Nenad Antonić & Krešimir Burazin & Marko Vrdoljak

Second-order equations as Friedrichs systems

Nonlinear analysis: real world applications

Abstract

On the basis of the recent progress in understanding the abstract setting for Friedrichs symmetric positive systems by Ern, Guermond and Caplain (2007), as well as Antonić and Burazin (2010), an attempt is made to relate these results to the classical Friedrichs theory.

A particular set of sufficient conditions for a boundary matrix field to define a boundary operator is given, and the applicability of this procedure is shown by examples of boundary/initial value problems for second-order partial differential equations written as symmetric systems.

Keywords: symmetric positive system, first-order system of partial differential equations, boundary operator, transonic flow

Mathematics subject classification: 35F45, 35M32, 47F05, 76H05, 76J20

Department of Mathematics
University of Zagreb
Bijenička cesta 30
Zagreb, Croatia
nenad@math.hr

Department of Mathematics
University of Osijek
Trg Ljudevita Gaja 6
Osijek, Croatia
kburazin@mathos.hr

Department of Mathematics
University of Zagreb
Bijenička cesta 30
Zagreb, Croatia
marko@math.hr

This work is supported in part by the Croatian MZOS trough projects 037–0372787–2795 and 037–1193086–3226.

Nonlinear Analysis B: Real World Applications
doi:10.1016/j.nonrwa.2011.08.031

23rd October 2013

1. Introduction

Various questions originating from gas dynamics, both of theoretical and practical nature, have been the source of interesting mathematical problems for quite some time. Stimulated by the appearance of a book [CF] on the subject *written by mathematicians seeking to understand in a rational way a fascinating field of physical reality*, a huge advance was made in the last half-century. However, a completely satisfactory theory still eludes the present state of analysis, and various specific approaches are used.

In quite general terms, the conservation laws can be written [Ma] in the form

$$\partial_t \mathsf{u} + \operatorname{div} \mathbf{F}(\mathsf{u}) = \mathsf{s}(\cdot, \mathsf{u}) \ ,$$

where u is the unknown vector function, and $\mathbf{F}$ a prescribed nonlinear matrix function, with s representing the source term. Such a system should be supplemented by appropriate initial/boundary conditions.

Assuming that the source term vanishes, any constant u_0 belonging to the domain of $\mathbf{F}$ provides a trivial solution. By linearisation (i.e. taking $\mathsf{u} := \mathsf{u}_0 + \mathsf{v}$ and inserting it into the equation), we get the linear system for new unknown v:

$$\partial_t \mathsf{v} + \sum_{k=1}^{d} \mathbf{A}^k(\mathsf{u}_0)\partial_k \mathsf{v} = 0 \ ,$$

where $\mathbf{A}^k$ are the gradients of the rows of $\mathbf{F}$. A reasonable requirement for a general system of conservation laws is that the corresponding linearised problem is well-posed.

Over fifty years ago, Friedrichs [F] realised that most of the equations of classical physics can be written as a form of first-order system of the above form; in fact, they can be symmetrised by a multiplication with a positive definite matrix function. However, the choice of such a multiplier was neither unique nor straightforward. An important consequence for the linearised problem, which is our main concern here, was well-posedness through an energy principle.

Furthermore, this framework can accommodate equations which change their type, such as the equations appearing in the mathematical models of transonic gas flow. Such models are inherently nonlinear, but often the equation can be transformed into a linear one, with the nonlinearity hidden in the unknown domain. Actually, these equations of mixed type were the main motivation for Friedrichs in the development of this theory.

To be specific, take $d, r \in \mathbf{N}$ and let $\Omega \subseteq \mathbf{R}^d$ be an open and bounded set with Lipschitz boundary Γ (we shall denote its closure by $\mathsf{Cl}\,\Omega = \Omega \cup \Gamma$). If real matrix functions $\mathbf{A}_k \in \mathrm{W}^{1,\infty}(\Omega; \mathrm{M}_r(\mathbf{R}))$, $k \in 1..d$, and $\mathbf{C} \in \mathrm{L}^\infty(\Omega; \mathrm{M}_r(\mathbf{R}))$ satisfy

$$\text{(F1)} \qquad \mathbf{A}_k \text{ is symmetric:} \quad \mathbf{A}_k = \mathbf{A}_k^\top \ ,$$

$$\text{(F2)} \qquad (\exists\, \mu_0 > 0) \quad \mathbf{C} + \mathbf{C}^\top + \sum_{k=1}^{d} \partial_k \mathbf{A}_k \geqslant 2\mu_0 \mathbf{I} \qquad \text{(a.e. on } \Omega) \ ,$$

then the first-order differential operator $\mathcal{L} : \mathrm{L}^2(\Omega; \mathbf{R}^r) \longrightarrow \mathcal{D}'(\Omega; \mathbf{R}^r)$ defined by

$$\mathcal{L}\mathsf{u} := \sum_{k=1}^{d} \partial_k(\mathbf{A}_k \mathsf{u}) + \mathbf{C}\mathsf{u}$$

is called *the Friedrichs operator* or *the symmetric positive operator*, while (for given $\mathsf{f} \in \mathrm{L}^2(\Omega; \mathbf{R}^r)$) the first-order system of partial differential equations $\mathcal{L}\mathsf{u} = \mathsf{f}$ is called *the Friedrichs system* or *the symmetric positive system*. Note that we have written both time and space variables together, as in this generality the difference between them is not clear, and that we have used the divergence form of the differential operator in order to allow coefficients with lower regularity (the difference $(\partial_k \mathbf{A}_k)\mathsf{u}$ can be included in the term $\mathbf{C}\mathsf{u}$).

In describing the boundary conditions, following Friedrichs [F] we first define

$$\mathbf{A}_{\boldsymbol{\nu}} := \sum_{k=1}^{d} \nu_k \mathbf{A}_k \,,$$

where $\boldsymbol{\nu} = (\nu_1, \nu_2, \cdots, \nu_d)^\top$ is the outward unit normal on Γ, which is, like $\mathbf{A}_{\boldsymbol{\nu}}$, of class L^∞ on Γ. For a given matrix field on the boundary $\mathbf{M} : \Gamma \longrightarrow \mathrm{M}_r(\mathbf{R})$, the boundary condition is prescribed by

$$(\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M})\mathsf{u}_{|_\Gamma} = 0 \,,$$

and by varying $\mathbf{M}$ one can enforce different boundary conditions. Friedrichs required the following two conditions (for a.e. $\mathbf{x} \in \Gamma$) to hold:

(FM1) $$(\forall \boldsymbol{\xi} \in \mathbf{R}^r) \quad \mathbf{M}(\mathbf{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geqslant 0 \,,$$

(FM2) $$\mathbf{R}^r = \ker\Big(\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x}) - \mathbf{M}(\mathbf{x})\Big) + \ker\Big(\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x}) + \mathbf{M}(\mathbf{x})\Big) \,;$$

and such an $\mathbf{M}$ he called *an admissible boundary condition*.

The boundary value problem thus reads: for given $\mathsf{f} \in \mathrm{L}^2(\Omega; \mathbf{R}^r)$ find u such that

(1) $$\begin{cases} \mathcal{L}\mathsf{u} = \mathsf{f} \\ (\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M})\mathsf{u}_{|_\Gamma} = 0 \end{cases} .$$

Of course, under such weak assumptions the existence of a classical solution (C^1 or $\mathrm{W}^{1,\infty}$) cannot be expected. It can be shown that, in general, the solution belongs only to the graph space of operator $\mathcal{L}$:

$$W = \Big\{ \mathsf{u} \in \mathrm{L}^2(\Omega; \mathbf{R}^r) : \mathcal{L}\mathsf{u} \in \mathrm{L}^2(\Omega; \mathbf{R}^r) \Big\} \,.$$

W is a separable Hilbert space (see e.g. [AB1]) with the inner product

$$\langle\, \mathsf{u} \mid \mathsf{v} \,\rangle_{\mathcal{L}} := \langle\, \mathsf{u} \mid \mathsf{v} \,\rangle_{\mathrm{L}^2(\Omega;\mathbf{R}^r)} + \langle\, \mathcal{L}\mathsf{u} \mid \mathcal{L}\mathsf{v} \,\rangle_{\mathrm{L}^2(\Omega;\mathbf{R}^r)} \,,$$

in which the restrictions of functions from $\mathrm{C}_c^\infty(\mathbf{R}^d; \mathbf{R}^r)$ to Ω are dense. The corresponding norm will be denoted by

$$\|\mathsf{u}\|_{\mathcal{L}} = \sqrt{\|\mathsf{u}\|^2_{\mathrm{L}^2(\Omega;\mathbf{R}^r)} + \|\mathcal{L}\mathsf{u}\|^2_{\mathrm{L}^2(\Omega;\mathbf{R}^r)}} \,.$$

However, with such a weak notion of a solution in a quite large space, the question arises of how to interpret the boundary condition. It is not a priori clear what would be the meaning of $\mathsf{u}_{|_\Gamma}$ for functions u from the graph space. Recently (cf. [AB1, J]) it has been shown that $\mathsf{u}_{|_\Gamma}$ can be interpreted as an element of $\mathrm{H}^{-\frac{1}{2}}(\Gamma; \mathbf{R}^r)$, and the appropriate well-posedness results for the weak formulation of (1), under additional assumptions, have been proven [Ra, J].

More recently the Friedrichs theory has been rewritten in an abstract setting by Ern, Guermond and Caplain [EG, EGC], in terms of operators acting on Hilbert spaces, such that the traces on the boundary have not been explicitly used. Instead, the boundary conditions have been represented in an intrinsic way. In fact, the trace operator has been replaced by the *boundary operator* $D \in \mathcal{L}(W; W')$ defined by

$$_{W'}\langle\, D\mathsf{u}, \mathsf{v} \,\rangle_W := \langle\, \mathcal{L}\mathsf{u} \mid \mathsf{v} \,\rangle_{\mathrm{L}^2(\Omega;\mathbf{R}^r)} - \langle\, \mathsf{u} \mid \tilde{\mathcal{L}}\mathsf{v} \,\rangle_{\mathrm{L}^2(\Omega;\mathbf{R}^r)} \,, \qquad \mathsf{u}, \mathsf{v} \in W \,,$$

where $\tilde{\mathcal{L}} : \mathrm{L}^2(\Omega; \mathbf{R}^r) \longrightarrow \mathcal{D}'(\Omega; \mathbf{R}^r)$, the formally adjoint operator to $\mathcal{L}$, is defined by

$$\tilde{\mathcal{L}}\mathsf{v} := -\sum_{k=1}^{d} \partial_k(\mathbf{A}_k^\top \mathsf{v}) + \Big(\mathbf{C}^\top + \sum_{k=1}^{d} \partial_k \mathbf{A}_k^\top\Big)\mathsf{v} \,.$$

Furthermore, it has been shown that operator D has better properties than the trace operator. One of them that we shall use later is given by this lemma.

Lemma 1. *The operator D is symmetric, i.e.*

$$(\forall\, \mathsf{u},\mathsf{v}\in W)\qquad {}_{W'}\langle\, D\mathsf{u},\mathsf{v}\,\rangle_W = {}_{W'}\langle\, D\mathsf{v},\mathsf{u}\,\rangle_W\,.$$

∎

In [EGC] the following *weak well–posedness result* has been shown as well.

