}
$$

Therefore, by the triangle inequality we conclude that the sequence $\mathsf{v}^n := \rho_{m(n)} * \tau_n \mathsf{u}$ approximates u in the norm of the graph space W. For example, we have

$$
\begin{aligned}
\|\mathsf{div}\, \big(\rho_{m(n)} * \tau_n \mathsf{u}\big) - \mathsf{div}\, \mathsf{u}\|_{\mathrm{L}^2(\Omega_T)} &\leqslant \|\mathsf{div}\, \big(\rho_{m(n)} * \tau_n \mathsf{u}\big) - \tau_n \mathsf{div}\, \mathsf{u}\|_{\mathrm{L}^2(\Omega_T)} + \|\tau_n \mathsf{div}\, \mathsf{u} - \mathsf{div}\, \mathsf{u}\|_{\mathrm{L}^2(\Omega_T)} \\
&= \|\rho_{m(n)} * \tau_n \mathsf{div}\, \mathsf{u} - \tau_n \mathsf{div}\, \mathsf{u}\|_{\mathrm{L}^2(\Omega_T)} + \|\tau_n \mathsf{div}\, \mathsf{u} - \mathsf{div}\, \mathsf{u}\|_{\mathrm{L}^2(\Omega_T)}\,,
\end{aligned}
$$

which tends to zero by construction.

The last component v^n_{d+1} has the desired properties, so a simple adjustment is needed only for v^n_d, as it does not belong to $\mathrm{H}^1(\Omega_T; \mathbf{R}^d)$. As for this part we have no boundary condition, we can use the classical approximation by a sequence of smooth functions approximating v^n_d and $\mathsf{div}_{\mathbf{x}} \mathsf{v}^n_d$ in L^2 norm (notice that the former belongs to $\mathrm{L}^2(\Omega_T)$, since both $\mathsf{div}\, \mathsf{v}^n$ and $\partial_t v^n_{d+1}$ do) and again use the Cantor diagonal procedure.

Q.E.D.

The following theorem assures that conditions (V) are satisfied by the above choice of V and $\widetilde{V}$.

Theorem 5. *The following statements hold:*

$$
\begin{aligned}
&\widetilde{V} = D(V)^0 \quad \textit{and} \quad V = D(\widetilde{V})^0\,, \\
&(\forall\, \mathsf{u} \in V) \quad {}_{W'}\langle\, D\mathsf{u}, \mathsf{u}\,\rangle_W \geqslant 0\,, \\
&(\forall\, \mathsf{v} \in \widetilde{V}) \quad {}_{W'}\langle\, D\mathsf{v}, \mathsf{v}\,\rangle_W \leqslant 0\,.
\end{aligned}
$$

Dem. For the first statement, let us first take arbitrary $\mathsf{v} \in \widetilde{V}$ and $\mathsf{u} \in V$ and denote by v^n and u^n the corresponding approximating sequences in S_b obtained by Lemma 4. Using the continuity of D and formula (7) we obtain

$$
\begin{aligned}
{}_{W'}\langle\, D\mathsf{u}, \mathsf{v}\,\rangle_W &= \lim_n {}_{W'}\langle\, D\mathsf{u}^n, \mathsf{v}^n\,\rangle_W \\
&= \lim_n \Big(\int_0^T \int_\Gamma \big(u^n_{d+1}\, \boldsymbol{\nu}_d \cdot \mathsf{v}^n_d + v^n_{d+1}\, \boldsymbol{\nu}_d \cdot \mathsf{u}^n_d \, dS_{\mathbf{x}} \big)\, dt \\
&\qquad + \int_\Omega u^n_{d+1}(\cdot, T) v^n_{d+1}(\cdot, T)\, d\mathbf{x} - \int_\Omega u^n_{d+1}(\cdot, 0) v^n_{d+1}(\cdot, 0)\, d\mathbf{x} \Big)\,.
\end{aligned}
$$

Since u^n_{d+1} and v^n_{d+1} are zero on Γ_d, the first two integrals in the last expression vanish. In the last two integrals we can pass to the limit since the convergence in W implies the convergence of $u^n_{d+1}(\cdot,0)$ and $v^n_{d+1}(\cdot,T)$ to $u_{d+1}(\cdot,0)=v_{d+1}(\cdot,T)=0$ in $\mathrm{L}^2(\Omega)$. Therefore, ${}_{W'}\langle D\mathsf{u},\mathsf{v}\rangle_W=0$, which implies $\mathsf{v}\in D(V)^0$, as $\mathsf{u}\in V$ is arbitrary.

Conversely, for some $\mathsf{v}\in D(V)^0$, let $(\mathsf{v}^n)\in \mathrm{H}^1(\Omega_T;\mathbf{R}^{d+1})$ be a sequence approximating v in the W norm. As the first step, let us take $\mathsf{u}=(\mathsf{u}_d,u_{d+1})\in V\cap \mathrm{H}^1(\Omega_T;\mathbf{R}^{d+1})$ such that $\mathsf{u}_d=\mathsf{0}$ and $u_{d+1}(t,\mathbf{x})=g(t)h(\mathbf{x})$ with $g\in \mathrm{C}^\infty_c(\langle 0,T])$, $g(T)=1$ and $h\in\mathrm{H}^1_0(\Omega)$. Then by the symmetry of D and Theorem 4 we have

$$
{}_{W'}\langle D\mathsf{u},\mathsf{v}\rangle_W={}_{W'}\langle D\mathsf{v},\mathsf{u}\rangle_W=\int_\Omega u_{d+1}(\cdot,T)v_{d+1}(\cdot,T)\,d\mathbf{x}=\int_\Omega hv_{d+1}(\cdot,T)\,d\mathbf{x}\,,
$$

which is zero by assumption $\mathsf{v}\in D(V)^0$. As h is arbitrary, it follows $v_{d+1}(\cdot,T)=0$.

