STRONGLY POSITIVE REPRESENTATIONS OF $GSpin_{2n+1}$ AND THE JACQUET MODULE METHOD WITH AN APPENDIX, 'STRONGLY POSITIVE REPRESENTATIONS IN AN EXCEPTIONAL RANK-ONE REDUCIBILITY CASE', BY IVAN MATIC

YEANSU KIM

ABSTRACT. We construct explicitly the structure of Jacquet modules of parabolically induced representations of $GSpin_{2n+1}$ over a p-adic field F of any characteristic. Using this construction of Jacquet module, we construct a classification of strongly positive representations of $GSpin_{2n+1}$ over F and describe the general discrete series representations of $GSpin_{2n+1}$ over F, assuming the half-integer conjecture. One of the applications of this paper will be to show the equality of L-functions from Langlands-Shahidi method and Artin L-functions through local Langlands correspondence [Y. Kim, Langlands-Shahidi L-functions for $GSpin$ groups and the generic Arthur packet conjecture, preprint].

1. INTRODUCTION

The classification of discrete series representations of connected reductive groups G over non-archimedean local field F is one of the important steps in local Langlands correspondence. Briefly, the local Langlands correspondence asserts that there exists a 'natural' bijection between two different sets of objects: Arithmetic (Galois or Weil-Deligne) side and analytic (representation theoretic) side. In the analytic side, the objects are irreducible admissible representations of connected reductive group over local field. To study admissible representations, we have the following filtration of admissible representation according to growth properties of matrix coefficients:

$$\text{supercuspidal} \implies \text{discrete series} \implies \text{tempered} \implies \text{admissible}. \tag{1.1}$$

Representations in each class are described in terms of representations induced from the previous class. In this paper, we study the first step (from supercuspidal representations to discrete series representations) which is called 'classification of discrete series representations'. This step was first proved in [33] for the general linear groups by Bernstein and Zelevinsky. After that, Moeglin and Tadic established the first step for classical groups [19, 20]. Recently, Ivan Matic has constructed the strongly positive representations of metaplectic groups using a purely algebraic approach [17].

The main purpose of the paper is to construct the classification of so-called strongly positive representations of odd $GSpin$ groups over non-archimedean local

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field F any characteristic, assuming half-integer conjecture ((HI) of [20], page 771 or Section 5.3). Remark that we can remove this assumption in the generic case due to Shahidi ([26]). This classification result generalizes Matic's algebraic approach to GSpin case. More precisely, let R (resp. R^{gen}) be the Grothendieck group of the category of all admissible representations of finite length of odd $GSpin$ groups (resp. GL) over F. We construct the following bijective mapping (Theorem 5.15):

Theorem A. There exists an bijective mapping between the set of all strongly positive representations in R and the set of induced representations of the following form:

$$(\prod_{i=1}^{k}\prod_{j=1}^{k_i}\delta([\nu^{a_{\rho_i}-k_i+j}\rho_i,\nu^{b_j^{(i)}}\rho_i]))\rtimes\sigma'$$

where

- σ' is an irreducible supercuspidal representation of $\mathbf{GSpin}_{2n+1}(F)$ in R,
- $\{\rho_1,\rho_2,\ldots,\rho_k\}\subset R^{gen}$ is a (possibly empty) set of mutually non-isomorphic irreducible essentially self-dual supercuspidal unitary representations of $\mathbf{GL}(F)$ such that $\mathrm{Ind}(\nu^{a_{\rho_i}}\rho_i\otimes\sigma')$ reduces for $a_{\rho_i}>0$ (this defines a_{ρ_i}),
- $k_i=\lceil a_{\rho_i}\rceil$,
- for each $i=1,2,\ldots,k, b_1^{(i)},b_2^{(i)},\ldots,b_{k_i}^{(i)}$ is a sequence of real numbers such that $a_{\rho_i}-b_j^{(i)}\in\mathbb{Z}$, for $j=1,2,\ldots,k_i$, and $-1<b_1^{(i)}<b_2^{(i)}<\cdots<b_{k_i}^{(i)}$.

Strongly positive representations are important class of discrete series representations and can be viewed as the basic building blocks for discrete series. We prove (Theorem 6.2)

Theorem B. Let σ denote a discrete series representation of $\mathbf{GSpin}_{2n+1}(F)$ in R. Then there exists an embedding of the form

$$\sigma\hookrightarrow\mathrm{Ind}(\delta([\nu^{a_1}\rho_1,\nu^{b_1}\rho_1])\otimes\delta([\nu^{a_2}\rho_2,\nu^{b_2}\rho_2])\otimes\cdots\otimes\delta([\nu^{a_r}\rho_r,\nu^{b_r}\rho_r])\otimes\sigma_{sp})$$

where $a_i\le 0, a_i+b_i>0$ and $\rho_i\in R^{gen}$ is an irreducible unitary supercuspidal representation of $\mathbf{GL}(F)$ for $i=1,2,\ldots,r$, with $\sigma_{sp}\in R$ a strongly positive representation of $\mathbf{GSpin}_{2n+1}(F)$ (we allow $k=0$).

The classification of strongly positive representations could be used in many problems in Langlands program. One of the them is to show the equality of L-functions through local Langlands correspondence ([16]). More precisely, in each side of local Langlands correspondence (i.e., arithmetic side and analytic side) we can define the L-functions. The L-functions from analytic side are defined by Shahidi (Langlands-Shahidi method) ([22, 23, 24, 25, 26, 27]) and the L-functions from arithmetic side are called Artin L-functions. The natural question is whether those two corresponding L-functions are equal through the local Langlands correspondence. Let us briefly explain the applications of the equality of L-functions. One of the applications of the equality of L-functions is the generic Arthur packet conjecture. The generic Arthur packet conjecture states that if the L-packet attached to Arthur parameter has a generic member, then it is tempered. This conjecture is first formulated in [28] for any connected reductive group and strengthened for classical groups and $GSpin$ groups in [16]. This conjecture can be considered as local version of Generalized Ramanujan conjecture.

The second purpose of the paper is to construct explicitly Tadic's structure formula using Jacquet module method which is one of the main tools to construct the classification of strongly positive representations of odd $GSpin$ groups over F. Tadic's structure formula study the Jacquet module of parabolically induced representations. More precisely, let $\mathbf{P}_{(k)} := \mathbf{M}_{(k)}\mathbf{N}_{(k)}$, where $\mathbf{M}_{(k)} \cong \mathbf{GL}_k \times \mathbf{G}_{n-k}$, be the standard maximal parabolic subgroup of $\mathbf{G}_n := GSpin_{2n+1}$ and let σ denote an irreducible representation of $\mathbf{G}_n(F)$ and let $r_{(k)}(\sigma)$ be the normalized Jacquet module with respect to $\mathbf{P}_{(k)}$. For such σ, we can also define $\mu^*(\sigma) \in R^{gen} \otimes R$ by $\mu^*(\sigma) = \sum_{k=0}^{n} s.s.(r_{(k)}(\sigma))$ (s.s. denotes the semisimplification) and extend μ^* linearly to the whole of R. We describe $\mu^*(\mathrm{Ind}(\pi \otimes \sigma))$ explicitly in Theorem 3.4. This description (Jacquet modules method) is very useful in the study of parabolically induced representations of connected reductive groups over p-adic field F especially when we construct the classification of strongly positive representations of odd $GSpin$ groups. Furthermore, the Jacquet module method can be used when we prove the irreducibility of certain induced representations (Section 4.1).

The paper is organized as follows. In Section 2, we recall the standard notation and preliminaries. In Section 3, we construct the Tadic's structure formula for odd $GSpin_{2n+1}$ (Theorem 3.4) which gives the explicit structure of Jacquet module of parabolically induced representation of $GSpin_{2n+1}$. In Section 4 we study the reducibility of certain parabolically induced representation of $GSpin_{2n+1}$ that we need for the classification of strongly positive representations and we also study the Weyl group action on the induced representations.

In Section 5, we construct the classification of strongly positive representations for $GSpin_{2n+1}$ (Theorem A). For simplicity, let us explain the special case $D(\rho; \sigma_{cusp})$ which denotes the set of strongly positive representations whose supercuspidal supports are the representation σ_{cusp} of $\mathbf{G}_n(F)$ and twists of the representation ρ of $\mathbf{GL}(F)$ by positive valued characters (Section 5.2). We first construct an injective mapping from $D(\rho; \sigma_{cusp})$ into the set of induced representation of the following form (Theorem 5.4):

$$\delta([\nu^{a_1}\rho, \nu^{b_1}\rho]) \times \delta([\nu^{a_2}\rho, \nu^{b_2}\rho]) \times \cdots \times \delta([\nu^{a_k}\rho, \nu^{b_k}\rho]) \rtimes \sigma_{cusp} \tag{1.2}$$

where

$$\begin{cases} a_i = a - k + i, \ b_1 < \ldots < b_k \ and \ k \leq \lceil a \rceil & when \quad a \in \frac{1}{2}\mathbb{Z} - \frac{1}{2}; \\ a_i = a \ and \ b_1 \leq \ldots \leq b_k & when \quad a = \frac{1}{2}. \end{cases}$$

(Here, a is the reducibility point determined by ρ and σ_{cusp}, i.e., $\mathrm{Ind}(\nu^s\rho \otimes \sigma_{cusp})$ is reducible if and only if $|s| = a$). We then show that the exact image of this injective mapping is given by the induced representation of the following form:

$$\delta([\nu^{a_1}\rho, \nu^{b_1}\rho]) \times \delta([\nu^{a_2}\rho, \nu^{b_2}\rho]) \times \cdots \times \delta([\nu^{a_k}\rho, \nu^{b_k}\rho]) \rtimes \sigma_{cusp} \tag{1.3}$$

where $a_i = a - k + i, \ b_1 < \ldots < b_k$ and $k \leq \lceil a \rceil$ for any $a \in \frac{1}{2}\mathbb{Z}$.

In other words, the proof of Theorem A in the special case $D(\rho; \sigma_{cusp})$ is equivalent to the commutativity of the following diagram:

$$\begin{array}{ccc} & & Jord^*_{(\rho,a)} \\ & \nearrow & \cup \\ D(\rho; \sigma_{cusp}) & \Longleftrightarrow & Jord_{(\rho,a)} \end{array}$$

where $Jord^*_{(\rho,a)}$ (resp. $Jord_{(\rho,a)}$) be the set of data that corresponds to induced representation of the form (1.2) (resp. (1.3)) (See page 17 of Section 5.2 for more detail).

In Section 6, We describe the general discrete series representations using Casselman's square integrability criterion [14] and Theorem A (Theorem B).

2. Notation and preliminaries

2.1. **Notation.** Let F be a non-archimedean local field of characteristic zero and let $\mathbf{G}_n$ be a general spin group $GSpin_{2n+1}$ of semisimple rank n defined over F. The Grothendieck group of the category of all admissible representations of finite length of $\mathbf{G}_n(F)$, i.e., a free abelian group over the set of all irreducible representations of $\mathbf{G}_n(F)$ (resp. $\mathbf{GL}_n(F)$) is denoted by $R(n)$ (resp. $R(GL_n)^{gen}$) and set $R = \underset{n\geq 0}{\oplus} R(n)$, $R^{gen} = \underset{n\geq 0}{\oplus} R(GL_n)^{gen}$. Let $\mathbf{s} = (n_1, n_2, \dots, n_k)$ be an ordered partition of some n' such that $n' \leq n$. Let $\mathbf{P_s} = \mathbf{M_s N_s}$ denote the standard parabolic subgroup of $\mathbf{G}_n$ that corresponds to the partition $\mathbf{s}$. The Levi factor $\mathbf{M_s}$ is isomoprhic to $\mathbf{GL}_{n_1} \times \mathbf{GL}_{n_2} \times \cdots \times \mathbf{GL}_{n_k} \times \mathbf{G}_{n-n'}$ (see [1]). Let ν be a character of $\mathbf{GL}_n(F)$ defined by $|\det|_F$. We denote the induced representation $\mathrm{Ind}_{\mathbf{P}}^{\mathbf{G}_n}(\rho_1 \otimes \cdots \otimes \rho_k \otimes \tau)$ by

$$\rho_1 \times \cdots \times \rho_k \rtimes \tau$$

where each ρ_i (resp. τ) is a representation of some $\mathbf{GL}_{n_i}(F)$ (resp. $\mathbf{G}_n(F)$). In particular, $\mathrm{Ind}_{\mathbf{P_s}}^{\mathbf{G}_n}$ is a functor from admissible representations of $\mathbf{M_s}(F)$ to admissible representations of $\mathbf{G}_n(F)$ that sends unitary representations to unitary representations. We also denote the normalized Jacquet module with respect to $\mathbf{P_s}$ by $r_{\mathbf{s}}(\tau)$. In particular, $r_{\mathbf{s}}$ is a functor from admissible representations of $\mathbf{G}_n(F)$ to admissible representations of $\mathbf{M_s}(F)$.

In the case of $\mathbf{GL}$, we denote the induced representation $\mathrm{Ind}_{\mathbf{P}'}^{\mathbf{GL}_n}(\rho_1 \otimes \cdots \otimes \rho_k)$ by

$$\rho_1 \times \cdots \times \rho_k$$

where $\mathbf{P}' = \mathbf{M}'\mathbf{N}'$ is the standard parabolic subgroup of $\mathbf{GL}_n$ where $\mathbf{M}' \cong \mathbf{GL}_{n_1} \times \mathbf{GL}_{n_2} \times \cdots \times \mathbf{GL}_{n_k}$ and each ρ_i is a representation of some $\mathbf{GL}_{n_i}(F)$. We also follow the notation in [7]. Let ρ is an irreducible unitary supercuspidal representation of some $\mathbf{GL}_p(F)$. We define the segment, $\Delta := [\nu^a\rho, \nu^{a+k}\rho] = \{\nu^a\rho, \nu^{a+1}\rho, \dots \nu^{a+k}\rho\}$ where $a \in \mathbb{R}$ and $k \in \mathbb{Z}_{\geq 0}$. If $a > 0$, we call the segment Δ strongly positive.

2.2. **Preliminaries.** Let us first introduce the strongly positive representations which is the main object in this paper.

Definition 2.1 (Strongly positive)**.** An irreducible representation $\sigma \in R$ is called strongly positive if for each representation $\nu^{s_1}\rho_1 \times \nu^{s_2}\rho_2 \times \cdots \times \nu^{s_k}\rho_k \rtimes \sigma_{cusp}$, where $\rho_i, i = 1, 2, \cdots, k$, are irreducible supercuspidal unitary representations of some $\mathbf{GL}_{n_i}(F)$, $\sigma_{cusp} \in R$ an irreducible supercuspidal representation and $s_i \in \mathbb{R}$, $i = 1, 2, \cdots, k$, such that

$$\sigma \hookrightarrow \nu^{s_1}\rho_1 \times \nu^{s_2}\rho_2 \times \cdots \times \nu^{s_k}\rho_k \rtimes \sigma_{cusp},$$

we have $s_i > 0$ for each i.

Remark 2.2. It is easy to see that strongly positive representations is discrete series using Casselman's square integrability criterion in [14]. Therefore, strongly positive representations are often called strongly positive discrete series representations.

One of the main tools of this paper is Jacquet module method (Tadic's structure formula, Theorem 3.4). In this paper, we consider the following Jacquet module with respect to maximal parabolic subgroups.

Definition 2.3 (Jacquet module)**.** Let $\mathbf{P}_{(k)} := \mathbf{M}_{(k)}\mathbf{N}_{(k)}$, where $\mathbf{M}_{(k)} \cong \mathbf{GL}_k \times \mathbf{G}_{n-k}$, be the standard maximal parabolic subgroup of $\mathbf{G}_n$ and let σ denote an irreducible representation of $\mathbf{G}_n(F)$. We denote by $r_{(k)}(\sigma)$ the normalized Jacquet module with respect to $\mathbf{P}_{(k)}$. The Jacquet module $r_{(k)}(\sigma)$ can be interpreted as a representation of $\mathbf{GL}_k(F) \times \mathbf{G}_{n-k}(F)$, i.e., is an element of $R^{gen} \otimes R$.

Let us recall structure theory for $GSpin$ groups which are studied by Asgari and Shahidi [1, 2, 3].

Definition 2.4. [1, 2, 3] The odd $GSpin$ groups $\mathbf{G}_n := GSpin_{2n+1}$ are reductive algebraic groups of type B_n whose derived groups are double coverings of special orthogonal groups. Furthermore, the connected component of their Langlands dual groups are $\mathbf{GSp}_{2n}(\mathbb{C})$.