Theorem 1. *Let (F1)–(F2) be valid for matrix functions $\mathbf{A}_k\in \mathrm{W}^{1,\infty}(\Omega;\mathrm{M}_r(\mathbf{R}))$, $k\in 1..d$, and $\mathbf{C}\in \mathrm{L}^\infty(\Omega;\mathrm{M}_r(\mathbf{R}))$. Further assume that there exists an operator $M\in\mathcal{L}(W;W')$ satisfying*

(*M1*) $$(\forall\,\mathsf{u}\in W)\quad {}_{W'}\langle\, M\mathsf{u},\mathsf{u}\,\rangle_W \geqslant 0\,,\qquad \textit{and}$$

(*M2*) $$W = \ker(D-M)+\ker(D+M)\,.$$

Then the restricted operators

$$\mathcal{L}_{|_{\ker(D-M)}}:\ker(D-M)\longrightarrow \mathrm{L}^2(\Omega;\mathbf{R}^r)\qquad \textit{and}\qquad \tilde{\mathcal{L}}_{|_{\ker(D+M^*)}}:\ker(D+M^*)\longrightarrow \mathrm{L}^2(\Omega;\mathbf{R}^r)$$

are isomorphisms.

∎

The operator M from the theorem is also called the boundary operator, as $\ker M=\ker D = W_0$. In the sequel we shall refer to both properties (M1) and (M2) as (M); similarly we shall use (F) and (FM).

In the abstract setting, Ern, Guermond and Caplain [EGC] considered, besides (M), two additional forms of the boundary conditions and their mutual relationship, raising a number of open questions. In the papers [AB1, AB2, AB3] we closed the most important question by proving that those abstract conditions are, in fact, all equivalent. The new development was based on the fact that the theory can be expressed in terms of Kreĭn spaces (particular kinds of indefinite inner product spaces). This approach allowed us to simplify a number of earlier proofs as well.

The above simplification of abstract theory paved the way to new investigations of the precise relationship between the classical Friedrichs theory and its abstract counterpart.

The analogy between the properties (M) for operator M and the conditions (FM) for matrix boundary condition $\mathbf{M}$ is apparent. A natural question to be investigated is that of the nature of the relationship between the matrix field $\mathbf{M}$ and the boundary operator M. More precisely, our goal is to find additional conditions on the matrix field $\mathbf{M}$ with properties (FM) which will guarantee the existence of a *suitable operator* $M\in\mathcal{L}(W;W')$ with properties (M).

For a given matrix field $\mathbf{M}$, which M will be a *suitable operator*? The condition is satisfied by such an operator M that the result of Theorem 1 really presents the *weak well–posedness result* for problem (1) in the following sense: if for given $\mathsf{f}\in \mathrm{L}^2(\Omega;\mathbf{R}^r)$, $\mathsf{u}\in\ker(D-M)$ is such that $\mathcal{L}\mathsf{u}=\mathsf{f}$, where we additionally have $\mathsf{u}\in \mathrm{C}^1(\Omega;\mathbf{R}^r)\cap \mathrm{C}(\mathsf{Cl}\,\Omega;\mathbf{R}^r)$, then u satisfies (1) in the classical sense.

With such a connection between $\mathbf{M}$ and the boundary operator M, applications of the abstract theory to some equations of particular interest will become easier, as calculations with matrices are simpler than those with operators. We also take it as a first step towards a better understanding of the relation between the existence and uniqueness results for the Friedrichs systems as in [EGC, AB2] and the earlier *classical* results [F, J, Ra].

In order to establish this connection between $\mathbf{M}$ and M, we first note that boundary operator D can be expressed [AB1, EGC] via matrix function $\mathbf{A}_{\boldsymbol{\nu}}$:

$$(2)\qquad (\forall\,\mathsf{u},\mathsf{v}\in \mathrm{C}^\infty_c(\mathbf{R}^d;\mathbf{R}^r))\qquad {}_{W'}\langle\, D\mathsf{u},\mathsf{v}\,\rangle_W = \int_\Gamma \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\mathsf{u}_{|_\Gamma}(\mathbf{x})\cdot \mathsf{v}_{|_\Gamma}(\mathbf{x})dS(\mathbf{x})\,.$$

In fact, the above can easily be extended to $\mathsf{u},\mathsf{v}\in \mathrm{H}^1(\Omega;\mathbf{R}^r)$, provided that the restriction to Γ is replaced by the trace operator $\mathcal{T}_{\mathrm{H}^1}:\mathrm{H}^1(\Omega;\mathbf{R}^r)\longrightarrow \mathrm{H}^{\frac12}(\Gamma;\mathbf{R}^r)$. Of course, for M we expect to have the following form (see [EG])

$$(3)\qquad (\forall\,\mathsf{u},\mathsf{v}\in \mathrm{C}^\infty_c(\mathbf{R}^d;\mathbf{R}^r))\qquad {}_{W'}\langle\, M\mathsf{u},\mathsf{v}\,\rangle_W = \int_\Gamma \mathbf{M}(\mathbf{x})\mathsf{u}_{|_\Gamma}(\mathbf{x})\cdot \mathsf{v}_{|_\Gamma}(\mathbf{x})dS(\mathbf{x})\,,$$

where we naturally assume that $\mathbf{M}$ is bounded, i.e. $\mathbf{M} \in \mathrm{L}^\infty(\Omega; \mathrm{M}_r(\mathbf{R}))$. As the properties (FM) do not guarantee that the preceding formula defines a continuous operator $M : W \longrightarrow W'$, we have found [AB4] some additional conditions under which we have continuity of M, as well as (M). They are stated in Theorem 2 below, where some well known properties (precisely stated in the next lemma, also taken from [AB4]) of matrix field $\mathbf{M}$ are used.

Lemma 2. *Let matrix function* $\mathbf{M}$ *satisfy (FM1). Then the following statements are equivalent:*

a) $\mathbf{M}$ *satisfies (FM2).*

b) For almost every $\mathbf{x} \in \Gamma$ *there is a projector* $\mathbf{S}(\mathbf{x})$ *such that*

$$\mathbf{M}(\mathbf{x}) = (\mathbf{I} - 2\mathbf{S}^\top(\mathbf{x}))\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x}) .$$

c) For almost every $\mathbf{x} \in \Gamma$ *there is a projector* $\mathbf{P}(\mathbf{x})$ *such that*

$$\mathbf{M}(\mathbf{x}) = \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})(\mathbf{I} - 2\mathbf{P}(\mathbf{x})) .$$

∎

For the boundedness of operator M defined by (3), the fact that $\mathbf{A}_{\boldsymbol{\nu}}$ via formula (2) defines a continuous operator D, as well as the representation of field $\mathbf{M}$ by $\mathbf{A}_{\boldsymbol{\nu}}$ from the above lemma, was used.

Theorem 2. *Let the matrix field* $\mathbf{M} \in \mathrm{L}^\infty(\Gamma; \mathrm{M}_r(\mathbf{R}))$ *satisfy (FM), and let* $\mathbf{S}$ *be as in Lemma 2. Additionally assume that* $\mathbf{S}$ *can be extended to a measurable matrix function* $\mathbf{S}_p : \mathrm{Cl}\,\Omega \longrightarrow \mathrm{M}_r(\mathbf{R})$ *satisfying*

(S1) The multiplication operator $\mathcal{S}_p$ *defined by* $\mathcal{S}_p(\mathsf{v}) := \mathbf{S}_p\mathsf{v}$ *for* $\mathsf{v} \in W$ *is in* $\mathcal{L}(W)$*.*

(S2) $(\forall \mathsf{v} \in \mathrm{H}^1(\Omega; \mathbf{R}^r)) \qquad \mathbf{S}_p\mathsf{v} \in \mathrm{H}^1(\Omega; \mathbf{R}^r) \quad \& \quad \mathcal{T}_{\mathrm{H}^1}(\mathbf{S}_p\mathsf{v}) = \mathbf{S}\mathcal{T}_{\mathrm{H}^1}\mathsf{v}$.

Then formula (3) defines a bounded operator $M \in \mathcal{L}(W; W')$*.* ∎

The paper is organised as follows. In the second section we show that the method described in [AB4], after a careful examination of matrix multipliers, suffices for the treatment of elliptic equations. However, in the following section it is shown that this method fails on the simplest hyperbolic equation, and an extension is proposed to overcome this shortcoming. On the part of the boundary where $\mathbf{A}_{\boldsymbol{\nu}}$ is singular, the matrix $\mathbf{P}$ appearing in Lemma 2(c) is not necessarily a projector. This allows the treatment of hyperbolic equations, which is demonstrated in the fourth section by writing the wave equation in three different ways as a Friedrichs system, and supplementing it with some standard initial/boundary conditions. Finally, in the last section we consider the equations of mixed type, and sketch the possibility of application of the procedure described in this case as well.

2. The elliptic case

Continuous linear operators on graph space

For the applications of our method (cf. [AB4]), it would be important to describe all possible matrix functions $\mathbf{S}$ such that $\mathbf{S}\mathsf{u}$ belongs to the graph space W for any $\mathsf{u} \in W$. More precisely, for the multiplication operator $\mathcal{S}\mathsf{u} := \mathbf{S}\mathsf{u}$ we would like to characterise the set of all matrix functions $\mathbf{S}$ such that the corresponding linear operator $\mathcal{S}$ belongs to $\mathcal{L}(W)$.

Let us first consider the case of constant matrices $\mathbf{A}_k \in \mathrm{M}_r(\mathbf{R})$, $k \in 1..d$, and $\mathbf{S} \in \mathrm{M}_r(\mathbf{R})$. After introducing matrices $\mathbf{B}_k = \mathbf{A}_k\mathbf{S}$ for $k \in 1..d$, the inclusion $\mathcal{S}W \subseteq W$ can be written as

$$(\forall \mathsf{u} \in \mathrm{L}^2(\Omega; \mathbf{R}^r)) \quad \sum_{k=1}^d \mathbf{A}_k\partial_k\mathsf{u} \in \mathrm{L}^2(\Omega; \mathbf{R}^d) \quad \Longrightarrow \quad \sum_{k=1}^d \mathbf{B}_k\partial_k\mathsf{u} \in \mathrm{L}^2(\Omega; \mathbf{R}^d) .$$

We shall use the following notation: let $\mathbf{A} \in \mathrm{M}_{r\times rd}(\mathbf{R})$ (and analogously for $\mathbf{B}$) denote the block matrix $\mathbf{A} = [\mathbf{A}_1\ \mathbf{A}_2\ \cdots \mathbf{A}_d]$; the columns of matrices $\mathbf{A}$ and $\mathbf{B}$ will be denoted by $\mathbf{a}^i$ and $\mathbf{b}^i$, for $i \in 1..rd$, respectively.

Lemma 3. *Let* $\mathbf{A}_k, \mathbf{B}_k \in \mathrm{M}_r(\mathbf{R})$, $k \in 1..d$. *Then the following three statements are equivalent:*

a) $(\forall\, \mathsf{u} \in \mathrm{L}^2(\Omega; \mathbf{R}^r))\ \sum_{k=1}^d \mathbf{A}_k \partial_k \mathsf{u} \in \mathrm{L}^2(\Omega; \mathbf{R}^d) \quad \Longrightarrow \quad \sum_{k=1}^d \mathbf{B}_k \partial_k \mathsf{u} \in \mathrm{L}^2(\Omega; \mathbf{R}^d)$.

b) $\ker \mathbf{A} \subseteq \ker \mathbf{B}$.

c) There exists $\mathbf{P} \in \mathrm{M}_r(\mathbf{R})$ *such that* $\mathbf{B}_k = \mathbf{P}\mathbf{A}_k$, *for* $k \in 1..d$.

Dem. It is clear that (c) implies (a).