To conclude the proof, we take arbitrary $\varphi\in\mathrm{L}^2(\Gamma)$ and a sequence (h_k) in $\mathrm{H}^1(\Omega;\mathbf{R}^d)$ such that the corresponding sequence of traces to Γ converges to $\varphi\boldsymbol{\nu}_d$ in $\mathrm{L}^2(\Gamma;\mathbf{R}^d)$. Finally, given any function $\psi\in\mathrm{C}^\infty_c(\langle 0,T\rangle)$, we take a sequence (of tensor products of a function in t and a function in $\mathbf{x}$) $\mathsf{u}^k=(\psi\mathsf{h}_k,0)^\top\in V\cap\mathrm{H}^1(\Omega;\mathbf{R}^{d+1})$, implying $0={}_{W'}\langle D\mathsf{u}^k,\mathsf{v}\rangle_W$, for any $k\in\mathbf{N}$. Similarly as above, using the symmetry of D and Theorem 4 we conclude that

$$
0={}_{W'}\langle D\mathsf{u}^k,\mathsf{v}\rangle_W=\int_0^T\int_\Gamma v_{d+1}\,\boldsymbol{\nu}_d\cdot\mathsf{u}^k_d\,dS_{\mathbf{x}}dt=\langle\, v_{d+1}\mid\boldsymbol{\nu}_d\cdot\mathsf{u}^k_d\,\rangle_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Gamma))}\,. \tag{13}
$$

By the Lebesgue dominated convergence theorem we have $\mathsf{u}^k_d\longrightarrow\psi\varphi\boldsymbol{\nu}_d$ in $\mathrm{L}^2(0,T;\mathrm{L}^2(\Gamma;\mathbf{R}^d))$. More precisely, we use

$$
\begin{aligned}
&\|\mathsf{u}^k_d(t,\cdot)\|_{\mathrm{L}^2(\Gamma;\mathbf{R}^d)}\longrightarrow\|\psi(t)\varphi\boldsymbol{\nu}_d\|_{\mathrm{L}^2(\Gamma;\mathbf{R}^d)}\,,\ t\in\langle 0,T\rangle\,,\\
&\|\mathsf{u}^k_d(t,\cdot)\|_{\mathrm{L}^2(\Gamma;\mathbf{R}^d)}\leqslant 2\psi(t)\|\varphi\|_{\mathrm{L}^2(\Gamma)}\in\mathrm{L}^2(\langle 0,T\rangle)\,.
\end{aligned}
$$

Therefore, from (13) we conclude

$$
\langle\, v_{d+1}\mid\psi\varphi\,\rangle_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Gamma))}=0\,,
$$

and by the density of tensor products in $\mathrm{L}^2(0,T;\mathrm{L}^2(\Gamma))$, we have $v_{d+1}=0$ a.e. on Γ_d, implying that $v_{d+1}\in\mathrm{L}^2(0,T;\mathrm{H}^1_0(\Omega))$.

The fact that $V=D(\widetilde{V})^0$ can be proved analogously.

The last two statements follow from Lemma 4. Namely, for the third one let us take $\mathsf{u}\in V$ and its approximating sequence $(\mathsf{u}^n)\subseteq S_b$ given by Lemma 4. Using the same reasoning as in the first part of this proof, we obtain

$$
{}_{W'}\langle D\mathsf{u},\mathsf{u}\rangle_W=\int_\Omega u_{d+1}(\cdot,T)u_{d+1}(\cdot,T)\,d\mathbf{x}\geqslant 0\,.
$$

Q.E.D.

Corollary 1. *The operator $\mathcal{L}$, given by (4), defines an isomorphism from V to* $\mathrm{L}^2(\Omega_T;\mathbf{R}^{d+1})$.

∎

Two-field theory with partial coercivity

One of the assumptions of the above corollary is the positivity condition (F2), which, for our system, reads $c-\frac14\mathbf{A}^{-1}\mathbf{b}\cdot\mathbf{b}\geqslant\gamma>0$ on Ω_T, for some constant γ. In particular, coefficient c in the heat equation should be uniformly positive on Ω_T, and thus the result of Corollary 1 does not apply if $c=0$. A similar situation occurs in the treatment of elliptic equations, such as the stationary diffusion equation. In order to overcome this problem for elliptic equations, in [EG2, EG3] the authors proposed the two-field theory of Friedrichs systems with partial coercivity, which enabled them to consider the case $c = 0$ as well.

We shall now show that their method can also be applied to the parabolic equation, namely the following initial boundary value problem (for simplicity here we shall consider only the case $\mathsf{b} = 0$)

$$\begin{cases} \partial_t u - \mathsf{div}_{\mathbf{x}}(\mathbf{A}\nabla_{\mathbf{x}} u) = f \text{ in } \Omega_T \\ \qquad\qquad u = 0 \text{ on } \Gamma_0 \cup \Gamma_d \,. \end{cases}$$

To be specific, in order to apply the two fields theory with partial coercivity, our coefficient matrix fields need to be of the following form

$$\mathbf{A}^k = \begin{bmatrix} \mathbf{0} & \mathsf{B}^k \\ (\mathsf{B}^k)^\top & a^k \end{bmatrix} \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} \mathbf{C}^d & 0 \\ 0^\top & c^{d+1} \end{bmatrix} ,$$

where $\mathsf{B}^k \in \mathbf{R}^d$ are constant vectors, $a^k \in \mathrm{W}^{1,\infty}(\Omega_T)$, $\mathbf{C}^d \in \mathrm{L}^\infty((\Omega_T; \mathrm{M}_d(\mathbf{R})))$ and $c^{d+1} \in \mathrm{L}^\infty(\Omega_T)$, $k \in 1..(d+1)$.

In our particular case (with $\mathsf{b} = 0$ and $c = 0$) we have $\mathsf{B}^k = \mathsf{e}_k$, $k \in 1..d$, $\mathsf{B}^{d+1} = 0$, $a^k = 0$ for $k \in 1..(d+1)$, $\mathbf{C}^d = \mathbf{A}^{-1}$ and $c^{d+1} = 0$.