Proposition 2.5. [1, 2, 3] *The root datum $(X, R, X^\vee, R^\vee)$ of $\boldsymbol{G}_n$ can be described as the following. $X = \mathbb{Z}e_0 \oplus \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n$, $X^\vee = \mathbb{Z}e_0^* \oplus \mathbb{Z}e_1^* \oplus \cdots \oplus \mathbb{Z}e_n^*$. (There is a standard $\mathbb{Z}$-pairing $<,>$ on $X \times X^\vee$.) And R and $R^\vee$ are generated, respectively, by $\Delta = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \cdots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_n\}$, $\Delta^\vee = \{\alpha_1^\vee = e_1^* - e_2^*, \alpha_2^\vee = e_2^* - e_3^*, \cdots, \alpha_{n-1}^\vee = e_{n-1}^* - e_n^*, \alpha_n^\vee = 2e_n^* - e_0^*\}$,*

Remark 2.6. [1, 2, 3] The root datum of $\mathbf{G}_n := GSpin_{2n+1}$ is the dual root datum to the one for the group $\mathbf{GSp}_{2n}$.

Let us recall the properties of discrete series representations in [1, 14, 15].

Proposition 2.7 ([1, 14, 15])**.** *Let $M = \boldsymbol{GL}_{n_1} \times \boldsymbol{GL}_{n_2} \times \cdots \times \boldsymbol{GL}_{n_k} \times \boldsymbol{G}_{n-n'} \subset \boldsymbol{G}_n$. Let ρ_i be a supercuspidal representation of $\boldsymbol{GL}_{n_i}(F)$ and let τ be a generic supercuspidal representation of $\boldsymbol{G}_{n-n'}(F)$. Write $\rho_i = \nu^{e(\rho_i)}\rho_i^u$ where $e(\rho_i) \in \mathbb{R}$ and ρ_i^u is unitary supercuspidal representation. If $\rho_1 \times \cdots \times \rho_k \rtimes \tau$ has a discrete series subrepresentation, then*

(i) $\rho_i^u \cong \widetilde{\rho_i}^u \otimes (\omega_\tau \circ \det)$.
(ii) $2e(\rho_i) \in \mathbb{Z}$ *for each* $i = 1, \ldots, k$.

Remark 2.8. The Proposition 2.7(i) is still true when τ is non-generic representations and the proof is exactly same as the proof of Proposition 2.7. However, to derive Proposition 2.7(ii) in the case when τ is non-generic representations, we need to assume the half-integer conjecture, i.e., the reducibility points are half-integers. Then slight variation of the proof of Proposition 2.7 applies to the non-generic case.

The following corollary is useful in Section 5:

Corollary 2.9. *If $\rho_1 \times \cdots \times \rho_k \rtimes \tau$ has a discrete series subrepresentation and $\rho_k \rtimes \tau$ is irreducible, then*

$$\nu^{e(\rho_k)}\rho_k^u \rtimes \tau \cong \nu^{-e(\rho_k)}\rho_k^u \rtimes \tau,$$

where $\rho_k \cong \nu^{e(\rho_k)}\rho_k^u$ and ρ_k^u is unitary.

Let us conclude this Section by recalling two results in the case of general linear groups. The following theorem is the classification of discrete series representations of general linear groups ([33]):

Theorem 2.10 (Bernstein, [33]). *Let ρ be an irreducible supercuspidal representation of $\boldsymbol{GL}(F)$. We note that the induced representation $\nu^{a+k}\rho\times\nu^{a+k-1}\rho\times\cdots\times\nu^{a}\rho$ has a unique irreducible subrepresentation, which we denote by $\delta(\Delta)$, and a unique irreducible quotient, which we denote by $\mathfrak{s}(\Delta)$. The $\delta(\Delta)$ is an essentially square-integrable representation attached to Δ (see [33], 3.1).*

Let us briefly review the Langlands classification for general linear groups. For every irreducible essentially square-integrable representation δ of some $\mathbf{GL}_n(F)$, there exists a unique $e(\delta)\in\mathbb{R}$ such that the representation $\nu^{-e(\delta)}\delta$ is unitarizable. Suppose $\delta_1,\delta_2,\ldots,\delta_k$ are irreducible essentially square-integrable representations of $\mathbf{GL}_{n_1}(F)$, $\mathbf{GL}_{n_2}(F)$, $\cdots$, $\mathbf{GL}_{n_k}(F)$ with $e(\delta_1)\le e(\delta_2)\le\cdots\le e(\delta_k)$. Then the induced representation $\delta_1\times\delta_2\times\cdots\times\delta_k$ has a unique irreducible subrepresentation, which we denote by $L(\delta_1,\delta_2,\ldots,\delta_k)$. This irreducible subrepresentation is called the Langlands subrepresentation, and it appears with the multiplicity one in $\delta_1\times\delta_2\times\cdots\times\delta_k$. Every irreducible representation π of $\mathbf{GL}_n(F)$ is isomorphic to some $L(\delta_1,\delta_2,\ldots,\delta_k)$. Given π, the representations $\delta_1,\delta_2,\ldots,\delta_k$ are unique up to a permutation. If $i_1,i_2,\ldots,i_k$ is a permutation of $1,2,\ldots,k$ such that the representations $\delta_{i_1}\times\cdots\times\delta_{i_k}$ and $\delta_1\times\cdots\times\delta_k$ are isomorphic, we also write $L(\delta_{i_1},\delta_{i_2},\ldots,\delta_{i_k})$ for $L(\delta_1,\delta_2,\ldots,\delta_k)$.

3. Tadic's construction on Jacquet module

In this section, we construct explicitly the structure of Jacquet modules of parabolically induced representations of $\mathbf{G}_n := GSpin_{2n+1}$ over F (Tadic's structure formula, Theorem 3.4).

Tadic's structure formula for $\mathbf{SO}_{2n+1}(F)$ ([30]) enables us to calculate Jacquet modules of an induced representation in our case. Let's construct Tadic's structure formula for $\mathbf{G}_n(F)$ by using the well known fact that the Weyl group of $\mathbf{G}_n(F)$ and that of $\mathbf{SO}_{2n+1}(F)$ are same and are isomorphic to $S_n\rtimes\{\pm1\}^n$.

Let $(p,\epsilon)\in S_n\rtimes\{\pm1\}^n$ with $\epsilon=(\epsilon_1,\ldots,\epsilon_n)\in\{\pm1\}^n$. We can identify (p,ϵ) with $p\cdot\epsilon\in W_{\mathbf{SO}_{2n+1}(F)}$ where the action by conjugation of p and $\epsilon\in W_{\mathbf{SO}_{2n+1}(F)}$ on the standard maximal torus in $\mathbf{SO}_{2n+1}(F)$ can be defined by $p\cdot diag(x_1,\ldots,x_n,1,x_n^{-1},\ldots,x_1^{-1})=diag(x_{p^{-1}(1)},\ldots,x_{p^{-1}(n)},1,x_{p^{-1}(n)}^{-1},\ldots,x_{p^{-1}(1)}^{-1})$ and $\epsilon\cdot diag(x_1,\ldots,x_n,1,x_n^{-1},\ldots,x_1^{-1})=diag(x_1^{\epsilon_1},\ldots,x_n^{\epsilon_n},1,x_n^{-\epsilon_n},\ldots,x_1^{-\epsilon_1})$.

Using the previous action of Weyl group elements on the maximal torus, we can also get the action of those on the roots (see also [11]).

Lemma 3.1. *Let $e_0,e_1,\cdots,e_n$ (resp. $e'_0,\cdots,e'_n$) be the standard basis of character lattice (resp. the cocharacter lattice) of $\boldsymbol{G}_n$ as in Proposition 2.5 and let $(p,\epsilon)\in S_n\rtimes\{\pm1\}^n$ be as above.*

Then

$$(p,\epsilon)\cdot e_i=\begin{cases} e_{p(i)} & \text{for } i>0,\epsilon_i=1;\\ -e_{p(i)} & \text{for } i>0,\epsilon_i=-1;\\ e_0+\sum\limits_{\epsilon_i=-1}e_{p(i)} & \text{for } i=0.\end{cases}$$

$$(p,\epsilon)\cdot e'_i = \begin{cases} e'_{p(i)} & \textit{for } i>0, \epsilon_i = 1; \\ e'_0 - e'_{p(i)} & \textit{for } i>0, \epsilon_i=-1; \\ e'_0 & \textit{for } i=0. \end{cases}$$

Proof. We can calculate $(p,\epsilon)\cdot e'_i$ directly from the matrix calculation since $e'_0, \cdots, e'_n$ are the character lattice of $\mathbf{GSp}$. We can easily calculate $(p,\epsilon)\cdot e_i$ for $i>0$ using previous action. For $(p,\epsilon)\cdot e_0$, we need to use the duality of e_i and e'_i. □

Let $\Delta_{\mathbf{G}_n} := \{\alpha_1 = e_1 - e_2, \cdots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_n\}$ (see [1] for more details) be the simple roots for $\mathbf{G}_n$. From Lemma 3.1, we can calculate the action of (p,ϵ) on the simple roots in R.

Corollary 3.2. *With notation as in Lemma 3.1.*

$$(p,\epsilon)\cdot\alpha_i = \begin{cases} \epsilon_i e_{p(i)} - \epsilon_{i+1}e_{p(i+1)} & \textit{for } 0\le i\le n-1; \\ \epsilon_n e_{p(n)} & \textit{for } i=n. \end{cases}$$

Remark 3.3. The Weyl group action on the simple roots for *GSpin* groups is exactly same as that for special orthogonal groups. In [30, chapter 4], the author characterize the representative element of the set $[W_{\Delta\backslash\alpha}\backslash W/W_{\Delta\backslash\beta}]$ and its explicit action on the simple roots for $\mathbf{SO}_{2n+1}$. We can also construct the same result, i.e., from Lemma 4.1 through Lemma 4.8 of [30] in the case of odd *GSpin* groups since those lemmas depend on the simple roots, Weyl group and its action on the simple roots. This result enables us to prove the Tadic's structure formula for $\mathbf{G}_n$.

Now we are ready to construct the Jacquet modules of induced representations for $\mathbf{G}_n$. I follow the notation in [30]. Let i_1, i_2 be integers which satisfy $1\le i_1, i_2\le n$. Take an integer d such that $0\le d\le min\{i_1,i_2\}$. Suppose that an integer k satisfies $max\{0,(i_1+i_2-n)-d\}\le k\le min\{i_1,i_2\}-d$. Let $p_n(d,k)_{i_1,i_2}\in S_n$ be defined by

$$p_n(d,k)_{i_1,i_2}(j) = \begin{cases} j & \text{for } 1\le j\le k; \\ j+i_1-k & \text{for } k+1\le j\le i_2-d; \\ (i_1+i_2-d+1)-j & \text{for } i_2-d+1\le j\le i_2; \\ j-i_2+k & \text{for } i_2+1\le j\le i_1+i_2-d-k; \\ j & \text{for } i_1+i_2-d-k+1\le j\le n. \end{cases}$$

Let $q_n(d,k)_{i_1,i_2}$ be $(p_n(d,k)_{i_1,i_2}, (\mathbf{1}_{i_2-d}, -\mathbf{1}_d, \mathbf{1}_{n-i_2}))$ where $\mathbf{1}_i = 1,\ldots,1$ (1 appears i times). Let $w = q_n(d,k)_{i_1,i_2}$. Then, for $(g_1,g_2,g_3,g_4,h)\in \mathbf{GL}_k(F)\times \mathbf{GL}_{i_2-d-k}(F)\times\mathbf{GL}_d(F)\times\mathbf{GL}_{i_1-d-k}(F)\times\mathbf{G}_{n-i_1-i_2+d+k}(F)$, note that $w\cdot(g_1,g_2,g_3,g_4,h) = (g_1, g_4, {}^\tau g_3^{-1}, g_2, \det(g_3)h)$.

Let π_i be an irreducible smooth representation of $\mathbf{GL}_{n_i}(F)$ for $i=1,2,3,4$. Let σ be an irreducible smooth representation of $\mathbf{G}_m$ and ω_σ is the central character of σ. By our previous calculation,

$$w^{-1}\cdot(\pi_1\otimes\pi_2\otimes\pi_3\otimes\pi_4\otimes\sigma) = \pi_1\otimes\pi_4\otimes(\widetilde{\pi_3}\otimes(\omega_\sigma\circ\det))\otimes\pi_2\otimes\sigma. \tag{3.1}$$

Set

$$(\pi_1\otimes\pi_2\otimes\pi_3)\,\widetilde{\rtimes}\,(\pi_4\otimes\sigma) = (\widetilde{\pi_1}\otimes(\omega_\sigma\circ\det))\times\pi_2\times\pi_4\otimes\pi_3\rtimes\sigma. \tag{3.2}$$

One extends $\widetilde{\rtimes}$ to a $\mathbb{Z}$-bilinear mapping $\widetilde{\rtimes} : (R^{gen}\otimes R^{gen}\otimes R^{gen})\times(R^{gen}\otimes R)$. We denote by $\mathbf{m}$ the linear extension to $R^{gen}\otimes R^{gen}$ of parabolic induction from a maximal parabolic subgroup. Let σ denote an irreducible representation of $\mathbf{G}_n(F)$. From definition $r_{(k)}(\sigma)$, the normalized Jacquet module of σ with respect to the standard maximal parabolic subgroup $\mathbf{P}_{(k)} = \mathbf{M}_{(k)}\mathbf{N}_{(k)}$, can be interpreted as a representation of $\mathbf{GL}_k(F) \times \mathbf{G}_{n-k}(F)$, i.e., is an element of $R^{gen} \otimes R$. For such σ we can define $\mu^*(\sigma) \in R^{gen} \otimes R$ by $\mu^*(\sigma) = \sum_{k=0}^{n} s.s.(r_{(k)}(\sigma))$ (s.s. denotes the semisimplification) and extend μ^* linearly to the whole of R. Let π be a representation of $\mathbf{GL}_k(F)$ and let σ_{sc} be a supercuspidal representation of $\mathbf{G}_n(F)$. Suppose that τ is a subquotient of $\pi \times \sigma_{sc}$. Then we shall denote $r_{(k)}(\tau)$ by $r_{GL}(\tau)$.

Using Jacquet modules with respect to the maximal parabolic subgroups of $\mathbf{GL}_n$, we can also define $\mathbf{m}^*(\pi) = \sum_{k=0}^{n} s.s.(r_k(\pi)) \in R^{gen} \otimes R^{gen}$, for an irreducible representation π of $\mathbf{GL}_n(F)$, and then extend $\mathbf{m}^*$ linearly to the whole of R^{gen}. Here $r_k(\pi)$ denotes Jacquet module of the representation π with respect to parabolic subgroup whose Levi subgroup is $\mathbf{GL}_k \times \mathbf{GL}_{n-k}$. We define $\mathbf{s} : R^{gen} \otimes R^{gen} \to R^{gen} \otimes R^{gen}$ by $\mathbf{s}(x \otimes y) = y \otimes x$. Let $\mathfrak{M}^* : R^{gen} \to R^{gen} \otimes R^{gen} \otimes R^{gen}$ be defined by $\mathfrak{M}^* = (1 \otimes \mathbf{m}^*) \circ \mathbf{s} \circ \mathbf{m}^*$.

The following theorem (Tadic's structure formula for odd $GSpin$ groups) is fundamental for our calculations with Jacquet modules:

Theorem 3.4. *For $\pi \in R(\boldsymbol{GL}_i)^{gen}$ and $\sigma \in R(n-i)$, the following structure formula holds*

$$\mu^*(\pi \rtimes \sigma) = \mathfrak{M}^*(\pi) \widetilde{\rtimes} \mu^*(\sigma).$$

Proof. Let us sketch the proof and explain how we can adapt the approach in [30] to our case. Write $\mu^*(\pi \rtimes \sigma) = \sum_{m=0}^{n} A'_m \in \sum_{m=0}^{n} R(\mathbf{GL}_m)^{gen} \otimes R(n-m)$ where $A'_m = s.s.(r_{(m)}(\pi \rtimes \sigma))$. Using the $GSpin$ version of [30, Proposition 4.6] (see Remark 3.3) and Weyl group action (3.1), we can calculate A'_m explicitly as in [30, page 25]. We remark that $\omega_\sigma \circ \det$ appears after $\widetilde{\pi_3}$ when the Weyl group element $w^{-1} = q_n(d,k)^{-1}_{i_1,i_2}$ acts on the representation (see (3.1)). Accordingly, we need to define $\widetilde{\rtimes}$ as (3.2). This forces $\mu^*(\pi \rtimes \sigma)$ be equal to $\mathfrak{M}^*(\pi) \widetilde{\rtimes} \mu^*(\sigma)$ after changing index several times as in the proof of [30, Theorem 5.2]. $\square$

Using the previous theorem, we obtain

Lemma 3.5. *Let ρ be an irreducible supercuspidal representation of $\boldsymbol{GL}_k(F)$ and $a, b \in \mathbb{R}$ be such that $b - a \in \mathbb{Z}_{\geq 0}$. Let σ be an admissible representation of finite length of $\boldsymbol{G}_n(F)$. Write $\mu^*(\sigma) = \sum_{\pi', \sigma'} \pi' \otimes \sigma'$. Then $\mathfrak{M}^*(\delta([\nu^a\rho, \nu^b\rho])) = \sum_{i=a-1}^{b} \sum_{j=i}^{b} \delta([\nu^a\rho, \nu^i\rho]) \otimes \delta([\nu^{j+1}\rho, \nu^b\rho]) \otimes \delta([\nu^{i+1}\rho, \nu^j\rho])$ and $\mu^*(\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma) = \sum_{i=a-1}^{b} \sum_{j=i}^{b} \sum_{\pi', \sigma'} \delta([\nu^{-i}\widetilde{\rho} \otimes (\omega_{\sigma'} \circ \det), \nu^{-a}\widetilde{\rho} \otimes (\omega_{\sigma'} \circ \det)]) \times \delta([\nu^{j+1}\rho, \nu^b\rho]) \times \pi' \otimes \delta([\nu^{i+1}\rho, \nu^j\rho]) \rtimes \sigma'$. We omit $\delta([\nu^x\rho, \nu^y\rho])$ if $x > y$.*

We also use $m^*(\delta([\nu^a\rho, \nu^b\rho])) = \sum_{i=a-1}^{b} \delta([\nu^{i+1}\rho, \nu^b\rho]) \otimes \delta([\nu^a\rho, \nu^i\rho])$ and $m^*(\prod_{j=1}^{n} \delta([\nu^{a_j}\rho_j, \nu^{b_j}\rho_j])) = \prod_{j=1}^{n} (\sum_{i_j=a_j-1}^{b_j} \delta([\nu^{i_j+1}\rho_j, \nu^{b_j}\rho_j]) \otimes \delta([\nu^{a_j}\rho_j, \nu^{i_j}\rho_j]))$. ((1.3) of [31]).