Let us next prove that (b) follows from (a) by contradiction: suppose that for some $\boldsymbol{\xi} \in \mathbf{R}^{rd}$ we have

$$\sum_{i=1}^{rd} \xi_i \mathbf{a}^i = 0 \qquad \text{and} \qquad \sum_{i=1}^{rd} \xi_i \mathbf{b}^i \neq 0 .$$

Assuming that the set Ω contains the origin (this argument can easily be adapted to the general case), we take a function $\phi \in \mathrm{C}(\mathbf{R}) \cap \mathrm{C}^1(\mathbf{R} \setminus \{0\})$ such that ϕ' is not square integrable on any interval around 0. For $j \in 1..r$ we set $\boldsymbol{\eta}^j = [\xi_j, \xi_{j+r} \ldots, \xi_{j+(d-1)r}]^\top \in \mathbf{R}^d$ and the function $u_j(\mathbf{x}) = \phi(\mathbf{x} \cdot \boldsymbol{\eta}^j)$, $\mathbf{x} \in \Omega$. Therefore, $\mathsf{u} = [u_1, \ldots, u_r]^\top$ belongs to $\mathrm{L}^2(\Omega; \mathbf{R}^r)$ and $\sum_{k=1}^d \mathbf{A}_k \partial_k \mathsf{u} = 0 \in \mathrm{L}^2(\Omega; \mathbf{R}^d)$, but $\sum_{k=1}^d \mathbf{B}_k \partial_k \mathsf{u} = \phi' \sum_i \xi_i \mathbf{b}^i \notin \mathrm{L}^2(\Omega; \mathbf{R}^d)$.

The proof of the remaining implication will be done by induction: we can easily construct matrix $\mathbf{P}$ satisfying $\mathbf{P}\mathbf{a}^i = \mathbf{b}^i$ for $i = 1..rd$. For the basis of induction, one can notice that if $\mathbf{a}^1 = \mathbf{0}$ then $\mathbf{b}^1 = \mathbf{0}$ by the assumption (b), so we can take any $\mathbf{P}$. Otherwise, the equality $\mathbf{P}\mathbf{a}^1 = \mathbf{b}^1$ defines linear operator $\mathbf{P}$ on $\mathrm{span}\{\mathbf{a}^1\}$.

Suppose that we have determined $\mathbf{P}$ on the subspace spanned by the first m columns of the matrix $\mathbf{A}$, mapping them to corresponding columns of the matrix $\mathbf{B}$. If the next column can be written as the linear combination $\mathbf{a}^{m+1} = \sum_{j=1}^m \lambda_j \mathbf{a}^j$ then, by assumption (b), we have $\mathbf{b}^{m+1} = \sum_{j=1}^m \lambda_j \mathbf{b}^j = \sum_{j=1}^m \lambda_j \mathbf{P}\mathbf{a}^j = \mathbf{P}\mathbf{a}^{m+1}$. Otherwise, we take the equality $\mathbf{b}_{m+1} = \mathbf{P}\mathbf{a}_{m+1}$ into account for the definition of $\mathbf{P}$ on $\mathrm{span}\{\mathbf{a}_1, \ldots, \mathbf{a}_{m+1}\}$. Notice that $\mathbf{P}$ is not uniquely determined if (and only if) $\mathbf{A}$ does not have maximal rank.

Q.E.D.

If matrix $\mathbf{S}$ fits the assumptions of the previous lemma, the continuity of the corresponding multiplication operator $\mathcal{S} : W \to W$ can easily be checked. We shall use the third characterisation from Lemma 3 since it seems to be easier than the second one to verify in examples.

Corollary 1. *For constant* $\mathbf{A}_k \in \mathrm{M}_r(\mathbf{R})$ *and* $\mathbf{S} \in \mathrm{M}_r(\mathbf{R})$ *the multiplication operator* $\mathcal{S} : \mathsf{u} \mapsto \mathbf{S}\mathsf{u}$ *belongs to* $\mathcal{L}(W)$ *if and only if there exists* $\mathbf{P} \in \mathrm{M}_r(\mathbf{R})$ *such that* $\mathbf{A}_k\mathbf{S} = \mathbf{P}\mathbf{A}_k$ *for* $k \in 1..d$. ∎

The case of non-constant matrix functions $\mathbf{A}_k$ and $\mathbf{S}$ is much more delicate, but still the sufficient condition from the previous corollary trivially holds.

Corollary 2. *Let* $\mathbf{A}_k, \mathbf{S} \in \mathrm{W}^{1,\infty}(\Omega; \mathrm{M}_r(\mathbf{R}))$, $k \in 1..d$. *If there exists* $\mathbf{P} \in \mathrm{W}^{1,\infty}(\Omega; \mathrm{M}_r(\mathbf{R}))$ *such that* $\mathbf{A}_k\mathbf{S} = \mathbf{P}\mathbf{A}_k$ *holds on* Ω *for any* $k \in 1..d$ *then the multiplication operator* $\mathcal{S} : \mathsf{u} \mapsto \mathbf{S}\mathsf{u}$ *belongs to* $\mathcal{L}(W)$. ∎

As was already noted in [AB4], the Lipschitz property of $\mathbf{S}_p$ in Theorem 2 implies (S2) (for the problem of Sobolev multipliers the reader might wish to consult a recent book [MŠ]). Combining this with the results of Lemma 2, Theorem 2 and Corollary 2 we get the following corollary, which is suitable for some applications. However, as we shall see later, this result does not enable us to treat some other important systems; this is the reason for some modifications described in the following section.

Corollary 3. *Let* $\mathbf{S} : \mathsf{Cl}\,\Omega \longrightarrow \mathrm{M}_r(\mathbf{R})$ *be a Lipschitz matrix function satisfying*

(L1) $(\exists\, \mathbf{P} \in \mathrm{W}^{1,\infty}(\Omega; \mathrm{M}_r(\mathbf{R})))(\forall\, k \in 1..d) \qquad \mathbf{A}_k\mathbf{S} = \mathbf{P}\mathbf{A}_k$.

(L2) For almost every $\mathbf{x} \in \Gamma$ *the matrix* $(\mathbf{I} - 2\mathbf{S}^\top(\mathbf{x}))\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})$ *is positive semidefinite.*

(L3) For almost every $\mathbf{x} \in \Gamma$ *matrix* $\mathbf{S}(\mathbf{x})$ *is a projection.*

Then formula (3), for $\mathbf{M}(\mathbf{x}) := (\mathbf{I} - 2\mathbf{S}^\top(\mathbf{x}))\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})$, $\mathbf{x} \in \Gamma$, *defines a bounded operator* $M \in \mathcal{L}(W; W')$ *that satisfies (M).* ∎

Application to scalar elliptic equations

Let $\Omega \subseteq \mathbf{R}^d$ be an open and bounded set with the Lipschitz boundary Γ, as before. We consider the following elliptic equation

$$-\mathsf{div}\,(\mathbf{A}\nabla u) + \mathbf{b}\cdot\nabla u + cu = f\,,$$

where $f \in \mathrm{L}^2(\Omega)$, $c \in \mathrm{L}^\infty(\Omega)$, $\mathbf{b} \in \mathrm{L}^\infty(\Omega;\mathbf{R}^d)$ and $\mathbf{A} \in \mathrm{L}^\infty(\Omega;\mathrm{M}_d(\mathbf{R}))$. Suppose that there exist constants $\beta \geq \alpha > 0$ such that $\mathbf{A}(\mathbf{x})$ is a symmetric matrix with eigenvalues between α and β, almost everywhere on Ω.

This equation can be rewritten as a Friedrichs system, for the vector function taking values in $\mathbf{R}^{d+1}$

$$\mathsf{u} = \begin{bmatrix} -\mathbf{A}\nabla u \\ u \end{bmatrix}\,,$$

with $\mathbf{A}_k = \mathbf{e}_k \otimes \mathbf{e}_{d+1} + \mathbf{e}_{d+1} \otimes \mathbf{e}_k \in \mathrm{M}_{d+1}(\mathbf{R})$, for $k \in 1..d$ (by $\mathbf{e}_1, \dots \mathbf{e}_{d+1}$ here we have denoted the standard basis for $\mathbf{R}^{d+1}$) and block matrix function

$$\mathbf{C} = \begin{bmatrix} \mathbf{A}^{-1} & \mathbf{0} \\ -(\mathbf{A}^{-1}\mathbf{b})^\top & c \end{bmatrix} \in \mathrm{M}_{d+1}(\mathbf{R})\,.$$

The positivity condition $\mathbf{C} + \mathbf{C}^\top \geqslant 2\mu_0\mathbf{I}$ reads

$$\begin{bmatrix} 2\mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{b} \\ -(\mathbf{A}^{-1}\mathbf{b})^\top & 2c \end{bmatrix}\begin{bmatrix} \mathbf{x} \\ y \end{bmatrix}\cdot\begin{bmatrix} \mathbf{x} \\ y \end{bmatrix} \geqslant 2\mu_0(|\mathbf{x}|^2 + y^2)\,,\ \mathbf{x} \in \mathbf{R}^d, y \in \mathbf{R}\,. \tag{4}$$

Notice that the first block $2\mathbf{A}^{-1}$ is positive definite, uniformly on Ω: $\mathbf{A}^{-1} \geq \frac{1}{\beta}\mathbf{I}$. Let us write (4) in a more elementary way by the use of Schur complement; by substitution

$$\begin{bmatrix} \mathbf{I} & \frac{1}{2}\mathbf{b} \\ \mathbf{0} & 1 \end{bmatrix}\begin{bmatrix} \mathbf{z} \\ w \end{bmatrix} = \begin{bmatrix} \mathbf{x} \\ y \end{bmatrix}$$

we introduce new variables $\mathbf{z} \in \mathbf{R}^d$ and $w \in \mathbf{R}$, obtaining thus

$$\begin{bmatrix} 2\mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{b} \\ -(\mathbf{A}^{-1}\mathbf{b})^\top & 2c \end{bmatrix}\begin{bmatrix} \mathbf{I} & \frac{1}{2}\mathbf{b} \\ \mathbf{0} & 1 \end{bmatrix}\begin{bmatrix} \mathbf{z} \\ w \end{bmatrix}\cdot\begin{bmatrix} \mathbf{I} & \frac{1}{2}\mathbf{b} \\ \mathbf{0} & 1 \end{bmatrix}\begin{bmatrix} \mathbf{z} \\ w \end{bmatrix} \geqslant 2\mu_0\Big(|\mathbf{z} + \frac{1}{2}w\mathbf{b}|^2 + w^2\Big)\,,$$

or equivalently that for arbitrary $\mathbf{z} \in \mathbf{R}^d$ and $w \in \mathbf{R}$ we have

$$\begin{bmatrix} 2\mathbf{A}^{-1} & \mathbf{0} \\ \mathbf{0} & 2c - \frac{1}{2}\mathbf{A}^{-1}\mathbf{b}\cdot\mathbf{b} \end{bmatrix}\begin{bmatrix} \mathbf{z} \\ w \end{bmatrix}\cdot\begin{bmatrix} \mathbf{z} \\ w \end{bmatrix} \geqslant 2\mu_0\Big(|\mathbf{z} + \frac{1}{2}w\mathbf{b}|^2 + w^2\Big)\,. \tag{5}$$

If there exists $\gamma > 0$ such that $2c - \frac{1}{2}\mathbf{A}^{-1}\mathbf{b}\cdot\mathbf{b} \geq \gamma$ on Ω, then the left hand side in (5) is greater than $c_1(|\mathbf{z}|^2 + w^2)$ with some positive c_1, while the right hand side is less than $\mu_0 c_2(|\mathbf{z}|^2 + w^2)$ with some positive c_2. Therefore, for $\mu_0 > 0$ small enough, the inequality (5) holds true, so we have obtained the positivity condition for the Friedrichs system.