To obtain the well-posedness result, one needs to verify that the following two conditions of Theorem 3.1 in [EG3] hold:

$$(\exists\, \mu_1 > 0)(\forall\, \boldsymbol{\xi} = (\boldsymbol{\xi}_d, \xi_{d+1}) \in \mathbf{R}^{d+1}) \quad \Big(\mathbf{C} + \mathbf{C}^\top + \sum_{k=1}^{d+1} \partial_k \mathbf{A}_k\Big)\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geqslant 2\mu_1 |\boldsymbol{\xi}_d|^2 \qquad \text{(a.e. on } \Omega) \,,$$

$$(\exists\, \mu_2 > 0)(\forall\, \mathsf{u} \in V \cup \widetilde{V}) \quad \sqrt{\langle\, \mathcal{L}\mathsf{u} \mid \mathsf{u}\,\rangle_{\mathrm{L}^2(\Omega_T;\mathbf{R}^{d+1})}} + \|\mathsf{B}u_{d+1}\|_{\mathrm{L}^2(\Omega_T;\mathbf{R}^d)} \geqslant \mu_2 \|u_{d+1}\|_{\mathrm{L}^2(\Omega_T)} \,,$$

where $\mathsf{B}u_{d+1} := \sum_{k=1}^{d+1} \mathsf{B}^k \partial_k u_{d+1} = \nabla_{\mathbf{x}} u_{d+1}$. It turns out that for our system both conditions are trivially fulfilled (the spaces V and $\widetilde{V}$ are the same as before): the former condition reduces to $\mathbf{A} \geqslant \mu_1^{-1}\mathbf{I}$ almost everywhere on Ω_T, while the latter is a simple consequence of the Poincaré inequality (it suffices to consider only the term $\|\mathsf{B}u_{d+1}\|_{\mathrm{L}^2(\Omega_T;\mathbf{R}^d)}$ on the left-hand side). Therefore, we have the well-posedness of the initial boundary value problem (10) even if c vanish:

Theorem 6. *The restriction of operator*

$$\mathcal{L} \begin{bmatrix} \mathsf{u}_d \\ u_{d+1} \end{bmatrix} = \begin{bmatrix} \nabla_{\mathbf{x}} u_{d+1} + \mathbf{A}^{-1}\mathsf{u}_d \\ \partial_t u_{d+1} + \mathsf{div}_{\mathbf{x}} \mathsf{u}_d \end{bmatrix}$$

to

$$V = \Big\{ \mathsf{u} = (\mathsf{u}_d, u_{d+1}) \in W : u_{d+1} \in \mathrm{L}^2(0,T; \mathrm{H}^1_0(\Omega)), \quad u_{d+1}(\cdot, 0) = 0 \text{ a.e. on } \Omega \Big\}$$

is an isomorphism from V *to* $\mathrm{L}^2(\Omega_T; \mathbf{R}^{d+1})$.

∎

5. Boundary operator M

In the previous section we have seen that subspaces V and $\widetilde{V}$ satisfy conditions (V), and thus we have the well-posedness result. We also know [AB2] that these conditions are equivalent to (M), in the sense that there exists at least one operator $M \in \mathcal{L}(W; W')$ satisfying (M), such that $V = \ker(D - M)$.

In this section we want to check whether one of these operators M satisfies (11). For given V and $\widetilde{V}$ satisfying (V), we are aware of two possible ways for construction of the corresponding M : the first one is given in [EGC, Theorem 4.3] and [AB2, Theorems 6 and 7], while the second is described in [AB2, Theorem 8].

Let us remark that the first construction can be performed only if $V + \widetilde{V}$ is closed in the graph space W. We recall results from [EGC, AB2] in the following theorem.

Theorem 7. *Let V and $\widetilde{V}$ be two subspaces of W satisfying (V).*

a) *Let us assume that there exist operators $P, Q \in \mathcal{L}(W)$ such that*

$$
\begin{aligned}
P^2 = P \quad &\text{and} \quad Q^2 = Q\,,\\
\operatorname{im} P = V \quad &\text{and} \quad \operatorname{im} Q = \widetilde{V}\,,\\
PQ &= QP\,.
\end{aligned}
\tag{14}
$$

If we define $M \in \mathcal{L}(W; W')$ by

$$
\begin{aligned}
{}_{W'}\langle M\mathsf{u}, \mathsf{v}\rangle_W &= {}_{W'}\langle DP\mathsf{u}, P\mathsf{v}\rangle_W - {}_{W'}\langle DQ\mathsf{u}, Q\mathsf{v}\rangle_W\\
&\quad + {}_{W'}\langle D(P+Q-PQ)\mathsf{u}, \mathsf{v}\rangle_W - {}_{W'}\langle D\mathsf{u}, (P+Q-PQ)\mathsf{v}\rangle_W\,,
\end{aligned}
$$

for $\mathsf{u}, \mathsf{v} \in W$, then $V = \ker(D - M)$, $\widetilde{V} = \ker(D + M^*)$, and M satisfies (M).

b) *Operators P and Q from (a) exist if and only if $V + \widetilde{V}$ is closed in W. In this case, P and Q can be constructed in the following way: let $V_0 := V \cap \widetilde{V}$, and let us denote by V_1 the $\langle\cdot \mid \cdot\rangle_{\mathcal{L}}$–orthogonal complement of V_0 in V, by V_2 the $\langle\cdot \mid \cdot\rangle_{\mathcal{L}}$–orthogonal complement of V_0 in $\widetilde{V}$, and by V_3 the $\langle\cdot \mid \cdot\rangle_{\mathcal{L}}$–orthogonal complement of $V + \widetilde{V}$ in W, so that W is the direct sum of V_i, $i \in 0..3$. If $\mathsf{w} = \mathsf{w}^0 + \mathsf{w}^1 + \mathsf{w}^2 + \mathsf{w}^3$ is a decomposition of an arbitrary $\mathsf{w} \in W$ corresponding to this direct sum, then P and Q can be defined by*

$$
P\mathsf{w} := \mathsf{w}^0 + \mathsf{w}^1 \quad \text{and} \quad Q\mathsf{w} := \mathsf{w}^0 + \mathsf{w}^2\,.
$$

∎

In order to apply the above theorem, we need the following lemma.

Lemma 5. *If V and $\widetilde{V}$ are as defined in section 4, then $V + \widetilde{V}$ is closed in W.*

Dem. It is obvious that

$$
V + \widetilde{V} = \{\mathsf{u} \in W : u_{d+1} \in \mathrm{L}^2(0, T; \mathrm{H}^1_0(\Omega))\}\,.
$$

The closedness of this space in W now easily follows from (5) and the closedness of $\mathrm{L}^2(0, T; \mathrm{H}^1_0(\Omega))$ in $\mathrm{L}^2(0, T; \mathrm{H}^1(\Omega))$.

Q.E.D.