Remark 3.6. In the case of even GSpin groups, Tadic's structure formula (Theorem 3.4) is different since Weyl group is different ($S_n \rtimes \{\pm 1\}^{n-1}$). This will be addressed in our future article soon using the structure of even special orthogonal groups ([4, 12]).

4. The reducibility of parabolic induction

In this Section, we study the reducibility of certain parabolically induced representations that are needed in Section 5.

4.1. **The reducibility of parabolic induction.** We first consider the representation $\nu^{\beta}\rho \rtimes \delta(\nu^{\beta}\rho, \sigma)$.

Lemma 4.1. *Let ρ be an irreducible unitarizable supercuspidal representation of $\boldsymbol{GL}_p(F)$ and let σ be an irreducible supercuspidal representation of $\boldsymbol{G}_m(F)$. Suppose that $\beta > 1/2$ is in $(1/2)\mathbb{Z}$ and that $\nu^{\beta}\rho \rtimes \sigma$ reduces. Then $\nu^{\beta}\rho \rtimes \delta(\nu^{\beta}\rho, \sigma)$ is irreducible.*

In the proof of Lemma 4.1 we shall need the following lemma:

Sublemma 4.2. Let ρ be an irreducible unitarizable supercuspidal representation of the group $\mathbf{GL}_p(F)$ and let σ be an irreducible supercuspidal representation of $\mathbf{G}_m(F)$. Suppose that $\nu^{\alpha}\rho \rtimes \sigma$ reduces for some $\alpha > 0$. Then

(i) $\rho \cong \widetilde{\rho}w_{\sigma}$.

(ii) The representation $\nu^{\alpha+n}\rho \times \nu^{\alpha+n-1}\rho \times \cdots \times \nu^{\alpha}\rho \rtimes \sigma$, $n \geq 0$, has a unique irreducible subrepresentation which we denote by $\delta([\nu^{\alpha}\rho, \nu^{\alpha+n}\rho], \sigma)$.

(iii) We have $r_{(p)^{n+1}}(\delta([\nu^{\alpha}\rho, \nu^{\alpha+n}\rho], \sigma)) = \nu^{\alpha+n}\rho \otimes \nu^{\alpha+n-1}\rho \otimes \cdots \otimes \nu^{\alpha}\rho \otimes \sigma$.

(iv) If τ is an irreducible representation of $\mathbf{G}_{p(n+1)+m}(F)$ such that $\nu^{\alpha+n}\rho \otimes \nu^{\alpha+n-1}\rho \otimes \cdots \otimes \nu^{\alpha+1}\rho \otimes \nu^{\alpha}\rho \otimes \sigma$ is a subquotient of $r_{(p)^{n+1}}(\tau)$, then $\tau \cong \delta([\nu^{\alpha}\rho, \nu^{\alpha+n}\rho], \sigma)$.

(v) We have $\mu^*(\delta([\nu^{\alpha}\rho, \nu^{\alpha+n}\rho], \sigma)) = \sum_{k=-1}^{n} \delta([\nu^{\alpha+k+1}\rho, \nu^{\alpha+n}\rho]) \otimes \delta([\nu^{\alpha}\rho, \nu^{\alpha+k}\rho], \sigma)$, where we assume that $\delta(\emptyset, \sigma) = \sigma$ in the above formula.

(vi) The representation $\nu^{\alpha+n}\rho \times \nu^{\alpha+n-1}\rho \times \cdots \times \nu^{\alpha+1}\rho \times \nu^{\alpha}\rho \rtimes \sigma$ is regular. Here, we shall say that $\mathrm{Ind}_P^G(\rho')$ for given supercuspidal representation ρ' of M is regular representation if the Jacquet module of $\mathrm{Ind}_P^G(\rho')$ with respect to P is a multiplicity one representation.

Proof. Since $r_{(p)}(\nu^{\alpha}\rho \rtimes \sigma) = \nu^{\alpha}\rho \otimes \sigma + \nu^{-\alpha}\widetilde{\rho}\omega_{\sigma} \otimes \sigma$, $\nu^{\alpha}\rho \rtimes \sigma$ has two irreducible subquotients π_1 and π_2 such that $r_{(p)}(\pi_1) = \nu^{\alpha}\rho \otimes \sigma$ and $r_{(p)}(\pi_2) = \nu^{-\alpha}\widetilde{\rho}\omega_{\sigma} \otimes \sigma$. Therefore, Casselman's square integrable criterion for *GSpin* groups ([14, Proposition 3.8 and 3.9]) implies that π_1 is discrete series representations. Furthermore, π_1 can be embedded into $\nu^{\alpha}\rho \rtimes \sigma$. Therefore, Proposition 2.7 and Remark 2.8 imply (i). Remark that Ban and Goldberg also prove (i) in [5]. Using the Tadic's structure formula for Jacquet module, i.e., Theorem 3.4, $s.s.(r_{(p(n+1))}(\nu^{\alpha+n}\rho \times \cdots \times \nu^{\alpha}\rho \rtimes \sigma)) = \sum_{(\epsilon_i) \in \{\pm 1\}^{n+1}} \nu^{\epsilon_n(\alpha+n)}\rho \times \cdots \times \nu^{\epsilon_0 \alpha}\rho \otimes \sigma$ and the transitivity of Jacquet modules and [33] implies $s.s.(r_{(p)^{n+1}}(\nu^{\alpha+n}\rho \times \cdots \times \nu^{\alpha}\rho \rtimes \sigma)) = \sum_{(\epsilon_i) \in \{\pm 1\}^{n+1}} \sum_{p \in S_{\{0,1,\ldots,n\}}} \nu^{\epsilon_n(\alpha+p(n))}\rho \otimes$

$\cdots\otimes\nu^{\epsilon_0(\alpha+p(0))}\rho\otimes\sigma$. This Jacquet module is of length $2^{n+1}(n+1)!$ and is a multiplicity one representation. This proves (vi). Every irreducible subrepresentation of $\nu^{\alpha+n}\rho\times\cdots\times\nu^{\alpha}\rho\rtimes\sigma$ has $\nu^{\alpha+n}\rho\otimes\cdots\otimes\nu^{\alpha}\rho\otimes\sigma$ for a subquotient of the corresponding Jacquet module by Frobenius reciprocity. The regularity of $\nu^{\alpha+n}\rho\times\cdots\times\nu^{\alpha}\rho\rtimes\sigma$ and exactness of Jacquet modules imply that $\nu^{\alpha+n}\rho\times\cdots\times\nu^{\alpha}\rho\rtimes\sigma$ has a unique irreducible subrepresentation which we denote by $\delta([\nu^{\alpha}\rho,\nu^{\alpha+n}\rho],\sigma)$. This proves (ii).

Suppose that an irreducible representation τ of $\mathbf{G}_{p(n+1)+m}(F)$ has $\nu^{\alpha+n}\rho\otimes\cdots\otimes\nu^{\alpha}\rho\otimes\sigma$ for a subquotient of $r_{(p)^{n+1}}(\tau)$. Frobenius reciprocity implies that τ is a subrepresentation of some $Ind_{P_\alpha}^{\mathbf{G}_{p(n+1)+m}(F)}(\sigma')$ where σ' and $\nu^{\alpha+n}\rho\otimes\cdots\otimes\nu^{\alpha}\rho\otimes\sigma$ are associate. Since σ' and $\nu^{\alpha+n}\rho\otimes\cdots\otimes\nu^{\alpha}\rho\otimes\sigma$ are associate, $Ind_{P_\alpha}^{\mathbf{G}_{p(n+1)+m}(F)}(\sigma')$ and $\nu^{\alpha+n}\rho\times\cdots\times\nu^{\alpha}\rho\rtimes\sigma$ have the same composition factors. This implies $\tau\cong\delta([\nu^{\alpha}\rho,\nu^{\alpha+n}\rho],\sigma)$.

We prove (iii) and (v) by induction on n.

When $n=0$, the Frobenius reciprocity implies that $s.s.(r_{(p)}(\delta(\nu^{\alpha}\rho,\sigma)))\geq\nu^{\alpha}\rho\otimes\sigma$. Since $\mu^*(\nu^{\alpha}\rho\rtimes\sigma)=(\nu^{-\alpha}\widetilde{\rho}\otimes(\omega_\sigma\circ\det))\otimes\sigma+\nu^{\alpha}\rho\otimes\sigma+\nu^{\alpha}\rho\rtimes\sigma$ and $\nu^{\alpha}\rho\rtimes\sigma$ is reducible, $r_{(p)}(\delta(\nu^{\alpha}\rho,\sigma))=\nu^{\alpha}\rho\otimes\sigma$. This proves (iii) for $n=0$. Then (v) follows from (iii) in the case $n=0$.

The Frobenius reciprocity implies that $\nu^{\alpha}\rho\otimes\sigma$ is a quotient of $r_{(p)}(\delta(\nu^{\alpha}\rho,\sigma))$. The regularity implies that $\delta(\nu^{\alpha}\rho,\sigma)$ is the only irreducible subquotient of $\nu^{\alpha}\rho\rtimes\sigma$ which has $\nu^{\alpha}\rho\otimes\sigma$ for a subquotient of the corresponding Jacquet module. We need the following claim to finish the proof:

Claim. Let $n\geq 0$ and assume that (iii) and (v) hold for $k\leq n$. Let $\pi_1:=\nu^{\alpha+n+1}\rho\rtimes\delta([\nu^{\alpha}\rho,\nu^{\alpha+n}\rho],\sigma)$ and $\pi_2:=\delta([\nu^{\alpha+n}\rho,\nu^{\alpha+n+1}\rho])\rtimes\delta([\nu^{\alpha}\rho,\nu^{\alpha+n-1}\rho],\sigma)$. The intersection of π_1 and π_2 is $\delta([\nu^{\alpha}\rho,\nu^{\alpha+n+1}\rho],\sigma)$.

Proof of the claim We can explicitly calculate $s.s.(r_{(p(n+2))}(\pi_1))$ and $s.s.(r_{(p(n+2))}(\pi_2))$ using Theorem 3.4 and the inductive assumption. Furthermore, a simple analysis of Jacquet modules for general linear groups implies that $s.s.(r_{(p)^{n+1}}(\pi_1))$ and $s.s.(r_{(p)^{n+1}}(\pi_2))$ have $\nu^{\alpha+n+1}\rho\otimes\nu^{\alpha+n}\rho\otimes\cdots\otimes\nu^{\alpha}\rho\otimes\sigma$ for subquotients. This is the only irreducible subquotient which appears in both Jacquet modules. Therefore, the intersection of π_1 and π_2, which is denoted by π, is nonzero and $r_{(p)^{n+1}}(\pi)=\nu^{\alpha+n+1}\rho\otimes\nu^{\alpha+n}\rho\otimes\cdots\otimes\nu^{\alpha}\rho\otimes\sigma$. This implies that π is irreducible. By (iv), we have $\pi=\delta([\nu^{\alpha}\rho,\nu^{\alpha+n+1}\rho],\sigma)$ and completes the proof of the claim. ■

The claim and its proof prove (iii).

Using Theorem 3.4, we can also calculate the intersection of $\mu^*(\pi_1)$ and $\mu^*(\pi_2)$. Furthermore, the claim implies that the intersection is $\sum_{k=-1}^{n-1}\delta([\nu^{\alpha+k+1}\rho,\nu^{\alpha+n+1}\rho])\otimes\delta([\nu^{\alpha}\rho,\nu^{\alpha+k}\rho],\sigma)+\nu^{\alpha+n+1}\rho\otimes\delta([\nu^{\alpha}\rho,\nu^{\alpha+n}\rho],\sigma)+\delta([\nu^{\alpha}\rho,\nu^{\alpha+n+1}\ \rho],\sigma)$. In sum, we have $\mu^*(\delta([\nu^{\alpha}\rho,\nu^{\alpha+n+1}\rho],\sigma))=\sum_{k=-1}^{n+1}\delta([\nu^{\alpha+k+1}\rho,\nu^{\alpha+n+1}\rho])\otimes\delta([\nu^{\alpha}\rho,\nu^{\alpha+k}\rho],\sigma)$ since $\delta([\nu^{\alpha}\rho,\nu^{\alpha+n+1}\rho],\sigma)$ is the intersection of π_1 and π_2. By induction on n, this proves (v). □

Proof of Lemma 4.1 Once we construct the basic structures on the Jacquet modules (Sublemma 4.2), the argument is exactly same as [31, Proposition 5.1]. Let us sketch the proof here. Suppose that $\nu^\beta\rho \rtimes \delta(\nu^\beta\rho, \sigma)$ is reducible. Note that $r_{GL}(\nu^\beta\rho \rtimes \delta(\nu^\beta\rho, \sigma)) = \nu^\beta\rho \times \nu^\beta\rho \otimes \sigma + \nu^{-\beta}\rho \times \nu^\beta\rho \otimes \sigma$ has length two. Therefore, there exists subquotient π of $\nu^\beta\rho \rtimes \delta(\nu^\beta\rho, \sigma)$ such that $r_{GL}(\pi) = \nu^\beta\rho \times \nu^\beta\rho \otimes \sigma$. We also have the following:

$$\delta([\nu^{-\beta+1}\rho, \nu^{\beta-1}\rho]) \rtimes \pi \leq \delta([\nu^{-\beta+1}\rho, \nu^{\beta-1}\rho]) \times \nu^{-\beta} \times \nu^\beta \rtimes \sigma \tag{4.1}$$

and

$$\delta([\nu^{-\beta}\rho, \nu^\beta\rho]) \rtimes \sigma \leq \delta([\nu^{-\beta+1}\rho, \nu^{\beta-1}\rho]) \times \nu^{-\beta} \times \nu^\beta \rtimes \sigma. \tag{4.2}$$

From Sublemma 4.2 and Theorem 3.4 we get

$$r_{GL}(\delta([\nu^{-\beta}\rho, \nu^\beta\rho]) \rtimes \sigma) = \sum_{k=-\beta-1}^{\beta} \delta([\nu^{-k}\rho, \nu^\beta\rho]) \times \delta([\nu^{k+1}\rho, \nu^\beta\rho]) \otimes \sigma, \tag{4.3}$$

$$r_{GL}(\nu^\beta\rho \times \nu^{-\beta}\rho \times \delta([\nu^{-\beta+1}\rho, \nu^{\beta-1}\rho]) \rtimes \sigma) \tag{4.4}$$

$$= (\nu^\beta\rho + \nu^{-\beta}\rho) \times (\nu^\beta\rho + \nu^{-\beta}\rho) \times \sum_{k=-\beta}^{\beta-1} \delta([\nu^{-k}\rho, \nu^{\beta-1}\rho]) \times \delta([\nu^{k+1}\rho, \nu^{\beta-1}\rho]) \otimes \sigma$$

and

$$r_{GL}(\delta([\nu^{-\beta+1}\rho, \nu^{\beta-1}\rho]) \rtimes \pi) = \nu^\beta\rho \times \nu^\beta\rho \times \sum_{k=-\beta}^{\beta-1} \delta([\nu^{-k}\rho, \nu^{\beta-1}\rho]) \times \delta([\nu^{k+1}\rho, \nu^{\beta-1}\rho]) \otimes \sigma. \tag{4.5}$$

Using (4.1) through (4.5), we show that $\delta([\nu^{-\beta}\rho, \nu^\beta\rho]) \rtimes \sigma$ and $\delta([\nu^{-\beta+1}\rho, \nu^{\beta-1}\rho]) \rtimes \pi$ have an irreducible subquotient. Let τ is such subquotient. Since $\delta([\nu^{-\beta}\rho, \nu^\beta\rho]) \rtimes \sigma$ is unitarizable, Frobenius reciprocity implies that $\delta([\nu^{-\beta}\rho, \nu^\beta\rho]) \otimes \sigma \leq r_{GL}(\tau)$. This is a contradiction since τ is also a subquotient of $\delta([\nu^{-\beta+1}\rho, \nu^{\beta-1}\rho]) \rtimes \pi$ and $\nu^{-\beta}\rho$ cannot appear in the equation (4.5). This completes the proof of Lemma 4.1. ∎

Proposition 4.3. *Let ρ and ρ_0 be irreducible unitarizable supercuspidal representations of $\boldsymbol{GL}_p(F)$ and $\boldsymbol{GL}_{p_0}(F)$ respectively. Let σ be an irreducible supercuspidal representation of $\boldsymbol{G}_m(F)$ and let β be positive. Suppose that $\nu^\beta\rho \rtimes \sigma$ reduces, and that $\nu^\alpha\rho \rtimes \sigma$ is irreducible for any $\alpha \in \mathbb{R} \setminus \{\pm\beta\}$. Let l be a non-negative integer and let $\alpha \in \mathbb{R}$. If $\nu^\alpha\rho \rtimes \delta([\nu^\beta\rho, \nu^{\beta+l}], \sigma)$ reduces, then $\alpha \in \{\pm(\beta-1), \pm(\beta+l+1), \pm\beta\}$. In particular, if we assume that $\beta \in (1/2)\mathbb{Z}$, then the reducibility of $\nu^\alpha\rho \rtimes \delta([\nu^\beta\rho, \nu^{\beta+l}], \sigma)$ implies that $\alpha \in \{\pm(\beta-1), \pm(\beta+l+1)\}$.*

Proof. Suppose that $\alpha \notin \{\pm(\beta-1), \pm(\beta+l+1), \pm\beta\}$. We use Lemma 3.7 of [31] to show that $\nu^\alpha\rho \rtimes \delta([\nu^\beta\rho, \nu^{\beta+l}], \sigma)$ is irreducible.