Let us now apply Corollary 2 and determine all possible $\mathbf{S} \in \mathrm{W}^{1,\infty}(\Omega;\mathrm{M}_r(\mathbf{R}))$ such that

$$(\exists\,\mathbf{P} \in\ \mathrm{W}^{1,\infty}(\Omega;\mathrm{M}_r(\mathbf{R})))(\forall\,k \in 1..d) \qquad \mathbf{A}_k\mathbf{S} = \mathbf{P}\mathbf{A}_k\,.$$

Since matrices $\mathbf{A}_k$ have a simple form, we see that $\mathbf{A}_k\mathbf{S}$ has only two non-vanishing rows: its kth row equals the last row of $\mathbf{S}$ and its last row equals the kth row of $\mathbf{S}$. A similar conclusion holds for $\mathbf{P}\mathbf{A}_k$: its only nontrivial columns are the kth, which is exactly the last column of $\mathbf{P}$, and the last column, which equals the kth column of $\mathbf{P}$. One can now conclude that kth and $d+1$st rows of $\mathbf{S}$ and kth and $d+1$st columns of $\mathbf{P}$ vanish, except eventually at the kth and the last component, where we have, for any $k \in 1..d$,

$$S_{k,k} = P_{d+1,d+1}\,,\ S_{d+1,d+1} = P_{k,k}\,,\ S_{k,d+1} = P_{d+1,k}\,,\ S_{d+1,k} = P_{k,d+1} = 0\,.$$

The last equality is obtained by applying the same reasoning for some other k. This leads to the following matrices

$$\mathbf{S} = \begin{bmatrix} a\mathbf{I} & \boldsymbol{\eta} \\ \mathbf{0}^\top & b \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} b\mathbf{I} & \mathbf{0} \\ \boldsymbol{\eta}^\top & a \end{bmatrix},$$

where a, b and $\boldsymbol{\eta}$ are Lipschitz functions on Ω.

Therefore, for the matrices of interest we have

$$\mathbf{A}_{\boldsymbol{\nu}} = \begin{bmatrix} \mathbf{0} & \boldsymbol{\nu} \\ \boldsymbol{\nu}^\top & 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{M} = (I - 2\mathbf{S}^\top)\mathbf{A}_{\boldsymbol{\nu}} = \begin{bmatrix} \mathbf{0} & (1-2a)\boldsymbol{\nu} \\ (1-2b)\boldsymbol{\nu}^\top & -2\boldsymbol{\eta}^\top\boldsymbol{\nu} \end{bmatrix}.$$

Using the fact that $\mathbf{S}$ is a projector, a simple calculation leads to two cases: either $a = 1$ and $b = 0$ or vice versa. Checking whether $\mathbf{M} \geq 0$ leads us to the condition $\boldsymbol{\eta} \cdot \boldsymbol{\nu} \leq 0$ (in both cases). Finally, we calculate $\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M}$ and obtain the following possibilities for the boundary conditions

a) The case $a = 1$ and $b = 0$ leads us to the Dirichlet boundary condition $u = 0$ on Γ.
b) The case $a = 0$ and $b = 1$ leads to the Robin boundary condition $\mathbf{A}\nabla u \cdot \boldsymbol{\nu} - \boldsymbol{\eta} \cdot \boldsymbol{\nu} u = 0$ on Γ which in particular gives the Neumann boundary condition $\mathbf{A}\nabla u \cdot \boldsymbol{\nu} = 0$ if $\boldsymbol{\eta} \cdot \boldsymbol{\nu} = 0$.

3. Projectors are not the only possibility

The representation of $\mathbf{M}$ as a product of $\mathbf{A}_{\boldsymbol{\nu}}$ with some matrix field $\mathbf{I} - 2\mathbf{S}^\top$ is the essential ingredient in the proof of Theorem 2. However, the requirement for $\mathbf{S}$ to be a projector appears overly restrictive for applications of Theorem 2 to particular equations of interest, as we can see from the next simple example.

Example. (transport equation) Let $\boldsymbol{\alpha} \in \mathrm{W}^{1,\infty}(\Omega; \mathbf{R}^d)$ and $\mu \in \mathrm{L}^\infty(\Omega)$, such that

$$(\exists\, \mu_0 > 0) \quad \mu(\mathbf{x}) - \frac{1}{2}\mathsf{div}\, \boldsymbol{\alpha}(\mathbf{x}) \geqslant \mu_0 \qquad (\text{a.e. } \mathbf{x} \in \Omega).$$

Then the scalar equation

$$\boldsymbol{\alpha} \cdot \nabla u + \mu u = f,$$

for given $f \in \mathrm{L}^2(\Omega)$, takes the form of a Friedrichs system (actually, it consists of only one equation)

$$\sum_{k=1}^d \partial_k(A_k u) + Cu = f$$

for

$$A_k = \alpha_k \qquad \text{and} \qquad C = \mu - \sum_{k=1}^d \partial_k A_k.$$

The graph space is given by

$$W = \{u \in \mathrm{L}^2(\Omega) : \boldsymbol{\alpha} \cdot \nabla u \in \mathrm{L}^2(\Omega)\},$$

while $A_{\boldsymbol{\nu}} = \boldsymbol{\alpha} \cdot \boldsymbol{\nu}$.

We define the *inflow boundary* Γ^- and the *outflow boundary* Γ^+ by

$$\Gamma^- = \{\mathbf{x} \in \Gamma : \boldsymbol{\alpha}(\mathbf{x}) \cdot \boldsymbol{\nu}(\mathbf{x}) < 0\} \qquad \text{and} \qquad \Gamma^+ = \{\mathbf{x} \in \Gamma : \boldsymbol{\alpha}(\mathbf{x}) \cdot \boldsymbol{\nu}(\mathbf{x}) > 0\},$$

and we additionally suppose that they are *well-separated*: $d(\Gamma^-, \Gamma^+) > 0$.

We would like to determine all possible boundary conditions that can be imposed by using Corollary 3. The fact that S (being a scalar here) needs to be a projector in almost every point of Γ, in combination with the requirement that it is a Lipschitz function, gives us only two possibilities: either $S_{|_\Gamma} = 0$ or $S_{|_\Gamma} = 1$. In the first case the property (FM1) gives $(\boldsymbol{\alpha}\cdot\boldsymbol{\nu})_{|_{\Gamma^-}} = 0$, while in the second case it reads $(\boldsymbol{\alpha}\cdot\boldsymbol{\nu})_{|_{\Gamma^+}} = 0$. Therefore, either Γ^- is empty, or this is the case with Γ^+, and neither of these situations is suitable for most practical examples (in particular, if $\boldsymbol{\alpha}$ is a constant vector, this cannot be achieved).

However, the abstract Theorem 1 can be applied to any $\boldsymbol{\alpha}$ (cf. [B, EGC]). Actually, the boundary condition that can be imposed in the framework of Theorem 1 is

$$u_{|_{\Gamma^-}} = 0\,,$$

and it corresponds to $M = |\boldsymbol{\alpha}\cdot\boldsymbol{\nu}|$. ∎

It appears that the conditions of Corollary 3 are overly restrictive for envisaged applications. This seems to be particularly true for hyperbolic equations, and motivates further investigations of possible improvements of Lemma 2, Theorem 2, and Corollary 3.

Our results will be based on the following simple lemma, which we prove for the sake of completeness.

Lemma 4. *Two matrices $\mathbf{A}_+, \mathbf{A}_- \in \mathrm{M}_r(\mathbf{R})$ satisfy $\ker\mathbf{A}_+ + \ker\mathbf{A}_- = \mathbf{R}^r$ if and only if there is a projector $\mathbf{P}\in\mathrm{M}_r(\mathbf{R})$ such that*

$$\mathbf{A}_+ = (\mathbf{A}_+ + \mathbf{A}_-)(\mathbf{I}-\mathbf{P}) \qquad \textit{and} \qquad \mathbf{A}_- = (\mathbf{A}_+ + \mathbf{A}_-)\mathbf{P}\,.$$

Dem. If there exists such $\mathbf{P}$, then from $\ker\mathbf{P}\subseteq\ker\mathbf{A}_-$, $\ker(\mathbf{I}-\mathbf{P})\subseteq\ker\mathbf{A}_+$, and $\ker\mathbf{P}+\ker(\mathbf{I}-\mathbf{P})=\mathbf{R}^r$ it immediately follows that $\ker\mathbf{A}_+ + \ker\mathbf{A}_- = \mathbf{R}^r$.

The other implication follows from the fact that if $\ker\mathbf{A}_+ + \ker\mathbf{A}_- = \mathbf{R}^r$, then there is a projector $\mathbf{P}$ such that $\mathrm{im}\,(\mathbf{I}-\mathbf{P})\subseteq\ker\mathbf{A}_-$ and $\mathrm{im}\,\mathbf{P}\subseteq\ker\mathbf{A}_+$. Such a $\mathbf{P}$ then satisfies

$$\begin{aligned}(\mathbf{A}_+ + \mathbf{A}_-)(\mathbf{I}-\mathbf{P}) &= \mathbf{A}_+(\mathbf{I}-\mathbf{P}) = \mathbf{A}_+\,,\\ (\mathbf{A}_+ + \mathbf{A}_-)\mathbf{P} &= \mathbf{A}_-\mathbf{P} = \mathbf{A}_-\,.\end{aligned}$$

Q.E.D.

By defining $\mathbf{A}_+ := (\mathbf{A}_{\boldsymbol{\nu}}+\mathbf{M})(\mathbf{x})$ and $\mathbf{A}_- := (\mathbf{A}_{\boldsymbol{\nu}}-\mathbf{M})(\mathbf{x})$, we obtain that $\mathbf{M}$ satisfies (FM2) (at $\mathbf{x}\in\Gamma$ which, for simplicity, will be omitted from expressions in the sequel) if and only if there is a projector $\mathbf{P}$ such that $\mathbf{M} = \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{I}-2\mathbf{P})$. With this representation of $\mathbf{M}$ the condition (FM2) becomes

$$\ker(\mathbf{A}_{\boldsymbol{\nu}}\mathbf{P}) + \ker(\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{I}-\mathbf{P})) = \mathbf{R}^r\,. \tag{6}$$

In order to relax the assumptions of Lemma 2 and Theorem 2, it is important to observe that it is not necessary that $\mathbf{P}$ is a projector for (6) to hold. To be more precise, there are two situations that can occur:

- If $\mathbf{A}_{\boldsymbol{\nu}}$ is a regular matrix, then $\ker(\mathbf{A}_{\boldsymbol{\nu}}\mathbf{P}) = \ker\mathbf{P}$ and $\ker((\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{I}-\mathbf{P})) = \ker(\mathbf{I}-\mathbf{P})$, and therefore (6) is equivalent to $\ker\mathbf{P}+\ker(\mathbf{I}-\mathbf{P})=\mathbf{R}^r$, which is equivalent to the statement that $\mathbf{P}$ is a projector.
- If $\mathbf{A}_{\boldsymbol{\nu}}$ is not regular, then there can be several matrices $\mathbf{P}$, which are not projectors, but nevertheless satisfy (6). For example, any matrix $\mathbf{P}$ such that $\mathrm{im}\,\mathbf{P}\subseteq\ker\mathbf{A}_{\boldsymbol{\nu}}$ or $\mathrm{im}\,(\mathbf{I}-\mathbf{P})\subseteq\ker\mathbf{A}_{\boldsymbol{\nu}}$, would satisfy (6), as for such a $\mathbf{P}$ either $\ker(\mathbf{A}_{\boldsymbol{\nu}}\mathbf{P})=\mathbf{R}^r$ or $\ker((\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{I}-\mathbf{P}))=\mathbf{R}^r$.

The above discussion can be formulated as a lemma.