For P and Q defined as in Theorem 7(b), we have

$$
(P + Q - PQ)\mathsf{w} = \mathsf{w}^0 + \mathsf{w}^1 + \mathsf{w}^2\,,
$$

which together with (V2) and the symmetry of D gives the following formula for M

$$
\begin{aligned}
{}_{W'}\langle M\mathsf{u}, \mathsf{v}\rangle_W &= {}_{W'}\langle D\mathsf{u}^1, \mathsf{v}^1\rangle_W - {}_{W'}\langle D\mathsf{u}^2, \mathsf{v}^2\rangle_W\\
&\quad + {}_{W'}\langle D(\mathsf{u}^0 + \mathsf{u}^1 + \mathsf{u}^2), \mathsf{v}^3\rangle_W - {}_{W'}\langle D\mathsf{u}^3, \mathsf{v}^0 + \mathsf{v}^1 + \mathsf{v}^2\rangle_W\,,
\end{aligned}
\tag{15}
$$

where $\mathsf{u} = \mathsf{u}^0 + \mathsf{u}^1 + \mathsf{u}^2 + \mathsf{u}^3$ and $\mathsf{v} = \mathsf{v}^0 + \mathsf{v}^1 + \mathsf{v}^2 + \mathsf{v}^3$ are decompositions of arbitrary $\mathsf{u}, \mathsf{v} \in W$ as in Theorem 7(b). Unfortunately, up to now we still do not know whether this M satisfies (11) even for smooth functions u and v. One can easily check that if the above M satisfies (11), and if every $\mathsf{u}^i, \mathsf{v}^i$ is smooth, $i \in 0..3$, then one has the equality

$$
\begin{aligned}
\int_0^T \int_\Gamma \left(-u^3_{d+1}\,\boldsymbol{\nu}_d \cdot \mathsf{v}^3_d + v^3_{d+1}\,\boldsymbol{\nu}_d \cdot \mathsf{u}^3_d\right) dS_{\mathbf{x}}\, dt + \int_\Omega u^3_{d+1}(\cdot, T)\Big(2v^1_{d+1}(\cdot, T) + v^3_{d+1}(\cdot, T)\Big)\, d\mathbf{x}&\\
+ \int_\Omega v^3_{d+1}(\cdot, 0)\Big(2u^2_{d+1}(\cdot, 0) + u^3_{d+1}(\cdot, 0)\Big)\, d\mathbf{x} = 0\,.&
\end{aligned}
$$

Since we have no argument for this formula to be valid, we suspect that operator given by (15) does not satisfy (11).

Next we turn our attention to the second construction of operator M. First, let us recall Theorem 8 from [AB2].

Theorem 8.

a) *If V and $\widetilde{V}$ are two subspaces of W that satisfy (V) and if there exists a closed subspace $W_2 \subseteq C^-$ of W, such that $V \dot{+} W_2 = W$, then there exists an operator $M \in \mathcal{L}(W; W')$ satisfying (M) and $V = \ker(D - M)$. If we define W_1 as $\langle \cdot \mid \cdot \rangle_{\mathcal{L}}$-orthogonal complement of W_0 in V, so that $W = W_1 \dot{+} W_0 \dot{+} W_2$, and denote by R_1, R_0, R_2 projectors corresponding to this direct sum, then one such operator is given by $M = D(R_1 - R_2)$.*

b) *Let $M \in \mathcal{L}(W; W')$ be an operator satisfying (M), and denote $V = \ker(D - M)$. If we denote by W_2 the orthogonal complement of W_0 in $\ker(D + M)$, then $W_2 \subseteq C^-$ is closed, $V \dot{+} W_2 = W$, and M coincides with the operator constructed in (a).* ∎

In the theorem above C^- stands for the set of all nonpositive vectors in W with respect to D, i. e.

$$C^- := \{\mathsf{u} \in W : {}_{W'}\langle D\mathsf{u}, \mathsf{u} \rangle_W \leqslant 0\} .$$

Remark. By Lemma 12 in [AB2], if $W_2'' \subseteq C^-$ is a subspace of W such that $V + W_2'' = W$, then there exists a closed subspace $W_2 \subseteq W_2''$ of W, such that $W_2 \subseteq C^-$ and $V \dot{+} W_2 = W$. ∎

Remark. In Theorem 8(a) the space W_1 can be any closed subspace of V, such that $W = W_1 \dot{+} W_0 \dot{+} W_2$, and not necessarily the $\langle \cdot \mid \cdot \rangle_{\mathcal{L}}$-orthogonal complement of W_0 in V. The similar statement holds for W_2 in Theorem 8(b): W_2 can be any closed subspace of $\ker(D + M)$, such that $V \dot{+} W_2 = W$. ∎

Let us for the moment assume that formula (11) defines an operator $M \in \mathcal{L}(W; W')$ satisfying (M). Then for any $\mathsf{u}, \mathsf{v} \in \mathrm{H}^1(\Omega_T; \mathbf{R}^{d+1})$ we have

$${}_{W'}\langle (D + M)\mathsf{u}, \mathsf{v} \rangle_W = 2 \int_0^T \int_\Gamma v_{d+1} \boldsymbol{\nu}_d \cdot \mathsf{u}_d \, dS_{\mathbf{x}} \, dt + 2 \int_\Omega u_{d+1}(\cdot, T) v_{d+1}(\cdot, T) \, d\mathbf{x} .$$

We would like to find the expression for $D + M$, valid for arbitrary $\mathsf{u}, \mathsf{v} \in W$.