Using Theorem 3.4, we get

$$\mu^*(\nu^\alpha\rho \rtimes \delta([\nu^\beta\rho, \nu^{\beta+l}], \sigma) = (1 \otimes \nu^\alpha\rho + \nu^\alpha\rho \otimes 1 + (\nu^{-\alpha}\widetilde{\rho} \otimes (\omega \circ \det)) \otimes 1) \tag{4.6}$$

$$\rtimes (\sum_{j=-1}^{l} \delta([\nu^{\beta+j+1}\rho, \nu^{\beta+l}\rho]) \otimes \delta([\nu^\beta\rho, \nu^{\beta+j}\rho], \sigma)).$$

Since σ is supercuspidal, we also get

$$r_{GL}(\nu^\alpha\rho \rtimes \delta([\nu^\beta\rho, \nu^{\beta+l}\rho], \sigma)) \tag{4.7}$$

$$= (\nu^{-\alpha}\widetilde{\rho} \otimes (\omega \circ \det)) \times \delta([\nu^\beta\rho, \nu^{\beta+l}\rho]) \otimes \sigma + \nu^\alpha\rho \times \delta([\nu^\beta\rho, \nu^{\beta+l}\rho]) \otimes \sigma,$$

$$r_{(lp)}(\nu^\alpha\rho \rtimes \delta([\nu^\beta\rho, \nu^{\beta+l}\rho], \sigma)) \geq \delta([\nu^\beta\rho, \nu^{\beta+l}\rho]) \otimes \nu^\alpha\rho \rtimes \sigma \tag{4.8}$$

and

$$r_{((l-1)p)}(\nu^\alpha\rho \rtimes \delta([\nu^\beta\rho, \nu^{\beta+l}\rho], \sigma)) \geq \delta([\nu^{\beta+1}\rho, \nu^{\beta+l}\rho]) \otimes \nu^\alpha\rho \rtimes \delta(\nu^\beta\rho, \sigma). \tag{4.9}$$

Since $\alpha \notin \{\pm(\beta-1), \pm(\beta+l+1)\}$, (4.7) has length 2, i.e., two representations in the right hand side are irreducible.

Let us consider the case when $\alpha \neq 0$. Let $\mathbf{P}' = \mathbf{M}'\mathbf{N}', \mathbf{P}'' = \mathbf{M}''\mathbf{N}''$ and $\mathbf{P}''' = \mathbf{M}'''\mathbf{N}'''$ be standard parabolic subgroups of $\mathbf{G}_{(l+1)p+m}$ such that $\mathbf{M}' = \mathbf{GL}_{lp} \times \mathbf{GL}_p \times \mathbf{G}_m$, $\mathbf{M}'' = \mathbf{GL}_{lp} \times \mathbf{G}_{p+m}$ and $\mathbf{M}''' = \mathbf{GL}_{(l+1)p} \times \mathbf{G}_m$. Let $\tau'' = \delta([\nu^\beta\rho, \nu^{\beta+l}\rho]) \otimes \nu^\alpha\rho \rtimes \sigma$, i.e., the term in the right hand side of (4.8) and $\tau''' = \nu^\alpha\rho \times \delta([\nu^\beta\rho, \nu^{\beta+l}\rho]) \otimes \sigma$ (Case1) or $(\nu^{-\alpha}\widetilde{\rho} \otimes (\omega \circ \det)) \times \delta([\nu^\beta\rho, \nu^{\beta+l}\rho]) \otimes \sigma$ (Case2), i.e., one of the terms in the right hand side of (4.7). In either case, we show $r^{M''}_{M'}(\tau'') + r^{M'''}_{M'}(\tau''') \nleq r^G_{M'}(\nu^\alpha\rho \rtimes \delta([\nu^\beta\rho, \nu^{\beta+l}\rho], \sigma))$.

(Case1) Let us first consider the case when $\tau''' = \nu^\alpha\rho \times \delta([\nu^\beta\rho, \nu^{\beta+l}\rho]) \otimes \sigma$. Then $r^{M''}_{M'}(\tau'') + r^{M'''}_{M'}(\tau''') = \delta([\nu^\beta\rho, \nu^{\beta+l}\rho]) \otimes r_{GL}(\nu^\alpha\rho \rtimes \sigma) + r_{(lp,p)}(\nu^\alpha\rho \times \delta([\nu^\beta\rho, \nu^{\beta+l}\rho])) \otimes \sigma = \delta([\nu^\beta\rho, \nu^{\beta+l}\rho]) \otimes \nu^\alpha\rho \otimes \sigma + \delta([\nu^\beta\rho, \nu^{\beta+l}\rho]) \otimes (\nu^{-\alpha}\widetilde{\rho} \otimes (\omega \circ \det)) \otimes \sigma + r_{(lp,p)}(\nu^\alpha\rho \times \delta([\nu^\beta\rho, \nu^{\beta+l}\rho])) \otimes \sigma \nleq r_{(lp,p)}((\nu^{-\alpha}\widetilde{\rho} \otimes (\omega \circ \det)) \times \delta([\nu^\beta\rho, \nu^{\beta+l}\rho])) \otimes \sigma + r_{(lp,p)}(\nu^\alpha\rho \times \delta([\nu^\beta\rho, \nu^{\beta+l}\rho])) \otimes \sigma = r^G_{M'}(\nu^\alpha\rho \rtimes \delta([\nu^\beta\rho, \nu^{\beta+l}\rho], \sigma))$. The $\nleq$ above is justified by the fact that the supercuspidal support $\nu^\alpha\rho$ does not appear in $r_{(lp,p)}((\nu^{-\alpha}\widetilde{\rho} \otimes (\omega \circ \det)) \times \delta([\nu^\beta\rho, \nu^{\beta+l}\rho])) \otimes \sigma$.

(Case2) Let us also consider the case when $\tau''' = (\nu^{-\alpha}\widetilde{\rho} \otimes (\omega \circ \det)) \times \delta([\nu^\beta\rho, \nu^{\beta+l}\rho]) \otimes \sigma$. Then, $r^{M''}_{M'}(\tau'') + r^{M'''}_{M'}(\tau''') \nleq r^G_{M'}(\nu^\alpha\rho \rtimes \delta([\nu^\beta\rho, \nu^{\beta+l}\rho], \sigma))$ since the supercuspidal support $\nu^{-\alpha}\widetilde{\rho} \otimes (\omega \circ \det)$ does not appear in $r_{(lp,p)}(\nu^\alpha\rho \times \delta([\nu^\beta\rho, \nu^{\beta+l}\rho])) \otimes \sigma$. Therefore, Lemma 3.7 of [31] implies the irreducibility of the representation $\nu^\alpha\rho \rtimes \delta([\nu^\beta\rho, \nu^{\beta+l}], \sigma)$.

If we further assume that $\beta \in \frac{1}{2}\mathbb{Z}$ and $\beta > \frac{1}{2}$, we can prove that $\nu^\alpha\rho \rtimes \delta([\nu^\beta\rho, \nu^{\beta+l}], \sigma)$ is irreducible using Lemma 3.7 of [31] and Lemma 4.1 with above (4.6) through (4.9). When $\beta = \frac{1}{2}$, the reducibility of $\nu^\alpha\rho \rtimes \delta([\nu^\beta\rho, \nu^{\beta+l}], \sigma)$ implies $\alpha \in \{\pm(\beta-1), \pm(\beta+l+1)\}$ since $\{\pm(\beta-1), \pm(\beta+l+1)\} = \{\pm(\beta-1), \pm(\beta+l+1), \pm\beta\}$. □

Remark 4.4. In general, the representation $\nu^\beta\rho \rtimes \delta([\nu^\beta\rho, \nu^{\beta+l}], \sigma)$ may not be irreducible since Lemma 4.1 might not be true when $\beta \notin \frac{1}{2}\mathbb{Z} - \{\frac{1}{2}\}$. For example, $\nu^{\frac{1}{2}}\rho \rtimes \delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{1}{2}+l}], \sigma)$ is reducible when we assume that $\nu^{\frac{1}{2}}\rho \rtimes \sigma$ reduces and $\nu^\alpha\rho \rtimes \sigma$ is irreducible for any $\alpha \in \mathbb{R} \setminus \{\pm\frac{1}{2}\}$ (see Theorem 8.2 (ii) of [31]).

4.2. **Weyl group action on the induced representations.** The following explicit calculation of Weyl group action on the induced representations are also useful when we apply Ivan Matic's idea ([17]) to the case of *GSpin* groups. Let M_θ be a Levi subgroup isomorphic to $\mathbf{GL}_k \times \mathbf{G}_{n-k}$ for $\theta = \Delta \backslash \alpha_k$. There is a unique Weyl group element w_0 such that $w_0(\alpha_k) < 0$ and $w_0(\theta) \subset \Delta$.

Lemma 4.5. *If we identify w_0 as (p, ϵ) as in Section 2.2. Then*

$$p(i) = \begin{cases} k+1-i & \text{for } 1 \leq i \leq k; \\ i & \text{for } k+1 \leq i \leq n. \end{cases} \quad \text{and } \epsilon_i = \begin{cases} -1 & \text{for } 1 \leq i \leq k; \\ 1 & \text{for } k+1 \leq i \leq n. \end{cases}$$

Proof. We can get this characterization of w_0 from the Weyl group action on the character lattice (Corollary 3.2). The condition $(p,\epsilon)\cdot\alpha_i = \epsilon_i e_{p(i)} - \epsilon_{i+1}e_{p(i+1)} \subset \Delta$ for $1 \le i \le k-1$ implies $\epsilon_{\mathbf{GL}} := \epsilon_1 = \epsilon_2 = \ldots = \epsilon_k$ and $p(i+1) = p(i) + \epsilon_i$ for $1 \le i \le k-1$. Similarly, the condition $(p,\epsilon)\cdot\alpha_j = \epsilon_j e_{p(j)} - \epsilon_{j+1}e_{p(j+1)} \subset \Delta$ for $k+1 \le j \le n-1$ implies $\epsilon_{\mathbf{GSpin}} := \epsilon_{k+1} = \epsilon_{k+2} = \ldots = \epsilon_n$ and $p(j+1) = p(j) + \epsilon_j$ for $k+1 \le j \le n-1$. The condition $(p,\epsilon)\cdot\alpha_n = \epsilon_n e_{p(n)} \subset \Delta$ implies $\epsilon_n = 1$ and $p(n) = n$. In sum, $\epsilon_{\mathbf{GSpin}} = 1$ and $p(j) = j$ for $k+1 \le j \le n-1$. Suppose that $\epsilon_{\mathbf{GL}} = 1$. Since $\{p(1), p(2), \ldots, p(k)\} = \{1, 2, \ldots, k\}$ and $p(1) < p(2) < \ldots < p(k)$, $p = id$ and $w_0 \cdot \alpha_k = e_k - e_{k+1} > 0$. This contradicts that $w_0 \cdot \alpha_k < 0$. We conclude that $\epsilon_{\mathbf{GL}} = -1$. Since $\{p(1), p(2), \ldots, p(k)\} = \{1, 2, \ldots, k\}$ and $p(1) > p(2) > \ldots > p(k)$, $p(i) = k+1-i$ for $1 \le i \le k$. In this case, $w_0 \cdot \alpha_k = -e_1 - e_{k+1} < 0$. □

Using Lemma 3.1 and Lemma 4.5, we can get the action of w_0 on the representation of the associated Levi subgroup.

Corollary 4.6. *Let σ be a representation of $\boldsymbol{GL}_k(F)$ and τ be one of $\boldsymbol{G}_{n-k}(F)$. Then*

$$w_0^{-1}e_i = \begin{cases} -e_{k+1-i} & \text{for } 1 \le i \le k; \\ e_i & \text{for } k+1 \le i \le n; \\ e_0 + \displaystyle\sum_{m=1}^{k} e_m & \text{for } i = 0. \end{cases}$$

and

$$(\sigma \otimes \tau)^{w_0} = (\widetilde{\sigma} \otimes (\omega_\tau \circ \det)) \otimes \tau,$$

where ω_τ is the central character of τ.

Corollary 4.7. *Let σ and τ be as in Corollary 4.6. Then $\rho \rtimes \tau$ and $(\widetilde{\rho} \otimes (\omega_\tau \circ \det)) \rtimes \tau$ are associate. Therefore, Lemma 5.4 (iii) of* [6] *implies that the set of irreducible composition factors of $\rho \rtimes \tau$ and $(\widetilde{\rho} \otimes (\omega_\tau \circ \det)) \rtimes \tau$ are same. Furthermore, if we assume that $\rho \rtimes \tau$ is irreducible, then $\rho \rtimes \tau \cong (\widetilde{\rho} \otimes (\omega_\tau \circ \det)) \rtimes \tau$ (cf.* [14, Lemma 3.7] *in the generic case).*

5. Classification of strongly positive representations for odd $GSpin$ groups

This section gives the classification of strongly positive representations of odd $GSpin$ groups. Our results parallel those of Ivan Matic for metaplectic groups. In this section, we apply ideas and adapt some proofs from [17] and also from [20, 31, 32] to our situation and $GSpin$ case. However let us remark that in the case when the reducibility point is $\frac{1}{2}$, more arguments are needed in [17] (Matic has kindly agreed to add arguments in detail about his paper as an appendix to this paper). In our paper, we use another idea to approach the case when the reducibility point is $\frac{1}{2}$ and focus more on this case and this approach can be applied to the case of classical groups and metaplectic groups.

5.1. **Embeddings of strongly positive representations.** In this subsection, we show that strongly positive representations can be embedded into parabolically induced representations of special type. More precisely, we consider the following type of parabolically induced representations:

$$\delta(\Delta_1) \times \delta(\Delta_2) \times \cdots \times \delta(\Delta_k) \rtimes \sigma_{cusp} \tag{5.1}$$

where $\Delta_1, \Delta_2, \ldots, \Delta_k$ a sequence of strongly positive segments satisfying $0 < e(\Delta_1) \leq e(\Delta_2) \leq \cdots \leq e(\Delta_k)$ (we allow $k = 0$ here), σ_{cusp} an irreducible supercuspidal representation of $\mathbf{G}_m(F)$. Note that the idea of certain embeddings of representations was initiated in [21] and further refined in [10].