Lemma 5. *For a matrix $\mathbf{M}\in\mathrm{M}_r(\mathbf{R})$ the following statements are equivalent.*
a) *$\mathbf{M}$ satisfies (FM2).*
b) *There is a projector $\mathbf{P}_1$ such that $\mathbf{M}=\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{I}-2\mathbf{P}_1)$.*
c) *There is a matrix $\mathbf{P}$ such that $\mathbf{M}=\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{I}-2\mathbf{P})$ and $\ker(\mathbf{A}_{\boldsymbol{\nu}}\mathbf{P})+\ker(\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{I}-\mathbf{P}))=\mathbf{R}^r$.* ∎

The key idea is to use the representation (c) of $\mathbf{M}$ from Lemma 5 in order to get better results than before. First we prove one technical lemma.

Lemma 6. *If $\mathbf{M}$ satisfies (FM), then for $\mathbf{P}$ as in the preceding lemma we have*

$$\mathbf{A}_{\boldsymbol{\nu}}\mathbf{P}(\mathbf{I}-\mathbf{P}) = \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{I}-\mathbf{P})\mathbf{P} = \mathbf{0}\,.$$

Dem. It is well known [F] that if $\mathbf{M}$ satisfies (FM), then $\mathbf{M}^\top$ also satisfies (FM). Therefore, from the preceding lemma (b) it follows that there can be found a projector $\mathbf{S}$ such that $\mathbf{M}^\top = \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{I}-2\mathbf{S})$. Since $\mathbf{A}_{\boldsymbol{\nu}}$ is symmetric,

$$\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{I}-2\mathbf{P}) = \mathbf{M} = (\mathbf{M}^\top)^\top = (\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{I}-2\mathbf{S}))^\top = \mathbf{A}_{\boldsymbol{\nu}} - 2\mathbf{S}^\top\mathbf{A}_{\boldsymbol{\nu}}\,,$$

and we get $\mathbf{A}_{\boldsymbol{\nu}}\mathbf{P} = \mathbf{S}^\top\mathbf{A}_{\boldsymbol{\nu}}$.

Any $\mathsf{w}\in\mathbf{R}^r$ can be decomposed as $\mathsf{w} = \boldsymbol{\xi}+\boldsymbol{\eta}$ such that $\boldsymbol{\xi}\in\ker(\mathbf{A}_{\boldsymbol{\nu}}\mathbf{P})$ and $\boldsymbol{\eta}\in\ker(\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{I}-\mathbf{P}))$. Now we easily get

$$\begin{aligned}\mathbf{A}_{\boldsymbol{\nu}}\mathbf{P}(\mathbf{I}-\mathbf{P})\mathsf{w} &= \mathbf{A}_{\boldsymbol{\nu}}\mathbf{P}(\mathbf{I}-\mathbf{P})\boldsymbol{\xi} + \mathbf{A}_{\boldsymbol{\nu}}\mathbf{P}(\mathbf{I}-\mathbf{P})\boldsymbol{\eta}\\ &= \mathbf{A}_{\boldsymbol{\nu}}\mathbf{P}\boldsymbol{\xi} - \mathbf{S}^\top\mathbf{A}_{\boldsymbol{\nu}}\mathbf{P}\boldsymbol{\xi} + \mathbf{S}^\top\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{I}-\mathbf{P})\boldsymbol{\eta} = 0\,,\end{aligned}$$

which concludes the proof.

Q.E.D.

Theorem 3. *Let the matrix field $\mathbf{M}\in\mathrm{L}^\infty(\Gamma;\mathrm{M}_r(\mathbf{R}))$ satisfy (FM), and let $\mathbf{P}$ be as in Lemma 5. Additionally assume that $\mathbf{P}$ can be extended to a measurable matrix function $\mathbf{P}_p : \mathsf{Cl}\,\Omega \longrightarrow \mathrm{M}_r(\mathbf{R})$ satisfying:*

(S1) The multiplication operator $\mathcal{P}_p$, defined by $\mathcal{P}_p(\mathsf{v}) := \mathbf{P}_p\mathsf{v}$ for $\mathsf{v}\in W$, is a bounded linear operator on W.

(S2) $(\forall\,\mathsf{v}\in\mathrm{H}^1(\Omega;\mathbf{R}^r))\qquad \mathbf{P}_p\mathsf{v}\in\mathrm{H}^1(\Omega;\mathbf{R}^r)\quad\&\quad \mathcal{T}_{\mathrm{H}^1}(\mathbf{P}_p\mathsf{v}) = \mathbf{P}\mathcal{T}_{\mathrm{H}^1}\mathsf{v}$.

Then formula (3) defines a bounded operator $M\in\mathcal{L}(W;W')$ satisfying (M).

Dem. Let us first prove that $M\in\mathcal{L}(W;W')$. For the formula (3) to define a unique bounded operator from $\mathcal{L}(W;W')$, by density it is necessary and sufficient that

$$(7)\qquad (\exists\,C>0)(\forall\,\mathsf{u},\mathsf{v}\in\mathrm{C}_c^\infty(\mathbf{R}^d;\mathbf{R}^r))\qquad \Big|\int_\Gamma \mathbf{M}(\mathbf{x})\mathsf{u}_{|_\Gamma}(\mathbf{x})\cdot\mathsf{v}_{|_\Gamma}(\mathbf{x})dS(\mathbf{x})\Big| \leqslant C\|\mathsf{u}\|_{\mathcal{L}}\|\mathsf{v}\|_{\mathcal{L}}\,.$$

For such u and v it holds that

$$\int_\Gamma \mathbf{M}\mathsf{u}_{|_\Gamma}\cdot\mathsf{v}_{|_\Gamma}dS = \int_\Gamma \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{I}-2\mathbf{P})\mathsf{u}_{|_\Gamma}\cdot\mathsf{v}_{|_\Gamma}dS = \int_\Gamma (\mathbf{I}-2\mathbf{P})\mathsf{u}_{|_\Gamma}\cdot\mathbf{A}_{\boldsymbol{\nu}}\mathsf{v}_{|_\Gamma}dS\,.$$

By (S2) it follows that $(\mathbf{I}-2\mathbf{P}_p)\mathsf{u}\in\mathrm{H}^1(\Omega;\mathbf{R}^r)$ and $\mathcal{T}_{\mathrm{H}^1}\Big((\mathbf{I}-2\mathbf{P}_p)\mathsf{u}\Big) = (\mathbf{I}-2\mathbf{P})\mathsf{u}_{|_\Gamma}$, so from (2) and Lemma 1 we can conclude that

$$(8)\qquad \begin{aligned}\int_\Gamma \mathbf{M}\mathsf{u}_{|_\Gamma}\cdot\mathsf{v}_{|_\Gamma}dS &= {}_{W'}\langle\, D\mathsf{v}, (\mathbf{I}-2\mathbf{P}_p)\mathsf{u}\,\rangle_W\\ &= {}_{W'}\langle\, D\mathsf{v}, (\mathcal{I}_W-2\mathcal{P}_p)\mathsf{u}\,\rangle_W = {}_{W'}\langle\, D(\mathcal{I}_W-2\mathcal{P}_p)\mathsf{u},\mathsf{v}\,\rangle_W\,,\end{aligned}$$

where $\mathcal{I}_W$ denotes the identity on W. Since all the operators appearing on the right hand side of the above equality are continuous, we conclude that

$$\Big|\int_\Gamma \mathbf{M}\mathsf{u}_{|_\Gamma}\cdot\mathsf{v}_{|_\Gamma}dS\Big| \leqslant \|D\|_{\mathcal{L}(W;W')}\cdot\|\mathcal{I}_W-2\mathcal{P}_p\|_{\mathcal{L}(W)}\cdot\|\mathsf{u}\|_W\cdot\|\mathsf{v}\|_W\,,$$

and therefore M defined by (3) belongs to $\mathcal{L}(W;W')$, so we have the equality $M = D(\mathcal{I}_W-2\mathcal{P}_p)$.

Since property (M1) obviously follows from (FM1) and (3), it remains to prove (M2). In order to do that let us first show that $D\mathcal{P}_p(\mathcal{I}_W-\mathcal{P}_p) = D(\mathcal{I}_W-\mathcal{P}_p)\mathcal{P}_p = 0$. Using Lemma 6, for $\mathsf{u},\mathsf{v}\in\mathrm{C}_c^\infty(\mathbf{R}^d;\mathbf{R}^r)$ we get

$$\begin{aligned}{}_{W'}\langle\, D\mathcal{P}_p(\mathcal{I}_W-\mathcal{P}_p)\mathsf{u},\mathsf{v}\,\rangle_W &= \int_\Gamma \mathbf{A}_{\boldsymbol{\nu}}\mathcal{T}_{\mathrm{H}^1}(\mathbf{P}_p(\mathbf{I}-\mathbf{P}_p)\mathsf{u})\cdot\mathsf{v}_{|_\Gamma}dS\\ &= \int_\Gamma \mathbf{A}_{\boldsymbol{\nu}}\mathbf{P}(\mathbf{I}-\mathbf{P})\mathsf{u}_{|_\Gamma}\cdot\mathsf{v}_{|_\Gamma}dS = 0\,,\end{aligned}$$

and thus $D\mathcal{P}_p(\mathcal{I}_W - \mathcal{P}_p) = D(\mathcal{I}_W - \mathcal{P}_p)\mathcal{P}_p = 0$ by density.

Finally, for an arbitrary $\mathsf{w} \in W$, let $\mathsf{u} = (\mathcal{I}_W - \mathcal{P}_p)\mathsf{w}$, and therefore $\mathsf{w} - \mathsf{u} = \mathcal{P}_p\mathsf{w}$. Now, from

$$(D - M)\mathsf{u} = 2D\mathcal{P}_p\mathsf{u} = 2D\mathcal{P}_p(\mathcal{I}_W - \mathcal{P}_p)\mathsf{w} = 0$$

and

$$(D + M)(\mathsf{w} - \mathsf{u}) = 2D(\mathcal{I}_W - \mathcal{P}_p)(\mathsf{w} - \mathsf{u}) = 2D(\mathcal{I}_W - \mathcal{P}_p)\mathcal{P}_p\mathsf{w} = 0\,,$$

(M2) follows.

Q.E.D.

At this point, just as before, it is natural to look for some sufficient conditions on $\mathbf{P}$ so that the assumptions of Theorem 2 will be fulfilled, and that these conditions can conveniently be verified in applications to equations of interest. Combining the results of Corollary 2, the previous theorem, and the already mentioned argument that the Lipschitz property of $\mathbf{P}_p : \mathsf{Cl}\,\Omega \longrightarrow \mathrm{M}_r(\mathbf{R})$ implies (S2), we have the following corollary.

Corollary 4. *Let $\mathbf{P} : \mathsf{Cl}\,\Omega \longrightarrow \mathrm{M}_r(\mathbf{R})$ be a Lipschitz matrix function satisfying:*

(P1) $(\exists\, \mathbf{R} \in \mathrm{W}^{1,\infty}(\Omega; \mathrm{M}_r(\mathbf{R})))(\forall\, k \in 1..d) \qquad \mathbf{A}_k\mathbf{P} = \mathbf{R}\mathbf{A}_k$,

(P2) for almost every $\mathbf{x} \in \Gamma$ the matrix $\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})(\mathbf{I} - 2\mathbf{P}(\mathbf{x}))$ is positive semidefinite, and

(P3) for almost every $\mathbf{x} \in \Gamma$ it holds that $\ker\Big(\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\mathbf{P}(\mathbf{x})\Big) + \ker\Big((\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})(\mathbf{I} - \mathbf{P}(\mathbf{x}))\Big) = \mathbf{R}^r$.