Lemma 6. *If formula (11) defines an operator $M \in \mathcal{L}(W; W')$ satisfying (M), then for $\mathsf{u}, \mathsf{v} \in W$ and $\mathsf{u}^n \in \mathrm{H}^1(\Omega_T; \mathbf{R}^{d+1})$ such that $\mathsf{u}^n \longrightarrow \mathsf{u}$ in W, it holds*

$${}_{W'}\langle (D + M)\mathsf{u}, \mathsf{v} \rangle_W = 2 \lim_n \int_0^T \int_\Gamma v_{d+1} \boldsymbol{\nu}_d \cdot \mathsf{u}_d^n \, dS_{\mathbf{x}} \, dt + 2 \int_\Omega u_{d+1}(\cdot, T) v_{d+1}(\cdot, T) \, d\mathbf{x} .$$

Dem. As the first step let us take $\mathsf{u} \in \mathrm{H}^1(\Omega_T; \mathbf{R}^{d+1})$, $\mathsf{v} \in W$ and $\mathsf{v}^n \in \mathrm{H}^1(\Omega_T; \mathbf{R}^{d+1})$ such that $\mathsf{v}^n \longrightarrow \mathsf{v}$ in W. Then by the continuity of $D + M$ one has

$${}_{W'}\langle (D + M)\mathsf{u}, \mathsf{v} \rangle_W = 2 \lim_n \left(\int_0^T \int_\Gamma v_{d+1}^n \boldsymbol{\nu}_d \cdot \mathsf{u}_d \, dS_{\mathbf{x}} \, dt + \int_\Omega u_{d+1}(\cdot, T) v_{d+1}^n(\cdot, T) \, d\mathbf{x} \right) .$$

By Lemma 2, we have $v_{d+1}^n \longrightarrow v_{d+1}$ in $W(0, T)$ and consequently

$$\begin{aligned} &v_{d+1}^n \longrightarrow v_{d+1} \quad \text{in} \quad \mathrm{L}^2(0, T; \mathrm{L}^2(\Omega)) , \\ &v_{d+1}^n(\cdot, T) \longrightarrow v_{d+1}(\cdot, T) \quad \text{in} \quad \mathrm{L}^2(\Omega) . \end{aligned} \tag{16}$$

Since $\mathsf{u} \in \mathrm{H}^1(\Omega_T; \mathbf{R}^{d+1})$, it follows that $\boldsymbol{\nu}_d \cdot \mathsf{u}_d \in \mathrm{L}^2(\partial\Omega_T)$, and the first convergence in (16) implies

$$\int_0^T \int_\Gamma v_{d+1}^n \boldsymbol{\nu}_d \cdot \mathsf{u}_d \, dS_{\mathbf{x}} \, dt \longrightarrow \int_0^T \int_\Gamma v_{d+1} \boldsymbol{\nu}_d \cdot \mathsf{u}_d \, dS_{\mathbf{x}} \, dt .$$

Similarly, the second convergence in (16) implies

$$\int_\Omega u_{d+1}(\cdot, T) v_{d+1}^n(\cdot, T) \, d\mathbf{x} \longrightarrow \int_\Omega u_{d+1}(\cdot, T) v_{d+1}(\cdot, T) \, d\mathbf{x} ,$$

and thus

$$_{W'}\langle\,(D+M)\mathsf{u},\mathsf{v}\,\rangle_W = 2\int_0^T\!\!\int_\Gamma v_{d+1}\,\boldsymbol{\nu}_d\cdot\mathsf{u}_d\,dS_{\mathbf{x}}\,dt + 2\int_\Omega u_{d+1}(\cdot,T)v_{d+1}(\cdot,T)\,d\mathbf{x}\,.$$

Let us now take $\mathsf{u},\mathsf{v}\in W$ and $\mathsf{u}^n\in \mathrm{H}^1(\Omega_T;\mathbf{R}^{d+1})$ such that $\mathsf{u}^n\longrightarrow\mathsf{u}$ in W. By the above formula and the continuity of $D+M$ we have

$$_{W'}\langle\,(D+M)\mathsf{u},\mathsf{v}\,\rangle_W = 2\lim_n\left(\int_0^T\!\!\int_\Gamma v_{d+1}\,\boldsymbol{\nu}_d\cdot\mathsf{u}^n_d\,dS_{\mathbf{x}}\,dt + \int_\Omega u^n_{d+1}(\cdot,T)v_{d+1}(\cdot,T)\right)\,d\mathbf{x}\,.$$

The convergence of the second integral in above limit can be achieved similarly as in the first step, and thus we have the claim.

Q.E.D.

Let us denote

$$W_2'':=\Big\{\mathsf{u}=\lim_n\mathsf{u}^n\in W:\mathsf{u}^n\in\mathrm{H}^1(\Omega_T;\mathbf{R}^{d+1})\,,\quad u_{d+1}(\cdot,T)=0\ \text{a.e. on}\ \Omega$$
$$\&\quad \boldsymbol{\nu}_d\cdot\mathsf{u}^n_d\rightharpoonup 0\ \text{in}\ \mathrm{L}^2(0,T;\mathrm{H}^{-\frac12}(\Gamma))\Big\}\,.$$

Lemma 7. *If formula (11) defines an operator $M\in\mathcal{L}(W;W')$ satisfying (M), then*

$$\ker(D+M)=W_2''\,.$$

Dem. Let us first note that $(D+M)\mathsf{u}=\mathsf{0}$ if and only if $_{W'}\langle\,(D+M)\mathsf{u},\mathsf{v}\,\rangle_W=0$, for every $\mathsf{v}\in\mathrm{H}^1(\Omega_T;\mathbf{R}^{d+1})$. By Lemma 6, $W_2''\subseteq\ker(D+M)$, so it remains to prove the converse inclusion.

Let $\mathsf{u}\in\ker(D+M)$ and $\mathsf{u}^n\in\mathrm{H}^1(\Omega_T;\mathbf{R}^{d+1})$ such that $\mathsf{u}^n\longrightarrow\mathsf{u}$ in W. Now we proceed similarly as before: we take $\mathsf{v}=(\mathsf{0},v_{d+1})^\top\in\mathrm{H}^1(\Omega_T;\mathbf{R}^{d+1})$ where $v_{d+1}(t,\mathbf{x})=g(t)h(\mathbf{x})$ with $g\in\mathrm{C}^\infty_c(\langle 0,T])$, $g(T)=1$ and $h\in\mathrm{H}^1_0(\Omega)$. Then by Lemma 6 we have

$$0={}_{W'}\langle\,(D+M)\mathsf{u},\mathsf{v}\,\rangle_W=2\int_\Omega u_{d+1}(\cdot,T)v_{d+1}(\cdot,T)\,d\mathbf{x}=2\int_\Omega u_{d+1}(\cdot,T)h\,d\mathbf{x}\,,$$

and as h is arbitrary, $u_{d+1}(\cdot,T)$ vanishes.