Theorem 5.1. *Let Δ_i, $i = 1, \ldots, k$ and σ_{cusp} be as above. Then the induced representation $\delta(\Delta_1) \times \delta(\Delta_2) \times \cdots \times \delta(\Delta_k) \rtimes \sigma_{cusp}$ has a unique irreducible subrepresentation which we denote by $\delta(\Delta_1, \ldots, \Delta_k; \sigma_{cusp})$.*

Proof. We briefly explain the main ideas of the proof and how we adapt the proof from [17] to the case of $GSpin$ groups. The case $k = 0$ is clear. We assume that $k > 0$ and let $\Delta_i = [\nu^{a_i}\rho_i, \nu^{b_i}\rho_i]$ for $i = 1, \ldots, k$. The strong positivity implies $0 < a_i \leq b_i$. Let $j_1 < j_2 < \cdots < j_s$ be the positive integers such that $e(\Delta_1) = \cdots = e(\Delta_{j_1}) < e(\Delta_{j_1+1}) = \cdots = e(\Delta_{j_2}) < \cdots < e(\Delta_{j_s+1}) = \cdots = e(\Delta_k)$. Then

$$\delta(\Delta_1) \times \cdots \times \delta(\Delta_{j_1}) \otimes \delta(\Delta_{j_1+1}) \times \cdots \times \delta(\Delta_{j_2}) \otimes \cdots \otimes \sigma_{cusp} \tag{5.2}$$

is irreducible.

Lemma 5.2. *The irreducible representation (5.2) appears with multiplicity one in the Jacquet module of $\delta(\Delta_1) \times \delta(\Delta_2) \times \cdots \times \delta(\Delta_k) \rtimes \sigma_{cusp}$ with respect to the appropriate parabolic subgroup.*

Lemma 5.2 implies the theorem since the Jacquet module of every subrepresentation of $\delta(\Delta_1) \times \delta(\Delta_2) \times \cdots \times \delta(\Delta_k) \rtimes \sigma_{cusp}$ with respect to the appropriate parabolic subgroup contains (5.2). Therefore, the induced representation $\delta(\Delta_1) \times \delta(\Delta_2) \times \cdots \times \delta(\Delta_k) \rtimes \sigma_{cusp}$ has a unique irreducible subrepresentation. □

Proof of Lemma 5.2

The proof relies on the Jacquet module method (Tadic's structure formula). Since we fully construct the Tadic's formula in the case of odd $GSpin$ groups (Section 3), we can apply the arguments of [17] to the case of odd $GSpin$ groups and we omit the proof here. ■

Now, we consider strongly positive representation and show that it can be embedded into induced representations of the form (5.1).

Theorem 5.3. *Let $\sigma \in R(n)$ denote a strongly positive representation. Then σ can be embedded into certain induced representation of the form (5.1).*

Proof. We also briefly explain the main ideas of the proof. We start with Jacquet quotient theorem ([8, Theorem 5.1.2]). Jacquet quotient theorem implies that

$$\sigma \hookrightarrow \rho_1 \times \rho_2 \times \cdots \times \rho_k \rtimes \sigma_{cusp}. \tag{5.3}$$

where ρ_i is an irreducible supercuspidal representation of $\mathbf{GL}_{n_i}(F), i = 1, 2, \ldots, k$ and $\sigma_{cusp} \in R(n - n')$ is an irreducible supercuspidal representation.

Consider all possible embeddings of the following form:

$$\sigma \hookrightarrow \delta(\Delta_1) \times \delta(\Delta_2) \times \cdots \times \delta(\Delta_l) \rtimes \sigma_{cusp}, \tag{5.4}$$

where $\Delta_1 + \Delta_2 + \cdots \Delta_l = \{\rho_1, \rho_2, \ldots, \rho_k\}$, viewed as an equality of multisets (such embedding exists; for example we can take $l = k, \Delta_1 = \{\rho_1\}, \ldots \Delta_l = \{\rho_l\}$).

Each $\delta(\Delta_i)$ is an irreducible representation of some GL_{n_i} (this defines n_i) for $i = 1, 2, \ldots, l$. To every such embedding we attach an n'-tuple

$$(5.5) \qquad (e(\Delta_1), \ldots, e(\Delta_1), e(\Delta_2), \ldots, e(\Delta_2), \ldots, e(\Delta_l), \ldots, e(\Delta_l)) \in \mathbb{R}^{n'},$$

where $e(\Delta_i)$ appears n_i times and $n' = n_1 + \cdots + n_l$ (see Section 3 of [10]). Clearly, the set of all embeddings (5.4) is finite. Then we can assume that (5.4) is such that (5.5) is minimal with respect to the lexicographic ordering on $\mathbb{R}^{n'}$. The strong positivity of σ implies that $e(\Delta_i) > 0$ for $i = 1, 2, \ldots, l$. Then we use the minimality of (5.5) to show that $e(\Delta_1) \leq e(\Delta_2) \leq \cdots \leq e(\Delta_l)$ as in [17] and we omit the proof here. □

5.2. **Classification of strongly positive representations:** $D(\rho; \sigma_{cusp})$**.** Let ρ be an irreducible supercuspidal representation of $\mathbf{GL}_{n_\rho}(F)$ and σ_{cusp} be an irreducible supercuspidal representation of $\mathbf{G}_{n_{\sigma_{cusp}}}(F)$. Let $D(\rho; \sigma_{cusp})$ be the set of strongly positive representations whose supercuspidal supports are the representation σ_{cusp} and twists of the representation ρ by positive valued characters. Let $a \geq 0$ be the unique non-negative real number such that $\nu^a \rho \rtimes \sigma_{cusp}$ reduces ([29]). Furthermore, we assume that this reducibility point a is in $\frac{1}{2}\mathbb{Z}$ (see (HI) of [20], page 771). Let k_ρ denote $\lceil a \rceil$, the smallest integer which is not smaller than a. In this section, we construct the classification of strongly positive representations in $D(\rho; \sigma_{cusp})$. Remark that the approach in the case $a = \frac{1}{2}$ is different from other cases ($a \in \frac{1}{2}\mathbb{N} - \{\frac{1}{2}\}$).

In a previous section, we show that every strongly positive representation can be viewed as the unique irreducible subrepresentation of induced representation of the form (5.1). Therefore, there exists an mapping from the set of strongly positive representations of $\mathbf{G}_n(F)$ into the set of induced representations of the form (5.1).

Now we further refine the image of this map when we restrict the map to $D(\rho; \sigma_{cusp})$.

Theorem 5.4. *Let σ be an irreducible strongly positive representation in $D(\rho; \sigma_{cusp})$ and consider it as the unique irreducible subrepresentation of induced representation of the form (5.1). Write $\Delta_i = [\nu^{a_i}\rho, \nu^{b_i}\rho]$. Then,*

$$\begin{cases} a_i = a - k + i, \ b_1 < \ldots < b_k \text{ and } k \leq \lceil a \rceil & \text{when } \ a \in \frac{1}{2}\mathbb{Z} - \frac{1}{2}; \\ a_i = a \text{ and } b_1 \leq \ldots \leq b_k & \text{when } \ a = \frac{1}{2}. \end{cases}$$

Proof. We first consider the case when $a = \frac{1}{2}$. We use induction on k. The case $k = 0$ is clear. When $k = 1$, we have the embedding $\sigma \hookrightarrow \delta([\nu^{a_1}\rho, \nu^{b_1}\rho]) \rtimes \sigma_{cusp} \hookrightarrow \nu^{b_1}\rho \times \cdots \times \nu^{a_1}\rho \rtimes \sigma_{cusp}$. If $a_1 \neq a$, then $\nu^{a_1}\rho \rtimes \sigma_{cusp} \cong \nu^{-a_1}\rho \rtimes \sigma_{cusp}$ as in the case $a = 0$. Using this isomorphism, we have $\sigma \hookrightarrow \nu^{b_1}\rho \times \cdots \times \nu^{-a_1}\rho \rtimes \sigma_{cusp}$ which contradicts the strong positivity of σ. This implies $a_1 = a$.

Suppose that theorem holds for all $m \in \mathbb{Z}$ such that $0 \leq m < k$, where $k \geq 2$. We prove it for k. If $a_k \neq a$, we get the inclusion $\sigma \hookrightarrow \nu^{b_1}\rho \times \cdots \times \nu^{a_1}\rho \times \cdots \times \nu^{b_k}\rho \times \cdots \times \nu^{a_{k-1}} \times \nu^{-a_k}\rho \rtimes \sigma_{cusp}$ as in the case $a = 0$. This contradicts that σ is strongly positive. This implies $a_k = a$.

Since $\delta(\Delta_2, \ldots, \Delta_k; \sigma_{cusp})$ is a subrepresentation of $\delta(\Delta_2) \times \cdots \times \delta(\Delta_k) \rtimes \sigma_{cusp}$, induction in stages gives the embedding $\delta(\Delta_1) \rtimes \delta(\Delta_2, \ldots, \Delta_k; \sigma_{cusp}) \hookrightarrow \delta(\Delta_1) \times \cdots \times \delta(\Delta_k) \rtimes \sigma_{cusp}$. Since σ is the unique irreducible subrepresentation of $\delta(\Delta_1) \times \cdots \times \delta(\Delta_k) \rtimes \sigma_{cusp}$, we deduce that $\sigma \hookrightarrow \delta(\Delta_1) \rtimes \delta(\Delta_2, \ldots \Delta_k; \sigma_{cusp})$. The strong positivity of σ implies the strong positivity of $\delta([\nu^{a_2}\rho, \nu^{b_2}\rho], \ldots, [\nu^{a_k}\rho, \nu^{b_k}\rho]; \sigma_{cusp})$. The inductive assumption implies $a_i = \frac{1}{2}$ for all $i \geq 2$ and $b_2 \leq \ldots \leq b_k$. It remains to show $a_1 = \frac{1}{2}$ and $b_1 \leq b_2$. We first show $a_1 \in \frac{1}{2} + \mathbb{Z}$. If not, we get the following embedding using the fact that each $\nu^{a_1} \times \delta([\nu^{a_i}\rho, \nu^{b_i}\rho])$ is irreducible for all $i \geq 2$:

$$\begin{aligned}
\sigma &\hookrightarrow \delta([\nu^{a_1+1}\rho, \nu^{b_1}\rho]) \times \nu^{a_1}\rho \times \delta([\nu^{a_2}\rho, \nu^{b_2}\rho]) \times \cdots \times \delta([\nu^{a_k}\rho, \nu^{b_k}\rho]) \rtimes \sigma_{cusp} \\
&\cong \delta([\nu^{a_1+1}\rho, \nu^{b_1}\rho]) \times \delta([\nu^{a_2}\rho, \nu^{b_2}\rho]) \times \cdots \times \delta([\nu^{a_k}\rho, \nu^{b_k}\rho]) \times \nu^{a_1}\rho \rtimes \sigma_{cusp} \\
&\cong \delta([\nu^{a_1+1}\rho, \nu^{b_1}\rho]) \times \delta([\nu^{a_2}\rho, \nu^{b_2}\rho]) \times \cdots \times \delta([\nu^{a_k}\rho, \nu^{b_k}\rho]) \times \nu^{-a_1}\rho \rtimes \sigma_{cusp}
\end{aligned}$$

which contradicts the strong positivity of σ.

It is enough to show that $a_1 < \frac{3}{2}$ since $a_1 \in \frac{1}{2} + \mathbb{Z}$. Suppose that $a_1 \geq \frac{3}{2}$. The inequality $e(\Delta_1) \leq e(\Delta_i)$ for all $i \geq 2$ implies that $b_1 < b_i$. This implies that each $\delta(\Delta_1) \times \delta(\Delta_i)$ is irreducible for all $i \geq 2$. Then we get the following embedding using the fact that $a_1 \neq \frac{1}{2}$:

$$\begin{aligned}
\sigma &\hookrightarrow \delta(\Delta_1) \times \cdots \times \delta(\Delta_k) \rtimes \sigma_{cusp} \\
&\cong \delta(\Delta_2) \times \cdots \times \delta(\Delta_k) \times \delta([\nu^{a_1}\rho, \nu^{b_1}\rho]) \rtimes \sigma_{cusp} \\
&\hookrightarrow \delta(\Delta_2) \times \cdots \times \delta(\Delta_k) \times \delta([\nu^{a_1+1}\rho, \nu^{b_1}\rho]) \times \nu^{a_1}\rho \rtimes \sigma_{cusp} \\
&\cong \delta(\Delta_2) \times \cdots \times \delta(\Delta_k) \times \delta([\nu^{a_1+1}\rho, \nu^{b_1}\rho]) \times \nu^{-a_1}\rho \rtimes \sigma_{cusp}
\end{aligned}$$

which contradicts the strong positivity of σ. Finally, the fact $a_1 \in \frac{1}{2} + \mathbb{Z}$ implies that $a_1 = \frac{1}{2}$. And $b_1 \leq b_2$ follows from the inequality $e(\Delta_1) \leq e(\Delta_2)$. When $a \in \{\frac{1}{2}\mathbb{Z} - \frac{1}{2}\}$, we can apply the arguments of [17] to the odd $GSpin$ groups since we construct all the tools that we needed in Section 3 and 4 and we, therefore, omit here. □

We also show that the map from $D(\rho; \sigma_{cusp})$ to the set of induced representations of the form (5.1) is well defined in the following theorem:

Theorem 5.5. *Let σ be an irreducible strongly positive representation in $D(\rho; \sigma_{cusp})$. Then, there exist a unique set of strongly positive segments $\Delta_1, \Delta_2, \ldots, \Delta_k$, with $0 < e(\Delta_1) \leq e(\Delta_2) \leq \cdots \leq e(\Delta_k)$, and a unique irreducible supercuspidal representation $\sigma' \in R$ such that $\sigma \simeq \delta(\Delta_1, \Delta_2, \ldots, \Delta_k; \sigma')$.*

Proof. We first consider the case $a = \frac{1}{2}$. The uniqueness of the partial supercuspidal support implies that $\sigma' = \sigma_{cusp}$. Suppose that there are two sequences of strongly positive segments, $\Delta_1, \Delta_2, \ldots, \Delta_k$ and $\Delta'_1, \Delta'_2, \ldots, \Delta'_l$ which satisfy the conditions in Theorem 5.4, i.e., $\Delta_i = [\nu^{\frac{1}{2}}\rho, \nu^{b_i}\rho]$, $b_1 \leq \cdots \leq b_k$ and $\Delta'_j = [\nu^{\frac{1}{2}}\rho, \nu^{b'_j}\rho]$, $b'_1 \leq \cdots \leq b'_l$. We have the following two embeddings:

$$\sigma \hookrightarrow \delta(\Delta_1) \times \delta(\Delta_2) \times \cdots \times \delta(\Delta_k) \rtimes \sigma_{cusp} \tag{5.6}$$

$$\sigma \hookrightarrow \delta(\Delta'_1) \times \delta(\Delta'_2) \times \cdots \times \delta(\Delta'_l) \rtimes \sigma_{cusp}. \tag{5.7}$$

Note that $\delta(\Delta_1) \times \delta(\Delta_2) \times \cdots \times \delta(\Delta_k)$ and $\delta(\Delta'_1) \times \delta(\Delta'_2) \times \cdots \times \delta(\Delta'_l)$ are irreducible since the segments are not connected in the sense of Zelevinsky. The embedding (5.7) implies that the Jacquet module of σ with respect to the appropriate parabolic subgroup has to contain the irreducible representation $\delta(\Delta'_1) \times \delta(\Delta'_2) \times \cdots \times \delta(\Delta'_l) \otimes \sigma_{cusp}$. The transitivity and exactness of Jacquet modules, applied to (5.6), imply that $\delta(\Delta'_1) \times \delta(\Delta'_2) \times \cdots \times \delta(\Delta'_l) \otimes \sigma_{cusp}$ is an irreducible member of $\mu^*(\delta(\Delta_1) \times \delta(\Delta_2) \times \cdots \times \delta(\Delta_k) \rtimes \sigma_{cusp})$. Theorem 3.4 implies that there are $-\frac{1}{2} \leq x_i \leq y_i \leq b_i$

such that

$$\prod_{i=1}^{k}(\delta([\nu^{-x_i}\widetilde{\rho}\otimes(\omega_{\sigma_{cusp}}\circ\det),\nu^{-\frac{1}{2}}\widetilde{\rho}\otimes(\omega_{\sigma_{cusp}}\circ\det)])\times\delta([\nu^{y_i+1}\rho,\nu^{b_i}\rho]))\geq\prod_{i=1}^{l}\delta([\nu^{\frac{1}{2}}\rho,\nu^{b'_l}\rho]). \tag{5.8}$$

We compare the supercuspidal supports of both sides to get x_i and y_i for every $i=1,\ldots,k$. Since all the segments in the right hand side are strongly positive, we first get $x_i=-\frac{1}{2}$ for every $i=1,\ldots,k$ so that each segment $[\nu^{-x_i}\widetilde{\rho}\otimes(\omega_{\sigma_{cusp}}\circ\det),\nu^{-\frac{1}{2}}\ \widetilde{\rho}\otimes(\omega_{\sigma_{cusp}}\circ\det)]$ is empty for every $i=1,\ldots,k$. We also get $k\geq l$ by comparing the supercuspidal support $\nu^{\frac{1}{2}}\rho$. Reversing the roles, we also get $l\geq k$. Since $y_i+1\geq\frac{1}{2}$ and $k=l$, we get $y_i=-\frac{1}{2}$ for all $i=1,\ldots,k$ by comparing the supercuspidal support $\nu^{\frac{1}{2}}\rho$. Therefore, the inequality (5.8) becomes $\prod_{i=1}^{k}(\delta([\nu^{\frac{1}{2}}\rho,\nu^{b_i}\rho]))\geq\prod_{i=1}^{k}\delta([\nu^{\frac{1}{2}}\rho,\nu^{b'_k}\rho])$. Since $b_1\leq\ldots\leq b_k$, $b'_1\leq\ldots\leq b'_l$ and $k=l$, we also get $b_i=b'_i$ for every $i=1,\ldots,k$. This proves the uniqueness in this case. Again, in the case when $a\in\{\frac{1}{2}\mathbb{Z}-\frac{1}{2}\}$, the proof is similar to [17] and we construct all the tools that we needed in Section 3 and 4 and we, therefore, omit the proof in this case. □