Then formula (3), for $\mathbf{M}(\mathbf{x}) := \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})(\mathbf{I} - 2\mathbf{P}(\mathbf{x}))$ on Γ, defines a bounded operator $M \in \mathcal{L}(W; W')$ satisfying (M).

∎

If we try to apply this result to the already mentioned transport equation we get that (P1) is trivially satisfied with $R = P$ (which are scalars in this example). Since $A_{\boldsymbol{\nu}} = \boldsymbol{\alpha} \cdot \boldsymbol{\nu}$, the condition (P3) is equivalent to $P(\mathbf{x}) \in \{0, 1\}$, for a.e. $\mathbf{x} \in \Gamma^+ \cup \Gamma^-$, which in combination with (P2) finally gives that properties (P) are equivalent to the equality

$$P(\mathbf{x}) = \begin{cases} 0\,, & \text{for } \mathbf{x} \in \Gamma^+ \\ 1\,, & \text{for } \mathbf{x} \in \Gamma^- \end{cases}.$$

Since Γ^+ and Γ^- are well-separated, we can find a Lipschitz function $P : \mathsf{Cl}\,\Omega \longrightarrow \mathbf{R}$ with this property, and thus the conditions of Corollary 4 are satisfied.

Note that for such a P we have

$$A_{\boldsymbol{\nu}} - M = \begin{cases} 0\,, & \mathbf{x} \in \Gamma^+ \\ 2\boldsymbol{\alpha} \cdot \boldsymbol{\nu}\,, & \mathbf{x} \in \Gamma^- \end{cases},$$

and therefore the boundary condition

$$u_{|_{\Gamma^-}} = 0\,,$$

which is in agreement with the well known results for the transport equation.

4. Wave equation as a Friedrichs system

Two unknowns: u and $u_t + \gamma u_x$

The complexity of application of the theory of Friedrichs systems to particular equations was partially demonstrated in [AB4]. From some examples presented there we can see how the application depends on the representation of a particular equation as a Friedrichs system, which is usually not unique. Among other things, the representation depends on the choice of unknown vector function u (its components can be the original unknown u, some of its derivatives, perhaps a linear combination of them, etc.), on the auxiliary equations that one takes to supplement the original equation in order to get a formally determined system (i.e. one with the same number of unknowns as equations), and even on the order in which equations are written. All these have to be carefully chosen in order to get a positive symmetric system. The type of boundary condition that can be treated also depends on this representation [AB4], as sometimes one representation allows the treatment of one type of boundary condition, while some other allows the treatment of some other type. This complexity will also be illustrated here, as we try to apply the results of Corollary 4 to the wave equation

$$u_{tt} - \gamma^2 u_{xx} = f$$

in some bounded open set $\Omega \subseteq \mathbf{R}^2$. We shall propose a representation of this equation as a Friedrichs system, and then check what type of domain Ω and which boundary conditions we can treat in the framework of Corollary 4. Here, we assume that $\gamma > 0$ is a constant, and $f \in \mathrm{L}^2(\Omega)$ a given function.

In the first representation we take $u_1 = u$ and $u_2 = u_t + \gamma u_x$. Then our new unknown vector function $\mathsf{u} = (u_1, u_2)^\top$ satisfies this symmetric system

$$\partial_t\Big(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\mathsf{u}\Big) + \partial_x\Big(\begin{bmatrix} \gamma & 0 \\ 0 & -\gamma \end{bmatrix}\mathsf{u}\Big) + \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}\mathsf{u} = \begin{bmatrix} 0 \\ f \end{bmatrix},$$

where the first equation is just the definition of u_2, while the second one is our original wave equation. The above system is not positive, so we introduce a new unknown $\mathsf{v} := e^{-\lambda t}\mathsf{u}$, which then satisfies

$$\partial_t\Big(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\mathsf{v}\Big) + \partial_x\Big(\begin{bmatrix} \gamma & 0 \\ 0 & -\gamma \end{bmatrix}\mathsf{v}\Big) + \begin{bmatrix} \lambda & -1 \\ 0 & \lambda \end{bmatrix}\mathsf{v} = \begin{bmatrix} 0 \\ e^{-\lambda t} f \end{bmatrix},$$

again being a symmetric system, but also a positive one for $\lambda > 0$ large enough. Note that

$$\mathbf{A}_{\boldsymbol{\nu}} = \begin{bmatrix} \nu_1 + \gamma\nu_2 & 0 \\ 0 & \nu_1 - \gamma\nu_2 \end{bmatrix}.$$

If we try to apply Corollary 4, one can easily check that (P1) is equivalent to the statement that $\mathbf{P}$ is a diagonal matrix function

$$\mathbf{P} = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix},$$

and since then

$$\mathbf{M} = \begin{bmatrix} (1-2a)(\nu_1 + \gamma\nu_2) & 0 \\ 0 & (1-2d)(\nu_1 - \gamma\nu_2) \end{bmatrix},$$

the condition (P2) becomes

$$\begin{aligned} &(1-2a)(\nu_1 + \gamma\nu_2) \geqslant 0 \quad \text{on} \quad \Gamma\,, \\ &(1-2d)(\nu_1 - \gamma\nu_2) \geqslant 0 \quad \text{on} \quad \Gamma\,. \end{aligned} \tag{9}$$

To verify which $\mathbf{P}$ fulfils (P3), we distinguish several cases depending on the rank of $\mathbf{A}_{\boldsymbol{\nu}}$:

I. If $\mathbf{A}_{\boldsymbol{\nu}}$ is regular, then $\mathbf{P}$ must be a projector, and thus $a, d \in \{0, 1\}$. Combining this with (9) we get four sub-cases

$$\begin{array}{lllll} \textbf{I.a} & a = 0\,, & d = 1\,, & \nu_1 + \gamma\nu_2 > 0\,, & \nu_1 - \gamma\nu_2 < 0\,; \\ \textbf{I.b} & a = 1\,, & d = 0\,, & \nu_1 + \gamma\nu_2 < 0\,, & \nu_1 - \gamma\nu_2 > 0\,; \\ \textbf{I.c} & a = 1\,, & d = 1\,, & \nu_1 + \gamma\nu_2 < 0\,, & \nu_1 - \gamma\nu_2 < 0\,; \\ \textbf{I.d} & a = 0\,, & d = 0\,, & \nu_1 + \gamma\nu_2 > 0\,, & \nu_1 - \gamma\nu_2 > 0\,. \end{array}$$

II. If $\mathbf{A}_{\boldsymbol{\nu}}$ is singular, then we have $\det \mathbf{A}_{\boldsymbol{\nu}} = (\nu_1 + \gamma\nu_2)(\nu_1 - \gamma\nu_2) = 0$, and this corresponds to situations where some part of boundary Γ lays on the characteristics of the original wave equation.

By distinguishing whether $\nu_1+\gamma\nu_2=0$ or $\nu_1-\gamma\nu_2=0$, combining this with (9), using condition (P3) and the fact that the normal is a unit vector $\nu_1^2+\nu_2^2=1$, we get the following four sub-cases:

$$\begin{aligned}
&\textbf{II.a}\quad a\in\mathbf{R}\,,\quad d=0\,,\quad \nu_1=\frac{\gamma}{\sqrt{1+\gamma^2}}\,,\quad \nu_2=\frac{-1}{\sqrt{1+\gamma^2}}\,;\\
&\textbf{II.b}\quad a\in\mathbf{R}\,,\quad d=1\,,\quad \nu_1=\frac{-\gamma}{\sqrt{1+\gamma^2}}\,,\quad \nu_2=\frac{1}{\sqrt{1+\gamma^2}}\,;\\
&\textbf{II.c}\quad a=0\,,\quad d\in\mathbf{R}\,,\quad \nu_1=\frac{\gamma}{\sqrt{1+\gamma^2}}\,,\quad \nu_2=\frac{1}{\sqrt{1+\gamma^2}}\,;\\
&\textbf{II.d}\quad a=1\,,\quad d\in\mathbf{R}\,,\quad \nu_1=\frac{-\gamma}{\sqrt{1+\gamma^2}}\,,\quad \nu_2=\frac{-1}{\sqrt{1+\gamma^2}}\,.
\end{aligned}$$

After combining these eight cases we get that Γ has a specific form presented in Figure 1.

Figure 1. An example of the acceptable domain Ω

It remains to clarify that we can indeed choose a and d in such a way that these conditions are satisfied, and that both a and d are Lipschitz functions at the same time. We simply note that the part of the boundary where $a=0$ (I.a, I.d, II.c) is *separated* from the part of the boundary where $a=1$ (I.b, I.c, II.d) with parts where a can take an arbitrary value (II.a, II.b). Thus, we can choose a to be Lipschitz as long as parts of Γ (of nonzero length) lay on characteristics corresponding to cases II.a and II.b. Similarly, the same holds for d when parts of Γ lay on characteristics corresponding to cases II.c and II.d.

We can conclude that, as long as parts of Γ that correspond to any of cases I are appropriately well-separated by parts that correspond to cases II, we can apply Corollary 4. Also note that Γ need not have parts corresponding to any of cases I and can be made only of characteristics (Figure 2).

Let us now take a closer look at the separate boundary conditions that we impose on specific parts of the boundary; as

$$\mathbf{A}_{\boldsymbol{\nu}}-\mathbf{M}=\begin{bmatrix} a(\nu_1+\gamma\nu_2) & 0\\ 0 & d(\nu_1-\gamma\nu_2)\end{bmatrix}\,,$$

and $v_1=e^{-\lambda t}u_1=e^{-\lambda t}u$, $v_2=e^{-\lambda t}u_2=e^{-\lambda t}(u_t+\gamma u_x)$, we get the following boundary conditions for the original wave equation at parts of the boundary

I.a	$u_t + \gamma u_x = 0\,;$
I.b	$u = 0\,;$
I.c	$u = 0$ and $u_t + \gamma u_x = 0\,;$
I.d	no boundary condition is imposed on this part of Γ ;
II.a	no boundary condition is imposed on this part of Γ ;
II.b	$u_t + \gamma u_x = 0\,;$
II.c	no boundary condition is imposed on this part of Γ ;
II.d	$u = 0\,.$

For example if we take that the entirety of Γ lays on the characteristics, and we impose $u = 0$ on the characteristics II.d, and $u_t + \gamma u_x = 0$ on II.b, we get a well-posed problem (Figure 2).

Figure 2. Boundary conditions for the simplest domain

Finally, let us note that the above result for the wave equation could not be achieved in the framework of Corollary 3: if we required $\mathbf{P}$ to be a projector on the whole Γ, we would get only cases I, and thus that a and d are identically equal to either 0 or 1 on Γ, being Lipschitz functions. Say, for example, that we took $a = d = 0$, this being the case I.d; then the conditions $\nu_1 + \gamma\nu_2 \geqslant 0$ and $\nu_1 - \gamma\nu_2 \geqslant 0$ must be satisfied on the whole Γ, which is impossible for bounded Ω. In other cases we would have got a contradiction in an analogous way, and thus the conditions of Corollary 3 could not be satisfied by this Friedrichs system.

Two unknowns: u_t and u_x

Next, we present another representation of the wave equation as a Friedrichs system: if we take $\mathsf{u} = (u_1, u_2)^\top = (e^{-\lambda t}u_t, e^{-\lambda t}u_x)^\top$, the wave equation transforms to the system (for $\lambda > 0$)

$$\partial_t\Big(\begin{bmatrix} 1 & 0 \\ 0 & \gamma^2 \end{bmatrix}\mathsf{u}\Big) + \partial_x\Big(\begin{bmatrix} 0 & -\gamma^2 \\ -\gamma^2 & 0 \end{bmatrix}\mathsf{u}\Big) + \begin{bmatrix} \lambda & 0 \\ 0 & \lambda\gamma^2 \end{bmatrix}\mathsf{u} = \begin{bmatrix} e^{-\lambda t} f \\ 0 \end{bmatrix}\,,$$

with

$$\mathbf{A}_{\boldsymbol{\nu}} = \begin{bmatrix} \nu_1 & -\gamma^2\nu_2 \\ -\gamma^2\nu_2 & \gamma^2\nu_1 \end{bmatrix}\,.$$

As before, multiplication by $e^{-\lambda t}$ was performed in order to get a positive system.