Using again Lemma 6, we now have

$$0={}_{W'}\langle\,(D+M)\mathsf{u},\mathsf{v}\,\rangle_W=\lim_n\int_0^T\!\!\int_\Gamma v_{d+1}\,\boldsymbol{\nu}_d\cdot\mathsf{u}^n_d\,dS_{\mathbf{x}}\,dt\,,$$

for arbitrary $\mathsf{v}\in W$. Thus, the restriction of v_{d+1} to Γ_d is an arbitrary element of $\mathrm{L}^2(0,T;\mathrm{H}^{\frac12}(\Omega))$, which implies $\boldsymbol{\nu}_d\cdot\mathsf{u}^n_d\rightharpoonup 0$ in $\mathrm{L}^2(0,T;\mathrm{H}^{-\frac12}(\Omega))$.

Q.E.D.

From lemmas 3 and 7 it follows that if formula (11) defines an operator $M\in\mathcal{L}(W;W')$ satisfying (M), then (by (M2)) $V+W_2''=W$. Although the expressions describing functions from V and W_2'' appear to be relatively simple, we do not know whether this equality really holds. In the sequel we shall prove the converse statement as well, i.e. if $V+W_2''=W$, then M defined by (11) is *good*. In order to do that we shall make use of the first remark following Theorem 8. We start with a simple lemma.

Lemma 8. *The space W_2'' is nonpositive with respect to D, i.e. $W_2''\subseteq C^-$.*

Dem. If u and u^n are as in definition of W_2'', then, as before, we have

$$_{W'}\langle\,D\mathsf{u},\mathsf{u}\,\rangle_W=\lim_n{}_{W'}\langle\,D\mathsf{u}^n,\mathsf{u}^n\,\rangle_W=2\lim_n\int_0^T\!\!\int_\Gamma u^n_{d+1}\,\boldsymbol{\nu}_d\cdot\mathsf{u}^n_d\,dS_{\mathbf{x}}\,dt-\int_\Omega u^2_{d+1}(\cdot,0)\,d\mathbf{x}\,.$$

By Lemma 2 and Theorem 10 from Appendix we have $u^n_{d+1}\longrightarrow u_{d+1}$ in $\mathrm{L}^2(0,T;\mathrm{H}^{\frac12}(\Gamma))$, and as $\boldsymbol{\nu}_d\cdot\mathsf{u}^n_d\rightharpoonup 0$ in $\mathrm{L}^2(0,T;\mathrm{H}^{-\frac12}(\Gamma))$ by definition of W_2'', it follows that the above limit vanishes, so

$$_{W'}\langle\,D\mathsf{u},\mathsf{u}\,\rangle_W=-\int_\Omega u^2_{d+1}(\cdot,0)\,d\mathbf{x}\leqslant 0\,,$$

which proves the lemma.

Q.E.D.

Theorem 9. *If $V + W_2'' = W$, and if $W_2 \subseteq W_2''$ is a closed subspace of W satisfying $W_2 \subseteq C^-$ and $V \dot{+} W_2 = W$, then operator M constructed in Theorem 8(a) satisfies (11) for any smooth* u *and* v.

Dem. Let $\mathsf{u} = \mathsf{u}_1 + \mathsf{u}_0 + \mathsf{u}_2$ be the decomposition of an arbitrary $\mathsf{u} \in W$ that corresponds to the direct sum from Theorem 8(a), and let R_1, R_0, R_2 be the corresponding projectors. Since $R_2\mathsf{u} = \mathsf{u}_2 = (\mathsf{u}_{2,d}, u_{2,d+1})^\top \in W_2 \subseteq W_2''$, by definition of W_2'', there is a sequence $\mathsf{u}^n \in \mathrm{H}^1(\Omega_T; \mathbf{R}^{d+1})$ converging to u_2 in W, and satisfying $\boldsymbol{\nu}_d \cdot \mathsf{u}_d^n \rightharpoonup 0$ in $\mathrm{L}^2(0,T; \mathrm{H}^{-\frac{1}{2}}(\Gamma))$. By Theorem 4, for arbitrary $\mathsf{v} \in \mathrm{H}^1(\Omega_T; \mathbf{R}^{d+1})$ we then have

$$
\begin{aligned}
{}_{W'}\langle DR_2\mathsf{u}, \mathsf{v} \rangle_W = \lim_n \int_0^T \int_\Gamma v_{d+1} \boldsymbol{\nu}_d \cdot \mathsf{u}_d^n \, dS_{\mathbf{x}} \, dt + \int_0^T \int_\Gamma u_{2,d+1} \boldsymbol{\nu}_d \cdot \mathsf{v}_d \, dS_{\mathbf{x}} \, dt \\
+ \int_\Omega u_{2,d+1}(\cdot, T) v_{d+1}(\cdot, T) \, d\mathbf{x} - \int_\Omega u_{2,d+1}(\cdot, 0) v_{d+1}(\cdot, 0) \, d\mathbf{x} \,.
\end{aligned}
$$

Since $\mathsf{u}_2 \in W_2''$, the first and the third integral in the above expression vanish. As $\mathsf{u} - \mathsf{u}_2 = \mathsf{u}_0 + \mathsf{u}_1 \in W_0 + W_1 = V$, it follows that $u_{d+1} - u_{2,d+1} \in \mathrm{L}^2(0,T; \mathrm{H}_0^1(\Omega))$ and $u_{d+1}(\cdot, 0) - u_{2,d+1}(\cdot, 0) = 0$. Therefore, in the second and the fourth integral we can replace $u_{2,d+1}$ by u_{d+1}, which leads to

$$
{}_{W'}\langle DR_2\mathsf{u}, \mathsf{v} \rangle_W = \int_0^T \int_\Gamma u_{d+1} \boldsymbol{\nu}_d \cdot \mathsf{v}_d \, dS_{\mathbf{x}} \, dt - \int_\Omega u_{d+1}(\cdot, 0) v_{d+1}(\cdot, 0) \, d\mathbf{x} \,. \tag{17}
$$