We, in Theorem 5.4 and Theorem 5.5, construct an injective mapping from $D(\rho;\sigma_{cusp})$ into the set of induced representations of the form (5.1) with refinement on the unitary exponents as in Theorem 5.4. More precisely, let $Jord^*_{(\rho,a)}$, when $a\neq\frac{1}{2}$, stand for the set of all increasing sequences $b_1,b_2,\ldots,b_{k_\rho}$, where $b_i\in\mathbb{R}, b_i-a+k_\rho-i\in\mathbb{Z}_{\geq0}$ for $i=1,\ldots,k_\rho$ and $-1<b_1<b_2<\cdots<b_{k_\rho}$ and let $Jord^*_{(\rho,\frac{1}{2})}$ stands for the set of all increasing sequences $b'_1,b'_2,\ldots,b'_k$, where $b'_i\in\mathbb{R}, b'_i-\frac{1}{2}\in\mathbb{Z}_{\geq0}$ for $i=1,\ldots,k$ and $-1<b'_1\leq b'_2\leq\cdots\leq b'_k$ for any non-negative integer k. So far, we construct the following injective mapping:

$$D(\rho;\sigma_{cusp})\hookrightarrow Jord^*_{(\rho,a)}$$

Now, it remains to describe the exact image of this mapping. Let $Jord_{(\rho,a)}$ be same as $Jord^*_{(\rho,a)}$ when $a\neq\frac{1}{2}$ and let $Jord_{(\rho,\frac{1}{2})}$ be a subset of $Jord^*_{(\rho,\frac{1}{2})}$ with condition $k=1$. In what follows, we show that the image of this mapping is exactly $Jord_{(\rho,a)}$. In other words, we show the image is the set of induced representation of the form 5.1 with $a_i=a-k+i,\ b_1<\ldots<b_k$ *and* $k\leq\lceil a\rceil$ for any $a\in\frac{1}{2}\mathbb{Z}$. We first show that the image contains $Jord_{(\rho,a)}$. Let $b_1,b_2,\ldots,b_{k_\rho}$ denote an increasing sequence appearing in $Jord_{(\rho,a)}$. We showed in Section 5.1 that the induced representation

$$\delta([\nu^{a-k_\rho+1}\rho,\nu^{b_1}\rho])\times\delta([\nu^{a-k_\rho+2}\rho,\nu^{b_2}\rho])\times\cdots\times\delta([\nu^{a}\rho,\nu^{b_{k_\rho}}\rho])\rtimes\sigma_{cusp} \tag{5.9}$$

has a unique irreducible subrepresentation, which we denote by $\sigma_{(b_1,\ldots,b_{k_\rho};a)}$.

We apply induction argument in [17] to show that the above subrepresentation is strongly positive and we don't repeat the argument here.

Theorem 5.6. *The representation $\sigma_{(b_1,\ldots,b_{k_\rho};a)}$ is strongly positive.*

It remains to show that the image is contained in $Jord_{(\rho,a)}$. It is enough to consider the case $a=\frac{1}{2}$ since $Jord_{(\rho,a)}=Jord^*_{(\rho,a)}$ when $a\neq\frac{1}{2}$.

Let $b'_1, b'_2, \ldots, b'_k$ denote an increasing sequence appearing in $Jord^*_{(\rho,\frac{1}{2})}$. Again, we showed in Section 5.1 that the induced representation $\delta([\nu^{\frac{1}{2}}\rho, \nu^{b'_1}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{b'_2}\rho]) \times \cdots \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{b'_k}\rho]) \rtimes \sigma_{cusp}$ has a unique irreducible subrepresentation, which we denote by $\sigma^*_{(b'_1,\ldots,b'_k;\frac{1}{2})}$.

Lemma 5.7. *The representation $\sigma^*_{(\frac{1}{2},\frac{1}{2};\frac{1}{2})}$, the unique irreducible subrepresentation of $\nu^{\frac{1}{2}}\rho \times \nu^{\frac{1}{2}}\rho \rtimes \sigma_{cusp}$, is not strongly positive.*

Sublemma 5.8. Suppose that $\nu^{\frac{1}{2}}\rho \rtimes \sigma_{cusp}$ reduces. Then $\delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho]) \rtimes \sigma_{cusp}$ reduces into a sum of two inequivalent irreducible representations.

Proof. Theorem 3.4 implies that $r_{GL}(\delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho]) \rtimes \sigma_{cusp}) = 2(\delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho]) \otimes \sigma_{cusp}) + \nu^{\frac{1}{2}}\rho \times \nu^{\frac{1}{2}}\rho \otimes \sigma_{cusp}$. The Lemma 3.8 (b) of [31] implies that $\delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho]) \rtimes \sigma_{cusp}$ is either irreducible or a direct sum of two irreducible non-isomorphic representations since the multiplicity of $\delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho]) \otimes \sigma_{cusp}$ in $r_{GL}(\delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho]) \rtimes \sigma_{cusp})$ is exactly two. So, it is enough to show that $\delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho]) \rtimes \sigma_{cusp}$ reduces. We use Remark 3.2 of [31] to show this. The transitivity of induced representations and Corollary 4.7 imply that $\delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho]) \rtimes \sigma_{cusp} \leq \nu^{\frac{1}{2}}\rho \times \nu^{-\frac{1}{2}}\rho \rtimes \sigma_{cusp}$ and $\nu^{\frac{1}{2}}\rho \times \delta(\nu^{\frac{1}{2}}\rho, \sigma_{cusp}) \leq \nu^{\frac{1}{2}}\rho \times \nu^{\frac{1}{2}}\rho \rtimes \sigma_{cusp} = \nu^{\frac{1}{2}}\rho \times \nu^{-\frac{1}{2}}\rho \rtimes \sigma_{cusp}$.

Theorem 3.4 implies that the multiplicity of $\nu^{\frac{1}{2}}\rho \times \nu^{\frac{1}{2}}\rho \otimes \sigma_{cusp}$ in each of $r_{GL}(\delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho]) \rtimes \sigma_{cusp})$, $r_{GL}(\nu^{\frac{1}{2}}\rho \times \delta(\nu^{\frac{1}{2}}\rho, \sigma_{cusp}))$ and $r_{GL}(\nu^{\frac{1}{2}}\rho \times \nu^{-\frac{1}{2}}\rho \rtimes \sigma_{cusp})$ is one and that the multiplicity of $\delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho]) \otimes \sigma_{cusp}$ in each of $r_{GL}(\delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho]) \rtimes \sigma_{cusp})$ and $r_{GL}(\nu^{\frac{1}{2}}\rho \times \nu^{-\frac{1}{2}}\rho \rtimes \sigma_{cusp})$ is two and one respectively. This implies that $r_{GL}(\delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho]) \rtimes \sigma_{cusp}) + r_{GL}(\nu^{\frac{1}{2}}\rho \times \delta(\nu^{\frac{1}{2}}\rho, \sigma_{cusp})) \nleq r_{GL}(\nu^{\frac{1}{2}}\rho \times \nu^{-\frac{1}{2}}\rho \rtimes \sigma_{cusp})$ and $r_{GL}(\delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho]) \rtimes \sigma_{cusp}) \nleq r_{GL}(\nu^{\frac{1}{2}}\rho \times \nu^{-\frac{1}{2}}\rho \rtimes \sigma_{cusp})$. Now we conclude the reducibility from Remark 3.2 of [31]. This completes the proof of Sublemma 5.8. □

Remark 5.9. Since $r_{GL}(\delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho]) \rtimes \sigma_{cusp}) = 2\delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho]) \otimes \sigma_{cusp} + \nu^{\frac{1}{2}}\rho \times \nu^{\frac{1}{2}}\rho \otimes \sigma_{cusp}$, Frobenius reciprocity implies that the irreducible subrepresentations, say τ_1 and τ_2, satisfy $r_{GL}(\tau_1) = \delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho]) \otimes \sigma_{cusp} + \nu^{\frac{1}{2}}\rho \times \nu^{\frac{1}{2}}\rho \otimes \sigma_{cusp}$ and $r_{GL}(\tau_2) = \delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho]) \otimes \sigma_{cusp}$.

Sublemma 5.10. Let $\rho, \sigma_{cusp}, \tau_1$ and τ_2 be as in Sublemma 5.8 and Remark 5.9. In the Grothendieck group, we have $\nu^{\frac{1}{2}}\rho \rtimes \delta(\nu^{\frac{1}{2}}\rho, \sigma_{cusp}) = \mathfrak{s}(\nu^{\frac{1}{2}}\rho, \delta(\nu^{\frac{1}{2}}\rho, \sigma_{cusp})) + \tau_1$. In particular, τ_1 is the unique irreducible subrepresentation of $\nu^{\frac{1}{2}}\rho \rtimes \delta(\nu^{\frac{1}{2}}\rho, \sigma_{cusp})$ (Here, $\mathfrak{s}(\nu^{\frac{1}{2}}\rho, \delta(\nu^{\frac{1}{2}}\rho, \sigma_{cusp}))$ represents the unique irreducible quotient, i.e., Langlands quotient of $\nu^{\frac{1}{2}}\rho \rtimes \delta(\nu^{\frac{1}{2}}\rho, \sigma_{cusp})$).

Proof. Step1. $\tau_1 \leq \nu^{\frac{1}{2}}\rho \rtimes \delta(\nu^{\frac{1}{2}}\rho, \sigma_{cusp})$.

We consider $\nu^{-\frac{1}{2}}\rho \rtimes \delta(\nu^{\frac{1}{2}}\rho, \sigma_{cusp})$. We have $\nu^{-\frac{1}{2}}\rho \rtimes \delta(\nu^{\frac{1}{2}}\rho, \sigma_{cusp}) \hookrightarrow \nu^{-\frac{1}{2}}\rho \times \nu^{\frac{1}{2}}\rho \rtimes \sigma_{cusp}$. Suppose that $\nu^{-\frac{1}{2}}\rho \rtimes \delta(\nu^{\frac{1}{2}}\rho, \sigma_{cusp}) \leq \mathfrak{s}(\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho) \rtimes \sigma_{cusp}$. However, if we consider the following two Jacquet modules, this is not the case:

$$r_{GL}(\nu^{-\frac{1}{2}}\rho \rtimes \delta(\nu^{\frac{1}{2}}\rho, \sigma_{cusp})) = \nu^{-\frac{1}{2}}\rho \times \nu^{\frac{1}{2}}\rho \otimes \sigma_{cusp} + \nu^{\frac{1}{2}}\rho \times \nu^{\frac{1}{2}}\rho \otimes \sigma_{cusp} \tag{5.10}$$

and

$$r_{GL}(\mathfrak{s}(\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho) \rtimes \sigma_{cusp}) = (2\mathfrak{s}(\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho) + \nu^{-\frac{1}{2}}\rho \times \nu^{-\frac{1}{2}}\rho) \otimes \sigma_{cusp}. \tag{5.11}$$

This implies that there exists an irreducible subquotient $\pi \leq \nu^{-\frac{1}{2}}\rho \rtimes \delta(\nu^{\frac{1}{2}}\rho, \sigma_{cusp})$ such that $\pi \leq \delta(\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho) \rtimes \sigma_{cusp} = \tau_1 \oplus \tau_2$.

Suppose that $\tau_1 \nleq \nu^{-\frac{1}{2}}\rho \rtimes \delta(\nu^{\frac{1}{2}}\rho, \sigma_{cusp})$. This implies that the intersection of $\nu^{-\frac{1}{2}}\rho \rtimes \delta(\nu^{\frac{1}{2}}\rho, \sigma_{cusp})$ and $\delta(\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho) \rtimes \sigma_{cusp}$ is τ_2 and $\nu^{-\frac{1}{2}}\rho \rtimes \delta(\nu^{\frac{1}{2}}\rho, \sigma_{cusp}) - \tau_2 \leq \mathfrak{s}(\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho) \rtimes \sigma_{cusp}$ which is a contradiction because of (5.10) and (5.11). Therefore, we conclude that $\tau_1 \leq \nu^{-\frac{1}{2}}\rho \rtimes \delta(\nu^{\frac{1}{2}}\rho, \sigma_{cusp})$.

Step2. The length of $\nu^{\frac{1}{2}}\rho \rtimes \delta(\nu^{\frac{1}{2}}\rho, \sigma_{cusp})$ is two.

We consider $r_{GL}(\nu^{\frac{1}{2}}\rho \rtimes \delta(\nu^{\frac{1}{2}}\rho, \sigma_{cusp})) = \nu^{-\frac{1}{2}}\rho \times \nu^{\frac{1}{2}}\rho \otimes \sigma_{cusp} + \nu^{\frac{1}{2}}\rho \times \nu^{\frac{1}{2}}\rho \otimes \sigma_{cusp}$. From this, we know that the length of $\nu^{\frac{1}{2}}\rho \rtimes \delta(\nu^{\frac{1}{2}}\rho, \sigma_{cusp})$ is at most 3. Since $r_{GL}(\tau_1) = \delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho]) \otimes \sigma_{cusp} + \nu^{\frac{1}{2}}\rho \times \nu^{\frac{1}{2}}\rho \otimes \sigma_{cusp} \lneq r_{GL}(\nu^{\frac{1}{2}}\rho \rtimes \delta(\nu^{\frac{1}{2}}\rho, \sigma_{cusp}))$ and $r_{GL}(\nu^{\frac{1}{2}}\rho \rtimes \delta(\nu^{\frac{1}{2}}\rho, \sigma_{cusp})) - r_{GL}(\tau_1) = \mathfrak{s}([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho]) \otimes \sigma_{cusp}$ is irreducible, the length is exactly 2.

Finally, since $\mathfrak{s}(\nu^{\frac{1}{2}}\rho, \delta(\nu^{\frac{1}{2}}\rho, \sigma_{cusp}))$ is the unique irreducible quotient of $\nu^{\frac{1}{2}}\rho \rtimes \delta(\nu^{\frac{1}{2}}\rho, \sigma_{cusp})$, τ_1 is the unique irreducible subrepresentation of $\nu^{\frac{1}{2}}\rho \rtimes \delta(\nu^{\frac{1}{2}}\rho, \sigma_{cusp})$. □

Proof of Lemma 5.7 The embedding $\nu^{\frac{1}{2}}\rho \rtimes \delta(\nu^{\frac{1}{2}}\rho, \sigma_{cusp}) \hookrightarrow \nu^{\frac{1}{2}}\rho \times \nu^{\frac{1}{2}}\rho \rtimes \sigma_{cusp}$ implies the following embedding:

$$\sigma^*_{(\frac{1}{2}, \frac{1}{2}; \frac{1}{2})} \hookrightarrow \nu^{\frac{1}{2}}\rho \rtimes \delta(\nu^{\frac{1}{2}}\rho, \sigma_{cusp}).$$

Sublemma 5.10 implies that $\sigma^*_{(\frac{1}{2}, \frac{1}{2}; \frac{1}{2})} \cong \tau_1$ which is a subrepresentation of $\delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho]) \rtimes \sigma_{cusp}$ by Sublemma 5.8. Therefore, we have the following embedding:

$$\sigma^*_{(\frac{1}{2}, \frac{1}{2}; \frac{1}{2})} \hookrightarrow \delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho]) \rtimes \sigma_{cusp} \hookrightarrow \nu^{\frac{1}{2}}\rho \times \nu^{-\frac{1}{2}}\rho \rtimes \sigma_{cusp}.$$

This implies that $\sigma^*_{(\frac{1}{2}, \frac{1}{2}; \frac{1}{2})}$ is not strongly positive. ■

Lemma 5.11. *The induced representation* $\delta([\nu^{\frac{3}{2}}\rho, \nu^{b'_1}\rho]) \times \delta([\nu^{\frac{3}{2}}\rho, \nu^{b'_2}\rho]) \times \cdots \times \delta([\nu^{\frac{3}{2}}\rho, \nu^{b'_k}\rho]) \times \nu^{\frac{1}{2}}\rho \times \cdots \times \nu^{\frac{1}{2}}\rho \rtimes \sigma_{cusp}$ *has a unique irreducible subrepresentation.*