We proceed similarly to what was done in the previous example above: condition (P1) gives us the following form of the matrix:

$$\mathbf{P} = \begin{bmatrix} a & d\gamma^2 \\ d & a \end{bmatrix} ,$$

and then

$$\mathbf{M} = \begin{bmatrix} (1-2a)\nu_1 + 2d\gamma^2\nu_2 & -(1-2a)\gamma^2\nu_2 - 2d\gamma^2\nu_1 \\ -(1-2a)\gamma^2\nu_2 - 2d\gamma^2\nu_1 & (1-2a)\gamma^2\nu_1 + 2d\gamma^4\nu_2 \end{bmatrix} ,$$

which turns (P2) into the following inequalities valid on Γ:

$$\begin{aligned} (1-2a)\nu_1 + 2d\gamma^2\nu_2 &\geqslant 0 \,, \\ (\nu_1^2 - \gamma^2\nu_2^2)[(1-2a)^2 - 4d^2\gamma^2] &\geqslant 0 \,. \end{aligned} \tag{10}$$

As before, to check (P3) we distinguish several cases.
I. When $\mathbf{A}_{\boldsymbol{\nu}}$ is regular, then $\mathbf{P}$ is projector, which in combination with (10) gives the following sub-cases

$$\begin{array}{ll} \textbf{I.a} & a = \frac{1}{2} \,, \quad d = \frac{1}{2\gamma} \,, \quad \nu_1 + \gamma\nu_2 > 0 \,, \quad \nu_1 - \gamma\nu_2 < 0 \,; \\ \textbf{I.b} & a = \frac{1}{2} \,, \quad d = -\frac{1}{2\gamma} \,, \quad \nu_1 + \gamma\nu_2 < 0 \,, \quad \nu_1 - \gamma\nu_2 > 0 \,; \\ \textbf{I.c} & a = 1 \,, \quad d = 0 \,, \quad \nu_1 + \gamma\nu_2 < 0 \,, \quad \nu_1 - \gamma\nu_2 < 0 \,; \\ \textbf{I.d} & a = 0 \,, \quad d = 0 \,, \quad \nu_1 + \gamma\nu_2 > 0 \,, \quad \nu_1 - \gamma\nu_2 > 0 \,. \end{array}$$

II. The situation when $\mathbf{A}_{\boldsymbol{\nu}}$ is singular corresponds again to the characteristic boundary, as $\det \mathbf{A}_{\boldsymbol{\nu}} = \gamma^2(\nu_1 + \gamma\nu_2)(\nu_1 - \gamma\nu_2)$. Similarly to what was done before we get the following four sub-cases:

$$\begin{array}{ll} \textbf{II.a} & a = -\gamma d \,, \quad d \in \mathbf{R} \,, \quad \nu_1 = \frac{\gamma}{\sqrt{1+\gamma^2}} \,, \quad \nu_2 = \frac{-1}{\sqrt{1+\gamma^2}} \,; \\ \textbf{II.b} & a = 1 - \gamma d \,, \quad d \in \mathbf{R} \,, \quad \nu_1 = \frac{-\gamma}{\sqrt{1+\gamma^2}} \,, \quad \nu_2 = \frac{1}{\sqrt{1+\gamma^2}} \,; \\ \textbf{II.c} & a = \gamma d \,, \quad d \in \mathbf{R} \,, \quad \nu_1 = \frac{\gamma}{\sqrt{1+\gamma^2}} \,, \quad \nu_2 = \frac{1}{\sqrt{1+\gamma^2}} \,; \\ \textbf{II.d} & a = 1 + \gamma d \,, \quad d \in \mathbf{R} \,, \quad \nu_1 = \frac{-\gamma}{\sqrt{1+\gamma^2}} \,, \quad \nu_2 = \frac{-1}{\sqrt{1+\gamma^2}} \,. \end{array}$$

Analogously to what was done for the first representation, one can easily see that we can choose a and d to be Lipschitz, as long as all cases II corresponding to the characteristic boundary appear as parts of Γ (of nonzero length).

Furthermore, it is interesting to notice that by this representation of the wave equation as a Friedrichs system we can treat the same domains as we did by the first representation (Figure 1). Non-convex domains are also possible: for example, instead of the part of the boundary corresponding to the case I.a in Figure 1, we could put two concave curves belonging to the same case I.a.

Let us finally write down the boundary conditions that we have to impose on specific parts of the boundary. As

$$\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M} = 2 \begin{bmatrix} a\nu_1 - d\gamma^2\nu_2 & -a\gamma^2\nu_2 + d\gamma^2\nu_1 \\ -a\gamma^2\nu_2 + d\gamma^2\nu_1 & a\gamma^2\nu_1 - d\gamma^4\nu_2 \end{bmatrix} ,$$

and $\mathsf{u} = (u_1, u_2)^\top = (e^{-\lambda t}u_t, e^{-\lambda t}u_x)^\top$, we get that the following boundary conditions are admissible for the wave equation at various parts of the boundary (in some cases, one has to solve a system of linear equations in u_t and u_x)

$$
\begin{array}{rl}
\textbf{I.a} & u_t + \gamma u_x = 0\,; \\
\textbf{I.b} & u_t - \gamma u_x = 0\,; \\
\textbf{I.c} & u_t = u_x = 0\,; \\
\textbf{I.d} & \text{no boundary condition is imposed on this part of } \Gamma\,; \\
\textbf{II.a} & \text{no boundary condition is imposed on this part of } \Gamma\,; \\
\textbf{II.b} & u_t + \gamma u_x = 0\,; \\
\textbf{II.c} & \text{no boundary condition is imposed on this part of } \Gamma\,; \\
\textbf{II.d} & u_t - \gamma u_x = 0\,.
\end{array}
$$

For example, for the domain presented in Figure 2, the boundary conditions are not exactly the same: the boundary condition on the part II.d of the boundary changes to $u_t - \gamma u_x = 0$. However, this is fine, as solution (u_t, u_x) of the system can determine u only up to a constant, while along the characteristics II.b and II.d the tangential derivatives are zero, imposing the constancy of u.

Let us remark that, using the same argument as for the first representation of the wave equation, one can easily see that the above results cannot be achieved within the framework of Corollary 3.

Three unknowns: u, u_t and u_x

Note that in the preceding representation the unknown u was not part of u, but only its derivatives. If one would like to have boundary conditions that explicitly involve values of u, then it should also be included as part of u. It is natural to, besides the first representation, also try with three unknowns $\mathsf{u} = (u_1, u_2, u_3) = (u_t, u_x, u)$ and supplement the two equations from the second representation with one additional equation. Two natural choices for the third equation might be

$$
\partial_t u_3 - u_1 = 0 \qquad \text{or} \qquad \partial_x u_3 - u_2 = 0\,.
$$

In the first case we get the following symmetric system

$$
\partial_t\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & \gamma^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\mathsf{u}\right) + \partial_x\left(\begin{bmatrix} 0 & -\gamma^2 & 0 \\ -\gamma^2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\mathsf{u}\right) + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}\mathsf{u} = \begin{bmatrix} f \\ 0 \\ 0 \end{bmatrix},
$$

while in the second

$$
\partial_t\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & \gamma^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\mathsf{u}\right) + \partial_x\left(\begin{bmatrix} 0 & -\gamma^2 & 0 \\ -\gamma^2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\mathsf{u}\right) + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}\mathsf{u} = \begin{bmatrix} f \\ 0 \\ 0 \end{bmatrix}.
$$

Neither of these two systems is positive, but they can easily be transformed into positive ones by an appropriate change of unknown function u, as before (for the second system this is done below). However, while for the second system we can apply Corollary 4 (see below), it turns out that this cannot be done for the first one. This also shows the inherent complexity in the applications of Friedrichs system theory, as we do not know (prior to actually trying them both) that the equation $\partial_x u_3 - u_2 = 0$ is a better choice for the third equation than $\partial_t u_3 - u_1 = 0$.

After taking $\mathsf{v} = e^{-\lambda t - \mu x}\mathsf{u}$, the second system becomes a Friedrichs system (for an appropriate choice of $\lambda > 0$ and $\mu > 0$):

$$
\partial_t\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & \gamma^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\mathsf{u}\right) + \partial_x\left(\begin{bmatrix} 0 & -\gamma^2 & 0 \\ -\gamma^2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\mathsf{u}\right) + \begin{bmatrix} \lambda & -\nu\gamma^2 & 0 \\ -\nu\gamma^2 & \lambda\gamma^2 & 0 \\ 0 & -1 & \mu \end{bmatrix}\mathsf{u} = \begin{bmatrix} e^{-\lambda t - \mu x} f \\ 0 \\ 0 \end{bmatrix},
$$

with

$$\mathbf{A}_{\boldsymbol{\nu}} = \begin{bmatrix} \nu_1 & -\gamma^2\nu_2 & 0 \\ -\gamma^2\nu_2 & \gamma^2\nu_1 & 0 \\ 0 & 0 & \nu_2 \end{bmatrix} .$$

The condition (P1) is now equivalent to the requirement that $\mathbf{P}$ is of the following form

$$\mathbf{P} = \begin{bmatrix} a & d\gamma^2 & 0 \\ d & a & 0 \\ 0 & 0 & i \end{bmatrix} ,$$

and thus

$$\mathbf{M} = \begin{bmatrix} (1-2a)\nu_1 + 2d\gamma^2\nu_2 & -(1-2a)\gamma^2\nu_2 - 2d\gamma^2\nu_1 & 0 \\ -(1-2a)\gamma^2\nu_2 - 2d\gamma^2\nu_1 & (1-2a)\gamma^2\nu_1 + 2d\gamma^4\nu_2 & 0 \\ 0 & 0 & (1-2i)\nu_2 \end{bmatrix} ,$$

and (P2) transforms into the following inequalities, which should be valid on Γ:

$$(11) \qquad \begin{aligned} (1-2a)\nu_1 + 2d\gamma^2\nu_2 &\geqslant 0 , \\ (\nu_1^2 - \gamma^2\nu_2^2)[(1-2a)^2 - 4d^2\gamma^2] &\geqslant 0 , \\ (1-2i)\nu_2 &\geqslant 0 . \end{aligned}$$

Similarly to before, (P3) leads to several cases:
I. $\mathbf{A}_{\boldsymbol{\nu}}$ being regular implies that $\mathbf{P}$ is a projector, which in combination with (11) gives the following sub-cases:

I.a $a = 0 , \quad d = 0 , \quad i = 0 , \quad \nu_2 > 0 , \quad \nu_1 + \gamma\nu_2 > 0 , \quad \nu_1 - \gamma\nu_2 > 0 ;$
I.b $a = 0 , \quad d = 0 , \quad i = 1 , \quad \nu_2 < 0 , \quad \nu_1 + \gamma\nu_2 > 0 , \quad \nu_1 - \gamma\nu_2 > 0 ;$
I.c $a = 1 , \quad d = 0 , \quad i = 0 , \quad \nu_2 > 0 , \quad \nu_1 + \gamma\nu_2 < 0 , \quad \nu_1 - \gamma\nu_2 < 0 ;$
I.d $a = 1 , \quad d = 0 , \quad i = 1 , \quad \nu_2 < 0 , \quad \nu_1 + \gamma\nu_2 < 0 , \quad \nu_1 - \gamma\nu_2 < 0 ;$
I.e $a = \frac{1}{2} , \quad d = \frac{1}{2\gamma} , \quad i = 0 , \quad \nu_1 + \gamma\nu_2 > 0 , \quad \nu_1 - \gamma\nu_2 < 0 ;$
I.f $a = \frac{1}{2} , \quad d = -\frac{1}{2\gamma} , \quad i = 1 , \quad \nu_1 + \gamma\nu_2 < 0 , \quad \nu_1 - \gamma\nu_2 > 0 .$