In order to find a similar expression for DR_1, let us now denote by $\mathsf{u}^n \in \mathrm{H}^1(\Omega_T; \mathbf{R}^{d+1})$ a sequence converging to u_1 in W. Then for any $\mathsf{v} \in \mathrm{H}^1(\Omega_T; \mathbf{R}^{d+1})$ one has

$$
\begin{aligned}
{}_{W'}\langle DR_1\mathsf{u}, \mathsf{v} \rangle_W = \lim_n \int_0^T \int_\Gamma v_{d+1} \boldsymbol{\nu}_d \cdot \mathsf{u}_d^n \, dS_{\mathbf{x}} \, dt + \int_0^T \int_\Gamma u_{1,d+1} \boldsymbol{\nu}_d \cdot \mathsf{v}_d \, dS_{\mathbf{x}} \, dt \\
+ \int_\Omega u_{1,d+1}(\cdot, T) v_{d+1}(\cdot, T) \, d\mathbf{x} - \int_\Omega u_{1,d+1}(\cdot, 0) v_{d+1}(\cdot, 0) \, d\mathbf{x} \,.
\end{aligned}
$$

In this case, the second and the fourth integral vanish, and since $\mathsf{u} - \mathsf{u}_1 = \mathsf{u}_0 + \mathsf{u}_2 \in W_0 + W_2 \subseteq W_2''$, we have $u_{d+1}(\cdot, T) - u_{1,d+1}(\cdot, T) = 0$, and therefore it follows that

$$
\int_\Omega u_{1,d+1}(\cdot, 0) v_{d+1}(\cdot, 0) \, d\mathbf{x} = \int_\Omega u_{d+1}(\cdot, 0) v_{d+1}(\cdot, 0) \, d\mathbf{x} \,. \tag{18}
$$

In order to get a similar result for the first integral in the above sum, let us take a sequence $\mathsf{w}^n \in \mathrm{H}^1(\Omega_T; \mathbf{R}^{d+1})$ converging to $\mathsf{u}_0 + \mathsf{u}_2$ in W, and satisfying $\boldsymbol{\nu}_d \cdot \mathsf{w}_d^n \rightharpoonup 0$ in $\mathrm{L}^2(0,T; \mathrm{H}^{-\frac{1}{2}}(\Gamma))$. For arbitrary $\mathsf{v} \in \mathrm{H}^1(\Omega_T; \mathbf{R}^{d+1})$ we then have

$$
\lim_n \int_0^T \int_\Gamma v_{d+1} \boldsymbol{\nu}_d \cdot \mathsf{w}_d^n \, dS_{\mathbf{x}} \, dt = 0 \,,
$$

and therefore

$$
\lim_n \int_0^T \int_\Gamma v_{d+1} \boldsymbol{\nu}_d \cdot \mathsf{u}_d^n \, dS_{\mathbf{x}} \, dt = \lim_n \int_0^T \int_\Gamma v_{d+1} \boldsymbol{\nu}_d \cdot (\mathsf{u}_d^n + \mathsf{w}_d^n) \, dS_{\mathbf{x}} \, dt \,.
$$

This, together with (17) and (18), gives

$$
\begin{aligned}
{}_{W'}\langle M\mathsf{u}, \mathsf{v} \rangle_W &= {}_{W'}\langle D(R_1 - R_2)\mathsf{u}, \mathsf{v} \rangle_W \\
&= \lim_n \int_0^T \int_\Gamma v_{d+1} \boldsymbol{\nu}_d \cdot (\mathsf{u}_d^n + \mathsf{w}_d^n) \, dS_{\mathbf{x}} \, dt - \int_0^T \int_\Gamma u_{d+1} \boldsymbol{\nu}_d \cdot \mathsf{v}_d \, dS_{\mathbf{x}} \, dt \\
&\quad + \int_\Omega u_{d+1}(\cdot, T) v_{d+1}(\cdot, T) \, d\mathbf{x} + \int_\Omega u_{d+1}(\cdot, 0) v_{d+1}(\cdot, 0) \, d\mathbf{x}
\end{aligned}
$$

for $\mathsf{v} \in \mathrm{H}^1(\Omega_T; \mathbf{R}^{d+1})$. Since $\mathsf{u}^n + \mathsf{w}^n \longrightarrow \mathsf{u}_1 + \mathsf{u}_0 + \mathsf{u}_2 = \mathsf{u}$ in W, it follows that M satisfies (11).

Q.E.D.

Corollary 2. *Formula (11) defines an operator $M \in \mathcal{L}(W; W')$ satisfying (M) if and only if $V + W_2'' = W$.* ∎

6. Concluding remarks

It is known that the abstract theory of Friedrichs systems can be applied to elliptic equations [EGC, AB4, BDG], and some applications to second-order hyperbolic equations are also known [ABV1, ABV2]. In this paper we have investigated an application of this theory to a second-order parabolic equation, namely to the heat equation. In particular, we wanted to check whether the results of [ABV1], such as Theorem 3, can be applied. This motivated our choice of the representation for heat equation in a form of the Friedrichs system which is close to the representation of its stationary counterpart, for which results such as Theorem 3 can be applied.

However, we have proved that the result of Theorem 3 cannot be applied in this setting, although there is a natural choice of matrix field $\mathbf{M}$ that enforces the Dirichlet boundary condition for the starting equation.

By changing the approach and using intrinsic conditions of Ern, Guermond and Caplain [EGC], we were able to prove a well-posedness result for the Dirichlet boundary condition. The question whether our matrix field $\mathbf{M}$ generates a *good* operator M via (2) appears to be equivalent to the equality $W = V + W_2''$. We guess that this statement is true, although we are not able to prove it at the moment. There are some technical difficulties in the related calculations arising from the interplay of function spaces involved: the graph space as a natural framework for Friedrichs systems and the evolution spaces as a natural framework for the heat equation.

We also believe that other types of boundary conditions (e.g. the Neumann condition) can be treated in this setting as well. It is possible that some more complicated spaces, like the Lions-Magenes space $\mathrm{H}^{\frac{1}{2}}_{00}$, would appear here, which could make the calculations even more challenging. These spaces appear naturally when one wants to treat mixed type boundary conditions [LM, T]. Since our setting does not distinguish between the time variable and space variables, our zero initial condition together with the Neumann boundary condition on Γ_d may appear as a kind of *mixed* boundary condition for our system.

Perhaps the above technical difficulties could be resolved by a development of the theory of Friedrich systems which is more natural for applications to evolution equations, as it was done in the classical setting [Ra].

Appendix

In this appendix we summarise some basic facts regarding evolution spaces, which we have used in the paper (the details can be found in [GGZ]). First we recall the notion of measurability and Bôchner integrability for functions $\mathsf{u} : S \to X$, where X is a Banach space, and $S \subseteq \mathbf{R}$ is an open interval.