Proof. Since $\delta([\nu^{\frac{3}{2}}\rho, \nu^{b'_1}\rho]) \times \delta([\nu^{\frac{3}{2}}\rho, \nu^{b'_2}\rho]) \times \cdots \times \delta([\nu^{\frac{3}{2}}\rho, \nu^{b'_k}\rho]) \otimes \nu^{\frac{1}{2}}\rho \times \cdots \times \nu^{\frac{1}{2}}\rho \otimes \sigma_{cusp}$ is irreducible, it is enough to show that $\delta([\nu^{\frac{3}{2}}\rho, \nu^{b'_1}\rho]) \times \delta([\nu^{\frac{3}{2}}\rho, \nu^{b'_2}\rho]) \times \cdots \times \delta([\nu^{\frac{3}{2}}\rho, \nu^{b'_k}\rho]) \otimes \nu^{\frac{1}{2}}\rho \times \cdots \times \nu^{\frac{1}{2}}\rho \otimes \sigma_{cusp}$ appears with multiplicity one in the Jacquet module of $\delta([\nu^{\frac{3}{2}}\rho, \nu^{b'_1}\rho]) \times \delta([\nu^{\frac{3}{2}}\rho, \nu^{b'_2}\rho]) \times \cdots \times \delta([\nu^{\frac{3}{2}}\rho, \nu^{b'_k}\rho]) \times \nu^{\frac{1}{2}}\rho \times \cdots \times \nu^{\frac{1}{2}}\rho \rtimes \sigma_{cusp}$. Exactness and transitivity of Jacquet modules imply that there exists an irreducible representation π such that $\mu^*(\delta([\nu^{\frac{3}{2}}\rho, \nu^{b'_1}\rho]) \times \delta([\nu^{\frac{3}{2}}\rho, \nu^{b'_2}\rho]) \times \cdots \times \delta([\nu^{\frac{3}{2}}\rho, \nu^{b'_k}\rho]) \times \nu^{\frac{1}{2}}\rho \times \cdots \times \nu^{\frac{1}{2}}\rho \rtimes \sigma_{cusp}) \geq \delta([\nu^{\frac{3}{2}}\rho, \nu^{b'_1}\rho]) \times \delta([\nu^{\frac{3}{2}}\rho, \nu^{b'_2}\rho]) \times \cdots \times \delta([\nu^{\frac{3}{2}}\rho, \nu^{b'_k}\rho]) \otimes \pi$, where $r_{GL}(\pi) \geq \nu^{\frac{1}{2}}\rho \times \cdots \times \nu^{\frac{1}{2}}\rho \otimes \sigma_{cusp}$. Theorem 3.4 implies that there exist $\frac{1}{2} \leq x'_i \leq y'_i \leq b'_i$ and $-\frac{1}{2} \leq x_i \leq y_i \leq \frac{1}{2}$ for $i = 1, \ldots, k$ such that the inequality $\prod_{i=1}^{k} \delta([\nu^{-x'_i}\rho, \nu^{-\frac{3}{2}}\rho]) \times \delta([\nu^{y'_i+1}\rho, \nu^{b'_i}\rho]) \prod_{i=1}^{k} \delta([\nu^{-x_i}\rho, \nu^{-\frac{1}{2}}\rho]) \times \delta([\nu^{y_i+1}\rho, \nu^{\frac{1}{2}}\rho]) \geq \prod_{i=1}^{k} \delta([\nu^{\frac{3}{2}}\rho, \nu^{b'_i}\rho])$ holds. The positivity of segments in the right hand side implies that $x'_i = \frac{1}{2}$ and $x_i = -\frac{1}{2}$ for every $i = 1, \ldots, k$ so that $[\nu^{-x'_i}\rho, \nu^{-\frac{3}{2}}\rho]$ and $[\nu^{-x_i}\rho, \nu^{-\frac{1}{2}}\rho]$ are empty. Since $\nu^{\frac{1}{2}}\rho$ does not appear in the right hand side, $y_i = \frac{1}{2}$

for every $i = 1, \ldots, k$ so that $[\nu^{y_i+1}\rho, \nu^{\frac{1}{2}}\rho]$ is empty. Comparing the supercuspidal support $\nu^{\frac{3}{2}}\rho$ in both sides, we also get $y'_i = \frac{1}{2}$ for every $i = 1, \ldots, k$. Thus $\delta([\nu^{\frac{3}{2}}\rho, \nu^{b'_1}\rho]) \times \delta([\nu^{\frac{3}{2}}\rho, \nu^{b'_2}\rho]) \times \cdots \times \delta([\nu^{\frac{3}{2}}\rho, \nu^{b'_k}\rho]) \otimes \pi$ appears with multiplicity one in $\mu^*(\delta([\nu^{\frac{3}{2}}\rho, \nu^{b'_1}\rho]) \times \delta([\nu^{\frac{3}{2}}\rho, \nu^{b'_2}\rho]) \times \cdots \times \delta([\nu^{\frac{3}{2}}\rho, \nu^{b'_k}\rho]) \times \nu^{\frac{1}{2}}\rho \times \cdots \times \nu^{\frac{1}{2}}\rho \rtimes \sigma_{cusp})$. Also, Theorem 3.4 implies that $\pi \leq \nu^{\frac{1}{2}}\rho \times \cdots \times \nu^{\frac{1}{2}}\rho \rtimes \sigma_{cusp}$. Theorem 5.1 implies that $\nu^{\frac{1}{2}}\rho \times \cdots \times \nu^{\frac{1}{2}}\rho \rtimes \sigma_{cusp}$ has a unique subrepresentation which contains $\nu^{\frac{1}{2}}\rho \times \cdots \times \nu^{\frac{1}{2}}\rho \otimes \sigma_{cusp}$ in an appropriate Jacquet module. Since $r_{GL}(\pi) \geq \nu^{\frac{1}{2}}\rho \times \cdots \times \nu^{\frac{1}{2}}\rho \otimes \sigma_{cusp}$ and $\nu^{\frac{1}{2}}\rho \times \cdots \times \nu^{\frac{1}{2}}\rho \otimes \sigma_{cusp}$ appears with multiplicity one in the Jacquet module of $\nu^{\frac{1}{2}}\rho \times \cdots \times \nu^{\frac{1}{2}}\rho \rtimes \sigma_{cusp}$, π appears with multiplicity one in the Jacquet module of $\nu^{\frac{1}{2}}\rho \times \cdots \times \nu^{\frac{1}{2}}\rho \rtimes \sigma_{cusp}$ and $\pi \cong \delta(\nu^{\frac{1}{2}}\rho, \cdots, \nu^{\frac{1}{2}}\rho; \sigma_{cusp})$. Finally, we get

$$\begin{aligned}
&\delta([\nu^{\frac{3}{2}}\rho, \nu^{b'_1}\rho]) \times \delta([\nu^{\frac{3}{2}}\rho, \nu^{b'_2}\rho]) \times \cdots \times \delta([\nu^{\frac{3}{2}}\rho, \nu^{b'_k}\rho]) \otimes \nu^{\frac{1}{2}}\rho \times \cdots \times \nu^{\frac{1}{2}}\rho \otimes \sigma_{cusp} \\
&= \delta([\nu^{\frac{3}{2}}\rho, \nu^{b'_1}\rho]) \times \delta([\nu^{\frac{3}{2}}\rho, \nu^{b'_2}\rho]) \times \cdots \times \delta([\nu^{\frac{3}{2}}\rho, \nu^{b'_k}\rho]) \otimes r_{GL}(\delta(\nu^{\frac{1}{2}}\rho, \cdots, \nu^{\frac{1}{2}}\rho; \sigma_{cusp})) \\
&\cong \delta([\nu^{\frac{3}{2}}\rho, \nu^{b'_1}\rho]) \times \delta([\nu^{\frac{3}{2}}\rho, \nu^{b'_2}\rho]) \times \cdots \times \delta([\nu^{\frac{3}{2}}\rho, \nu^{b'_k}\rho]) \otimes r_{GL}(\pi)
\end{aligned}$$

which appears with multiplicity one in the Jacquet module of $\delta([\nu^{\frac{3}{2}}\rho, \nu^{b'_1}\rho]) \times \delta([\nu^{\frac{3}{2}}\rho, \nu^{b'_2}\rho]) \times \cdots \times \delta([\nu^{\frac{3}{2}}\rho, \nu^{b'_k}\rho]) \times \nu^{\frac{1}{2}}\rho \times \cdots \times \nu^{\frac{1}{2}}\rho \rtimes \sigma_{cusp}$. □

Theorem 5.12. *The representation $\sigma^*_{(b'_1, \ldots, b'_k; \frac{1}{2})}$ is not strongly positive when $k \geq 2$.*

Proof. Suppose that $\sigma^*_{(b'_1, \ldots, b'_k; \frac{1}{2})}$ is strongly positive. Since each representation $\nu^{\frac{1}{2}}\rho \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{b'_i}\rho])$ is irreducible for all $i = 1, \ldots, k$, we have the following embedding:

$$\begin{aligned}
&\sigma^*_{(b'_1, \ldots, b'_k; \frac{1}{2})} \hookrightarrow \delta([\nu^{\frac{1}{2}}\rho, \nu^{b'_1}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{b'_2}\rho]) \times \cdots \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{b'_k}\rho]) \rtimes \sigma_{cusp} \\
&\hookrightarrow \delta([\nu^{\frac{3}{2}}\rho, \nu^{b'_1}\rho]) \times \nu^{\frac{1}{2}}\rho \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{b'_2}\rho]) \times \cdots \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{b'_k}\rho]) \rtimes \sigma_{cusp} \\
&\cong \delta([\nu^{\frac{3}{2}}\rho, \nu^{b'_1}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{b'_2}\rho]) \times \nu^{\frac{1}{2}}\rho \times \cdots \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{b'_k}\rho]) \rtimes \sigma_{cusp} \\
&\qquad \vdots \\
&\cong \delta([\nu^{\frac{3}{2}}\rho, \nu^{b'_1}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{b'_2}\rho]) \times \cdots \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{b'_k}\rho]) \times \nu^{\frac{1}{2}}\rho \rtimes \sigma_{cusp} \\
&\cong \delta([\nu^{\frac{3}{2}}\rho, \nu^{b'_1}\rho]) \times \delta([\nu^{\frac{3}{2}}\rho, \nu^{b'_2}\rho]) \times \nu^{\frac{1}{2}}\rho \times \cdots \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{b'_k}\rho]) \times \nu^{\frac{1}{2}}\rho \rtimes \sigma_{cusp} \\
&\qquad \vdots \\
&\cong \delta([\nu^{\frac{3}{2}}\rho, \nu^{b'_1}\rho]) \times \delta([\nu^{\frac{3}{2}}\rho, \nu^{b'_2}\rho]) \times \cdots \times \delta([\nu^{\frac{3}{2}}\rho, \nu^{b'_k}\rho]) \times \nu^{\frac{1}{2}}\rho \times \cdots \times \nu^{\frac{1}{2}}\rho \rtimes \sigma_{cusp}.
\end{aligned}$$

Since $\sigma^*_{(\frac{1}{2}, \frac{1}{2}; \frac{1}{2})}$ is the unique subrepresentation of $\nu^{\frac{1}{2}}\rho \times \nu^{\frac{1}{2}}\rho \rtimes \sigma_{cusp}$, Lemma 5.11 implies the following embedding:

$$\sigma^*_{(b'_1, \ldots, b'_k; \frac{1}{2})} \hookrightarrow \delta([\nu^{\frac{3}{2}}\rho, \nu^{b'_1}\rho]) \times \cdots \times \nu^{\frac{1}{2}}\rho \rtimes \sigma^*_{(\frac{1}{2}, \frac{1}{2}; \frac{1}{2})}$$

which implies that $\sigma^*_{(\frac{1}{2}, \frac{1}{2}; \frac{1}{2})}$ is strongly positive. This is a contradiction by Lemma 5.7. □

5.3. **Classification of strongly positive representations.** Let ρ_i be an essentially self-dual irreducible supercuspidal representation of $\mathbf{GL}_{n_{\rho_i}}(F)$ for $i = 1, \ldots, k$ and $\sigma_{cusp} \in R(n')$ is an irreducible supercuspidal representation of $\mathbf{G}_{n_{\sigma_{cusp}}}(F)$. Let $D(\rho_1, \rho_2, \ldots, \rho_k; \sigma_{cusp})$ be the set of strongly positive representations whose supercuspidal supports are the representation σ_{cusp} and twists of the representations ρ_i by positive valued characters for $i = 1, \ldots, k$. Let $a_{\rho_i} \geq 0$ be the unique nonnegative real number such that $\nu^{a_{\rho_i}}\rho_i \rtimes \sigma_{cusp}$ reduces for each $i = 1, \ldots, k$ ([29]).

Furthermore, we assume that this reducibility point a_{ρ_i} is in $\frac{1}{2}\mathbb{Z}$ (see (HI) of [20], page 771).

Theorem 5.13. *Let σ be strongly positive representation in $D(\rho_1, \rho_2, \ldots, \rho_k; \sigma_{cusp})$. Then σ can be considered as the unique irreducible subrepresentation of the following induced representation:*

$$(\prod_{i=1}^{k}\prod_{j=1}^{k_i} \delta([\nu^{a_{\rho_i}-k_i+j}\rho_i, \nu^{b_j^{(i)}}\rho_i])) \rtimes \sigma_{cusp} \tag{5.12}$$

where $k_i \in \mathbb{Z}_{\geq 0}$, $k_i \leq \lceil a_{\rho_i} \rceil, b_j^{(i)} > 0$ such that $b_j^{(i)} - a_{\rho_i} \in \mathbb{Z}_{\geq 0}$, for $i = 1, \ldots, k$ $j = 1, \ldots, k_i$. Also, $b_j^{(i)} < b_{j+1}^{(i)}$ for $1 \leq j \leq k_i - 1$.

Proof. Theorem 5.3 and Theorem 5.1 implies that there exist strongly positive segments $\Delta_1, \Delta_2, \ldots, \Delta_q$ such that $0 < e(\Delta_1) \leq e(\Delta_2) \leq \cdots \leq e(\Delta_q)$ and $\sigma \cong \delta(\Delta_1, \Delta_2, \ldots, \Delta_q; \sigma_{cusp})$. We describe these segments more precisely. Let $I_i := \{n_1, \ldots, n_{k_i}\} \subset \{1, \ldots, q\}$, for $i = 1, \ldots, k$, be the index such that $\{\Delta_j \mid j \in I_i\} \subset \{\Delta_1, \ldots, \Delta_q\}$ is the set of segments whose partial supercuspidal supports are twists of ρ_i and $e(\delta(\Delta_{n_1})) \leq \cdots \leq e(\delta(\Delta_{n_{k_i}}))$. Since $\delta(\Delta_m) \times \delta(\Delta_n)$ is irreducible for $m \notin I_i$ and $n \in I_i$, the representation $\delta(\Delta_1) \times \delta(\Delta_2) \times \cdots \times \delta(\Delta_q) \rtimes \sigma_{cusp}$ is isomorphic to the representation $(\prod_{j\notin I_i} \delta(\Delta_j)) \times \delta(\Delta_{n_1}) \times \cdots \times \delta(\Delta_{n_{k_i}}) \rtimes \sigma_{cusp}$. Since $\delta(\Delta_{n_1}, \ldots, \Delta_{n_{k_i}}; \sigma_{cusp})$ is subrepresentation of $\delta(\Delta_{n_1}) \times \cdots \times \delta(\Delta_{n_{k_i}}) \rtimes \sigma_{cusp}$, σ is the unique irreducible subrepresentation of the representation $\prod_{j\notin I_i} \delta(\Delta_j) \rtimes \delta(\Delta_{n_1}, \ldots, \Delta_{n_{k_i}}; \sigma_{cusp})$. The strong positivity of σ implies the strong positivity of $\delta(\Delta_{n_1}, \ldots, \Delta_{n_{k_i}}; \sigma_{cusp})$. When $a_{\rho_i} \neq \frac{1}{2}$, Theorem 5.4 implies that $\prod_{j=1}^{k_i} \delta(\Delta_{n_j}) \cong \prod_{j=1}^{k_i} \delta([\nu^{a_{\rho_i}-k_i+j}\rho_i, \nu^{b_j^{(i)}}\rho_i])$ which is of the form (5.12). When $a_{\rho_i} = \frac{1}{2}$, we have $\prod_{j=1}^{k_i} \delta(\Delta_{n_j}) \cong \prod_{j=1}^{k_i} \delta([\nu^{a_{\rho_i}}\rho_i, \nu^{b_j^{(i)}}\rho_i])$ which can be of the form (5.12) only if $k_i = 1$. Since σ is strongly positive, Theorem 5.12 implies that $k_i = 1$ when $a_{\rho_i} = \frac{1}{2}$. Since i can be an arbitrary integer in $\{1, \ldots, k\}$, the theorem follows. $\square$

In the following theorem, we show the uniqueness of strongly positive segments in the above theorem (5.12) as in the special case (Theorem 5.5) in Section 5.2.