II. The situation when $\mathbf{A}_{\boldsymbol{\nu}}$ is singular does not correspond just to the case of characteristic boundary, as $\det \mathbf{A}_{\boldsymbol{\nu}} = \nu_2\gamma^2(\nu_1 + \gamma\nu_2)(\nu_1 - \gamma\nu_2)$. Here we actually have the following sub-cases:

II.a $a = -\gamma d , \quad d \in \mathbf{R} , \quad i = 1 , \quad \nu_1 = \frac{\gamma}{\sqrt{1+\gamma^2}} , \quad \nu_2 = \frac{-1}{\sqrt{1+\gamma^2}} ;$
II.b $a = 1 - \gamma d , \quad d \in \mathbf{R} , \quad i = 0 , \quad \nu_1 = \frac{-\gamma}{\sqrt{1+\gamma^2}} , \quad \nu_2 = \frac{1}{\sqrt{1+\gamma^2}} ;$
II.c $a = \gamma d , \quad d \in \mathbf{R} , \quad i = 0 , \quad \nu_1 = \frac{\gamma}{\sqrt{1+\gamma^2}} , \quad \nu_2 = \frac{1}{\sqrt{1+\gamma^2}} ;$
II.d $a = 1 + \gamma d , \quad d \in \mathbf{R} , \quad i = 1 , \quad \nu_1 = \frac{-\gamma}{\sqrt{1+\gamma^2}} , \quad \nu_2 = \frac{-1}{\sqrt{1+\gamma^2}} ;$
II.e $a = 0 , \quad d = 0 , \quad i \in \mathbf{R} , \quad \nu_1 = 1 , \quad \nu_2 = 0 ;$
II.f $a = 1 , \quad d = 0 , \quad i \in \mathbf{R} , \quad \nu_1 = -1 , \quad \nu_2 = 0 ;$

The cases II.a–II.d correspond to the characteristic boundary, while in II.e and II.f we have $\nu_2 = 0$, which is the case when the boundary is parallel to the x–axis.

An example of the acceptable domain is presented in Figure 3.

Figure 3. An example of the domain Ω (three unknowns)

Similarly to what was done in the previous two cases, one can conclude that Γ must have parts that correspond to the characteristic boundary (cases II.a–II.d).

We can see that in cases I and II.a–II.d we have $i \in \{0, 1\}$, and in particular $i = 0$ when $\nu_2 > 0$, while $i = 1$ when $\nu_2 < 0$. Since i must be a Lipschitz function, the boundary Γ must have parts that correspond to cases II.e and II.f, where i can take arbitrary values. Therefore, we cannot treat the domain presented in Figure 2, which suggests that this representation is inferior to the other two.

Finally, let us write down what boundary conditions we impose on specific parts of boundary: since

$$\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M} = 2 \begin{bmatrix} a\nu_1 - d\gamma^2\nu_2 & -a\gamma^2\nu_2 + d\gamma^2\nu_1 & 0 \\ -a\gamma^2\nu_2 + d\gamma^2\nu_1 & a\gamma^2\nu_1 - d\gamma^4\nu_2 & 0 \\ 0 & 0 & i\nu_2 \end{bmatrix},$$

and $\mathsf{v} = (v_1, v_2, v_3)^\top = (e^{-\lambda t - \nu x}u_t, e^{-\lambda t - \nu x}u_x, e^{-\lambda t - \nu x}u)^\top$, we get the following boundary conditions for the wave equation at parts of the boundary (in some cases, one has to solve a system of linear equations in u_t, u_x and u)

I.a no boundary condition is imposed on this part of Γ;
I.b $u = 0$;
I.c $u_t = u_x = 0$;
I.d $u = u_t = u_x = 0$;
I.e $u_t + \gamma u_x = 0$;
I.f $u_t - \gamma u_x = 0$;
II.a $u = 0$;
II.b $u_t + \gamma u_x = 0$;
II.c no boundary condition is imposed on this part of Γ;
II.d $u = u_t - \gamma u_x = 0$.
II.e no boundary condition is imposed on this part of Γ;
II.f $u_t = u_x = 0$;

5. Some remarks on equations of mixed type

We have shown that the framework introduced in the third section encompasses both elliptic and hyperbolic equations. Let us now consider a linear second-order equation with variable coefficients:

$$(\alpha(x,y)u_x)_x + (\beta(x,y)u_y)_y + \gamma(x,y)u_x = f(x,y) \;,$$

in some bounded open set $\Omega \subseteq \mathbf{R}^2$ with Lipschitz boundary Γ. Here $\alpha, \beta \in \mathrm{W}^{1,\infty}(\Omega)$, $\gamma \in \mathrm{L}^\infty(\Omega)$ and $f \in \mathrm{L}^2(\Omega)$ are given. We allow that α and β change their sign in order to cover some equations of mixed type. By substituting $\mathsf{u} = (u_1, u_2)^\top = (e^{-\lambda x}u_x, e^{-\lambda x}u_y)^\top$ for some constant λ, our equation can be written as a symmetric system

$$\partial_x\Bigl(\begin{bmatrix}\alpha & 0\\ 0 & -\beta\end{bmatrix}\mathsf{u}\Bigr) + \partial_y\Bigl(\begin{bmatrix}0 & \beta\\ \beta & 0\end{bmatrix}\mathsf{u}\Bigr) + \begin{bmatrix}\gamma+\lambda\alpha & 0\\ -\beta_y & \beta_x-\lambda\beta\end{bmatrix}\mathsf{u} = \begin{bmatrix}e^{-\lambda x}f\\ 0\end{bmatrix} .$$

The positivity condition (F2) for this system is given by

$$\begin{aligned} 2\gamma + 2\lambda\alpha + \alpha_x &\geqslant 2\mu_0 \;,\\ \beta_x - 2\lambda\beta &\geqslant 2\mu_0 \;, \end{aligned} \tag{12}$$

while

$$\mathbf{A}_{\boldsymbol\nu} = \begin{bmatrix}\alpha\nu_1 & \beta\nu_2\\ \beta\nu_2 & -\beta\nu_1\end{bmatrix} .$$

Since the coefficients of our equation are functions that can change their sign, the conditions of Corollary 4 are now more technical to check. One usually has to pay special attention to situations where one of coefficients α and β is zero. Anyway, the condition (P1) will be satisfied for

$$\mathbf{P} = \begin{bmatrix}a & b\gamma^2\\ c & a\end{bmatrix} ,$$

with $c\beta = -b\alpha$, and then

$$\mathbf{M} = \begin{bmatrix}(1-2a)\alpha\nu_1 + 2b\alpha\nu_2 & (1-2a)\beta\nu_2 - 2b\alpha\nu_1\\ (1-2a)\beta\nu_2 - 2b\alpha\nu_1 & -(1-2a)\beta\nu_1 - 2b\beta\nu_2\end{bmatrix} ,$$

which turns (P2) into the following inequalities valid on Γ:

$$\begin{aligned} &(1-2a)\alpha\nu_1 + 2b\alpha\nu_2 \geqslant 0 \;,\\ &(\alpha\nu_1^2 + \beta\nu_2^2)[(1-2a)^2\beta + 4b^2\alpha] \geqslant 0 \;. \end{aligned} \tag{13}$$

To check (P3) we distinguish between $\mathbf{A}_{\boldsymbol\nu}$ regular and singular; for this, note that here we have $\det \mathbf{A}_{\boldsymbol\nu} = \beta(\alpha\nu_1^2 + \beta\nu_2^2)$.

However, in this case a procedure similar to the one applied for the wave equation would result in fifteen different sub-cases, and at the moment it is not clear whether a different representation of the equation as a Friedrichs system might lead to a more feasible discussion.

Clearly, the equations which change their type have been attacked by different methods (see [M1, M2] and [Ku] for a more recent review). We believe that the procedure described here will, after some additional modifications, lead to interesting new results.

Let us note that the feasibility of the approach via the trace operator is investigated in [ABV].

References

[AB1] Nenad Antonić, Krešimir Burazin: *Graph spaces of first-order linear partial differential operators*, *Math. Communications* **14**(1) (2009) 135–155.

[AB2] Nenad Antonić, Krešimir Burazin: *Intrinsic boundary conditions for Friedrichs systems*, *Comm. Partial Diff. Eq.* **35** (2010) 1690–1715.

[AB3] Nenad Antonić, Krešimir Burazin: *On equivalent descriptions of boundary conditions for Friedrichs systems*, *Math. Montisnigri* (in press).

[AB4] Nenad Antonić, Krešimir Burazin: *Boundary operator from matrix field formulation of boundary conditions for Friedrichs systems*, *J. Diff. Eq* **250** (2011) 36303651.

[ABV] Nenad Antonić, Krešimir Burazin, Marko Vrdoljak: *Connecting classical and abstract theory of Friedrichs systems via trace operator*, *ISRN Mathematical Analysis*, Volume 2011, Article ID 469795, 14 pages, doi:10.5402/2011/469795.

[B] Krešimir Burazin: *Contributions to the theory of Friedrichs' and hyperbolic systems* (in Croatian), Ph.D. thesis, University of Zagreb, 2008.

[CF] Richard Courant, Kurt O. Friedrichs: *Supersonic flow and shock waves*, Springer, 1976 (a reprint of 1948 edition).

[EG] Alexandre Ern, Jean-Luc Guermond: *Discontinuous Galerkin methods for Friedrichs' systems. I. General theory, SIAM J. Numer. Anal.* **44**(2) (2006) 753–778.

[EGC] Alexandre Ern, Jean-Luc Guermond, Gilbert Caplain: *An intrinsic criterion for the bijectivity of Hilbert operators related to Friedrichs' systems, Comm. Partial Diff. Eq.* **32** (2007) 317–341.

[F] Kurt O. Friedrichs: *Symmetric positive linear differential equations, Comm. Pure Appl. Math.* **11** (1958) 333–418.

[J] Max Jensen: *Discontinuous Galerkin methods for Friedrichs systems with irregular solutions*, Ph.D. thesis, University of Oxford, 2004.
`http://www.comlab.ox.ac.uk/research/na/thesis/thesisjensen.pdf`

[Ku] Alexander G. Kuz'min: *Boundary-value problems for transonic flow*, Wiley, 2002.

[Ma] Andrew Majda: *Compressible fluid flow and systems of conservation laws in several space variables*, Springer, 1984.

[MŠ] Vladimir G. Maz'ja, Tatyana O. Shaposhnikova: *Theory of Sobolev multipliers*, Springer, 2009.

[M1] Cathleen S. Morawetz: *A weak solution for a system of equations of elliptic-hyperbolic type, Comm. Pure Appl. Math.* **11** (1958) 315-331.

[M2] Cathleen S. Morawetz: *The Dirichlet problem for the Tricomi equation, Comm. Pure Appl. Math.* **23** (1970) 587–601.

[Ra] Jeffrey Rauch: *Boundary value problems with nonuniformly characteristic boundary, J. Math. Pures Appl.* **73** (1994) 347–353.