A function $\mathsf{u} : S \to X$ is said to be *strongly measurable* if it can be strongly approximated (in the norm of Banach space X) almost everywhere on S by a sequence (s_n) of simple functions. A *simple* function is any function of the form

$$\mathsf{s}(t) = \sum_{i=1}^{n} \chi_{B_i}(t)\mathsf{a}_i\,,$$

for some $n \in \mathbf{N}$, $\mathsf{a}_1, \dots, \mathsf{a}_n \in X$ and measurable sets $B_1, \dots, B_n \subseteq S$ of finite measure. If every B_i is an interval, s is called the *step* function.

A strongly measurable function u is said to be Bôchner integrable if there exists its approximation by simple functions which additionally satisfies

$$\lim_n \int_X \|\mathsf{u}(t) - \mathsf{s}_n(t)\|_X \, dt = 0\,.$$

By the Bôchner theorem, a strongly measurable function is Bôchner integrable if and only if the real function $t \mapsto \|\mathsf{u}(t)\|_X$ is integrable on S. Therefore, for $1 \leqslant p < \infty$ one introduces

$$\mathcal{L}^p(S;X) := \Big\{\mathsf{u} : S \longrightarrow X : \mathsf{u} \text{ is strongly measurable and } \int_S \|\mathsf{u}\|_X^p \, dt < \infty\Big\},$$

and by $\mathrm{L}^p(S;X)$ all equivalence classes of almost everywhere equal elements of $\mathcal{L}^p(S;X)$.

The space $\mathrm{L}^p(S;X)$ is a Banach space, with norm

$$\|\mathsf{u}\|_{\mathrm{L}^p(S;X)} := \Big(\int_S \|\mathsf{u}\|_X^p \, dt\Big)^{\frac{1}{p}},$$

and if X is a Hilbert space, then $\mathrm{L}^2(S;X)$ is also a Hilbert space with inner product

$$\langle\, \mathsf{u} \mid \mathsf{v} \,\rangle_{\mathrm{L}^2(S;X)} := \int_S \langle\, \mathsf{u} \mid \mathsf{v} \,\rangle_X \, dt\,.$$

Step functions are dense in $\mathrm{L}^p(S;X)$, and as the characteristic function of an interval can be approximated by smooth functions in L^p the density of $\mathrm{C}_c^\infty(S;X)$ in $\mathrm{L}^p(S;X)$ follows. Furthermore, for a Banach space Y which is densely imbedded in X, the space $\mathrm{L}^p(S;Y)$ is also densely imbedded in $\mathrm{L}^p(S;X)$. For the special case $X = \mathrm{L}^2(\Omega)$, with $\Omega \subseteq \mathbf{R}^d$ open, S and Ω bounded, one can naturally identify spaces of continuous functions $\mathrm{C}(\mathsf{Cl}\, S \times \mathsf{Cl}\, \Omega)$ and $\mathrm{C}(\mathsf{Cl}\, S; \mathrm{C}(\mathsf{Cl}\, \Omega))$, which leads (by density) to isometric isomorphism between $\mathrm{L}^2(S \times \Omega)$ and $\mathrm{L}^2(S; \mathrm{L}^2(\Omega))$

Throughout the paper we use the following result (for $p = 1$ it is classical; the generalisation is straightforward).

Theorem 10. *If X and Y are two Banach spaces and $T \in \mathcal{L}(X;Y)$, then mapping $\mathcal{T}$ defined by*

$$(\mathcal{T}\mathsf{u})(t) := T(\mathsf{u}(t))$$

is a continuous linear operator from $\mathrm{L}^p(S;X)$ to $\mathrm{L}^p(S;Y)$. ∎

The generalised *Hölder inequality* for evolution spaces states that if $\mathsf{u} \in \mathrm{L}^p(S;X)$ and $\mathsf{v} \in \mathrm{L}^{p'}(S;X')$, with $\frac{1}{p} + \frac{1}{p'} = 1$, then real function $t \mapsto {}_{X'}\langle\, \mathsf{v}(t), \mathsf{u}(t) \,\rangle_X$ is integrable and

$$\int_S {}_{X'}\langle\, \mathsf{v}(t), \mathsf{u}(t) \,\rangle_X \, dt \leqslant \|\mathsf{u}\|_{\mathrm{L}^p(S;X)} \|\mathsf{v}\|_{\mathrm{L}^{p'}(S;X')}\,.$$

Moreover, the characterisation of the dual space also holds:

Theorem 11. *If X is a separable and reflexive Banach space, then for every $f \in \mathrm{L}^p(S;X)'$ there exists unique $\mathsf{v} \in \mathrm{L}^{p'}(S;X')$ such that*

$$_{\mathrm{L}^p(S;X)'}\langle\, f, \mathsf{u} \,\rangle_{\mathrm{L}^p(S;X)} = \int_S {}_{X'}\langle\, \mathsf{v}(t), \mathsf{u}(t) \,\rangle_X \, dt\,, \qquad \mathsf{u} \in \mathrm{L}^p(S;X)\,.$$

The mapping $f \mapsto \mathsf{v}$ is an isometric isomorphism from $\mathrm{L}^p(S;X)'$ to $\mathrm{L}^{p'}(S;X')$. ∎

We recall that the set

$$W(0,T) = \Big\{u \in \mathrm{L}^2(0,T;\mathrm{H}^1(\Omega)) : \partial_t u \in \mathrm{L}^2(0,T;\mathrm{H}^{-1}(\Omega))\Big\},$$

is a Banach space with norm

$$\|\mathsf{u}\|_{W(0,T)} = \sqrt{\|u\|^2_{\mathrm{L}^2(0,T;\mathrm{H}^1(\Omega))} + \|\partial_t u\|^2_{\mathrm{L}^2(0,T;\mathrm{H}^{-1}(\Omega))}}\,.$$

For further properties we refer to [LM]. In particular, in [LM, Chapter I, Theorem 3.1, Lemma 12.1] the following continuity property of such functions is proved.

Theorem 12. *The space $W(0,T)$ is continuously imbedded into $\mathrm{C}([0,T];\mathrm{L}^2(\Omega))$.* ∎

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