Theorem 5.14. *Suppose that the representation σ can be considered as the unique irreducible subrepresentations of both representations $(\prod_{i=1}^{k}\prod_{j=1}^{k_i} \delta([\nu^{a_{\rho_i}-k_i+j}\rho_i, \nu^{b_j^{(i)}}\rho_i])) \rtimes \sigma_{cusp}$ and $(\prod_{i=1}^{k'}\prod_{j=1}^{k'_i} \delta([\nu^{a_{\rho'_i}-k'_i+j}\rho'_i, \nu^{c_j^{(i)}}\rho'_i])) \rtimes \sigma'_{cusp})$ as in Theorem 5.13. Then $k =$*

$k', \sigma_{cusp} \cong \sigma'_{cusp}$ *and* $\{\prod_{j=1}^{k_i} \delta([\nu^{a_{\rho_i}-k_i+j}\rho_i, \nu^{b_j^{(i)}}\rho_i])|i = 1, \ldots, k\}$ *is a permutation of* $\{\prod_{j=1}^{k'_i} \delta([\nu^{a_{\rho'_i}-k'_i+j}\rho'_i, \nu^{c_j^{(i)}}\rho'_i])|i = 1, \ldots k\}$.

Proof. We sketch the proof without repeating the whole arguments. The uniqueness of the partial supercuspidal support implies that $\sigma'_{cusp} = \sigma_{cusp}$. We have the following two embeddings:

$$(5.13)\ \ \sigma \hookrightarrow \prod_{j=1}^{k_1} \delta([\nu^{a_{\rho_1}-k_1+j}\rho_1, \nu^{b_j^{(1)}}\rho_1]) \times (\prod_{i=2}^{k}\prod_{j=1}^{k_i} \delta([\nu^{a_{\rho_i}-k_i+j}\rho_i, \nu^{b_j^{(i)}}\rho_i])) \rtimes \sigma_{cusp}$$

$$(5.14)\ \ \sigma \hookrightarrow \prod_{j=1}^{k'_1} \delta([\nu^{a_{\rho'_1}-k'_1+j}\rho'_1, \nu^{c_j^{(1)}}\rho'_1]) \times (\prod_{i=2}^{k'}\prod_{j=1}^{k'_i} \delta([\nu^{a_{\rho'_i}-k'_i+j}\rho'_i, \nu^{c_j^{(i)}}\rho'_i])) \rtimes \sigma'_{cusp}.$$

In the same way as in the proof of Theorem 5.5, Theorem 3.4 implies that there are $a_{\rho_i} - k_i + j - 1 \leq x_j^{(i)} \leq y_j^{(i)} \leq b_j^{(i)}$ such that $\prod_{i=1}^{k}\prod_{j=1}^{k_i} (\delta([\nu^{-x_j^{(i)}}\rho_i, \nu^{-a_{\rho_i}+k_i-j}\rho_i]) \times \delta([\nu^{y_j^{(i)}+1}\rho_i, \nu^{b_j^{(i)}}\rho_i])) \geq \delta([\nu^{a_{\rho'_1}-k'_1+1}\rho'_1, \nu^{c_1^{(1)}}\rho'_1])$. There exists i such that $\rho_i \cong \rho'_1$. Without loss of generality, we assume $i = 1$. Since $\rho_l \not\cong \rho'_1$ for $l = 2, \ldots, k$, we have $\prod_{j=1}^{k_1} (\delta([\nu^{-x_j^{(1)}}\rho_1, \nu^{-a_{\rho_1}+k_1-j}\rho_1]) \times \delta([\nu^{y_j^{(1)}+1}\rho_1, \nu^{b_j^{(1)}}\rho_1])) \geq \delta([\nu^{a_{\rho'_1}-k'_1+1}\rho'_1, \nu^{c_1^{(1)}}\rho'_1])$.

Now we are in the same situation as in Theorem 5.5. Therefore, we conclude that first segments in (5.13) and (5.14) are same. Proceeding in the same way, we conclude that $\prod_{j=1}^{k_1} \delta([\nu^{a_{\rho_1}-k_1+j}\rho_1, \nu^{b_j^{(1)}}\rho_1]) = \prod_{j=1}^{k'_1} \delta([\nu^{a_{\rho'_1}-k'_1+j}\rho'_1, \nu^{c_j^{(1)}}\rho'_1])$. Since $\sigma \hookrightarrow \prod_{j=1}^{k'_l} \delta([\nu^{a_{\rho'_l}-k'_l+j}\rho'_l, \nu^{c_j^{(l)}}\rho'_l]) \times (\prod_{i\neq l}\prod_{j=1}^{k'_i} \delta([\nu^{a_{\rho'_i}-k'_i+j}\rho'_i, \nu^{c_j^{(i)}}\rho'_i])) \rtimes \sigma'_{cusp}$, in the same way as above, we conclude the theorem. □

Theorem 5.13 and Theorem 5.14 imply that there exists an injective mapping from $D(\rho_1, \rho_2, \ldots, \rho_k; \sigma_{cusp})$ into the set of induced representations of the form (5.12). Since any strongly positive representation in R can be considered as the elements in $D(\rho'_1, \rho'_2, \ldots, \rho'_k; \sigma'_{cusp})$ for some ρ'_i and σ'_{cusp}. We can extend this mapping to any strongly positive representation in R. Let SP be the set of all strongly positive representations in R. To see this mapping explicitly, let us collect the data from induced representation of the form (5.12). Let LJ be the set of $(Jord, \sigma')$ where $Jord = \bigcup_{i=1}^{k}\bigcup_{j=1}^{k_i}\{(\rho_i, b_j^{(i)})\}$ and σ' be an irreducible supercuspidal representation in R such that

(i) $\{\rho_1, \rho_2, \ldots, \rho_k\} \subset R^{gen}$ is a (possibly empty) set of mutually non-isomorphic irreducible essentially self-dual supercuspidal unitary representations such that $\nu^{a'_{\rho_i}}\rho_i \rtimes \sigma'$ reduces for $a'_{\rho_i} > 0$ (this defines a'_{ρ_i}),

(ii) $k_i = \lceil a'_{\rho_i} \rceil$,

(iii) for each $i = 1, 2, \dots, k, b_1^{(i)}, b_2^{(i)}, \dots, b_{k_i}^{(i)}$ is a sequence of real numbers such that $a'_{\rho_i} - b_j^{(i)} \in \mathbb{Z}$, for $j = 1, 2, \dots, k_i$, and $-1 < b_1^{(i)} < b_2^{(i)} < \cdots < b_{k_i}^{(i)}$.

Now, it remains to show that this mapping is surjective.

Theorem 5.15. *The maps described above give a bijective correspondence between the sets SP and LJ.*

Proof. Let $(Jord, \sigma')$ denote an element of LJ, where $Jord = \bigcup_{i=1}^{k} \bigcup_{j=1}^{k_i} \{(\rho_i, b_j^{(i)})\}$. Let σ be the unique irreducible subrepresentation of $(\prod_{i=1}^{k} \prod_{j=1}^{k_i} \delta([\nu^{a'_{\rho_i} - k_i + j} \rho_i, \nu^{b_j^{(i)}} \rho_i])) \rtimes \sigma'$. Suppose that σ is not strongly positive. Then, there exists an embedding $\sigma \hookrightarrow \nu^{s_1} \rho_{i_1} \times \cdots \times \nu^{s_{i_l}} \rho_{i_l} \times \cdots \times \nu^{s_m} \rho_{i_m} \rtimes \sigma'$ such that $s_{i_l} \leq 0$. Without loss of generality, we assume that $i_l = 1$ since $\rho_{i_l} \in \{\rho_1, \dots, \rho_n\}$. Frobenius reciprocity implies that σ contains $\nu^{s_1} \rho_{i_1} \otimes \cdots \otimes \nu^{s_{i_l}} \rho_1 \otimes \cdots \otimes \nu^{s_m} \rho_{i_m} \otimes \sigma'$ in its Jacquet module. Since $\rho_p \not\cong \rho_q$ for $p \neq q$ and $\sigma \hookrightarrow (\prod_{i=2}^{k} \prod_{j=1}^{k_i} \delta([\nu^{a'_{\rho_i} - k_i + j} \rho_i, \nu^{b_j^{(i)}} \rho_i])) \rtimes \delta([\nu^{a'_{\rho_1} - k_1 + 1} \rho_1, \nu^{b_1^{(1)}} \rho_1], \cdots, [\nu^{a'_{\rho_1}} \rho_1, \nu^{b_{k_1}^{(1)}} \rho_1]; \sigma')$, $\nu^{s_{i_l}} \rho_1$ appears in the Jacqeut module of $\delta([\nu^{a'_{\rho_1} - k_1 + 1} \rho_1, \nu^{b_1^{(1)}} \rho_1], \cdots, [\nu^{a'_{\rho_1}} \rho_1, \nu^{b_{k_1}^{(1)}} \rho_1]; \sigma')$. However, Theorem 5.6 implies that $\delta([\nu^{a'_{\rho_1} - k_1 + 1} \rho_1, \nu^{b_1^{(1)}} \rho_1], \cdots, [\nu^{a'_{\rho_1}} \rho_1, \nu^{b_{k_1}^{(1)}} \rho_1]; \sigma')$ is strongly positive which is a contradiction. $\square$

6. Embeddings of discrete series and its applications

The strongly positive representations can be considered as basic building blocks for all discrete series representations (Theorem 6.2). We apply ideas and adapt some proofs from [18, Chapter 3] to our situation and the $GSpin$ case. Theorem 6.2 gives partial result of the first step in the filtration of admissible representation (1.1) and it has an interesting application on the proof of the equality of L-functions through local Langlands correspondence ([16]).

Lemma 6.1. *Suppose that σ is an irreducible representation of $\boldsymbol{G}_n(F)$ which is not a discrete series representation. Then there exists an embedding of the form $\sigma \hookrightarrow \delta([\nu^a \rho, \nu^b \rho]) \rtimes \sigma'$ where $a + b \leq 0$, $\rho \in R^{gen}$ is an irreducible unitary supercuspidal representation and $\sigma' \in R$ is an irreducible representation.*

Proof. Suppose that σ is an irreducible representation of $\mathbf{G}_n(F)$ which is not a discrete series representation. In the same way as in the proof of Theorem 5.3, we conclude that there exist a sequence of segments $\Delta_1, \cdots, \Delta_k$ satisfying $e(\Delta_1) \leq \cdots \leq e(\Delta_k)$ and an irreducible supercuspidal representation $\sigma_{cusp} \in R$ such that we have $\sigma \hookrightarrow \delta(\Delta_1) \times \cdots \times \delta(\Delta_k) \rtimes \sigma_{cusp}$. A slight variation of Casselman's square integrable criterion for $GSpin$ groups ([14, Proposition 3.8 and 3.9]) implies that $e(\Delta_1) \leq 0$. Now Lemma 3.2 of [20] finishes the proof. $\square$

Theorem 6.2. *Let σ denote a discrete series representation of $\boldsymbol{G}_n(F)$. Then there exists an embedding of the form*

$$\sigma \hookrightarrow \delta([\nu^{a_1} \rho_1, \nu^{b_1} \rho_1]) \times \delta([\nu^{a_2} \rho_2, \nu^{b_2} \rho_2]) \times \cdots \times \delta([\nu^{a_r} \rho_r, \nu^{b_r} \rho_r]) \rtimes \sigma_{sp}$$

where $a_i \leq 0, a_i + b_i > 0$ *and* $\rho_i \in R^{gen}$ *is an irreducible unitary supercuspidal representation for* $i = 1, \ldots, r$*, where* $\sigma_{sp} \in R$ *is a strongly positive representation (we allow* $k = 0$*).*

Proof. We briefly explain the main ideas of the proof. Let σ be a discrete series representation of $\mathbf{G}_n(F)$. If σ is strongly positive, the theorem follows with $k = 0$ and $\sigma = \sigma_{sp}$. Suppose that σ is not strongly positive. In the same way as in the proof of Theorem 5.3, we conclude that there exist a sequence of segments $\Delta_1, \cdots, \Delta_k$ satisfying $e(\Delta_1) \leq \cdots \leq e(\Delta_k)$ and an irreducible supercuspidal representation $\sigma_{cusp} \in R$ such that we have $\sigma \hookrightarrow \delta(\Delta_1) \times \cdots \times \delta(\Delta_k) \rtimes \sigma_{cusp}$. Write $\Delta_i := [\nu^{a_i}\rho_i, \nu^{b_i}\rho_i]$, where $\rho_i \in R^{gen}$ is an irreducible unitary supercuspidal representation for $i = 1, \ldots, k$. Let $a := min\{a_i \mid 1 \leq i \leq k\}$. Since we assume that σ is not strongly positive, $a \leq 0$. Let $j := min\{i \in \{1, \ldots, k\} \mid a_i = a\}$. We have $b_l \leq b_j$ for $l = 1, \ldots, j-1$ since $e(\Delta_l) \leq e(\Delta_j)$ for $l = 1, \ldots, j-1$. This implies that Δ_l and Δ_j are not connected in the sense of Zelevinsky for $l = 1, \ldots, j-1$. Therefore, we obtain $\sigma \hookrightarrow \delta(\Delta_j) \times \delta(\Delta_1) \times \delta(\Delta_2) \times \cdots \times \delta(\Delta_{j-1}) \times \delta(\Delta_{j+1}) \times \cdots \times \delta(\Delta_k) \rtimes \sigma_{cusp}$. Lemma 3.2 of [20] implies that there exists an irreducible representation $\sigma_1 \in R$ such that $\sigma \hookrightarrow \delta(\Delta_j) \rtimes \sigma_1$. We show that σ_1 is discrete series. Suppose that σ_1 is not discrete series representation. Lemma 6.1 implies that there exists an embedding of the form $\sigma_1 \hookrightarrow \delta([\nu^{a'}\rho, \nu^{b'}\rho]) \rtimes \sigma'$, where $a' + b' \leq 0$. Therefore, $\sigma \hookrightarrow \delta([\nu^{a_j}\rho_j, \nu^{b_j}\rho_j]) \times \delta([\nu^{a'}\rho, \nu^{b'}\rho]) \rtimes \sigma'$. Since a_j is the minimum of unitary exponents, $a_j \leq a'$. The inequality $a' + b' \leq 0 < a_j + b_j$ implies that $[\nu^{a_j}\rho_j, \nu^{b_j}\rho_j]$ and $[\nu^{a'}\rho, \nu^{b'}\rho]$ are not connected in the sense of Zelevinsky. Therefore, we have $\sigma \hookrightarrow \delta([\nu^{a'}\rho, \nu^{b'}\rho]) \times \delta([\nu^{a_j}\rho, \nu^{b_j}\rho]) \rtimes \sigma'$ which is a contradiction since σ is discrete series. We conclude that σ_1 is also discrete series representation. If σ_1 is strongly positive, the theorem follows with $k = 1$ and $\sigma_{cusp} = \sigma_1$. If not, in the same way as above, σ_1 can be embedded into the representation $\delta([\nu^{a_{j'}}\rho_{j'}, \nu^{b_{j'}}\rho_{j'}]) \rtimes \sigma_2$, where $a_{j'} \leq 0$, $a_{j'} + b_{j'} > 0$ and σ_2 is discrete series. We repeat this argument until we get strongly positive. Then, theorem follows. □

Appendix A. Strongly positive representations in an exceptional rank-one reducibility case

by Ivan Matić

The purpose of this appendix is to provide a proper treatment of an exceptional case which appears in the investigation of strongly positive discrete series in [17]. As in [17], we choose to work with metaplectic groups, but the same arguments can be used in the classical or *GSpin* group case.

Let $\sigma \in D(\rho, \sigma_{cusp})$ denote a strongly positive discrete series of a metaplectic group over a non-archimedean local field F of a characteristic different than two, $\sigma \neq \sigma_{cusp}$. Also, we suppose that ρ is a self-dual irreducible genuine cuspidal representation of $\widetilde{GL(l, F)}$, a two-fold cover of the general linear group. Further, let $\sigma = \delta(\Delta_1, \Delta_2, \ldots, \Delta_k; \sigma_{cusp})$, i.e., we realize σ as a unique irreducible subrepresentation of the induced representation of the form

$$\delta(\Delta_1) \times \delta(\Delta_2) \times \cdots \times \delta(\Delta_k) \rtimes \sigma_{cusp},$$

where $\Delta_1, \Delta_2, \ldots, \Delta_k$ is a sequence of strongly positive genuine segments satisfying $0 < e(\Delta_1) \leq e(\Delta_2) \leq \cdots \leq e(\Delta_k)$. Let us write $\Delta_i = [\nu^{a_i}\rho, \nu^{b_i}\rho]$, $i = 1, 2, \ldots, k$. The following result complements Theorem 4.4 of [17].

Theorem A.1. *Suppose that $\nu^s\rho \rtimes \sigma_{cusp}$ reduces for $s = \frac{1}{2}$. Then $k = 1$ and $a_1 = \frac{1}{2}$.*

Proof. Strong positivity of σ and assumption of the theorem immediately give $a_k = \frac{1}{2}$.

Observe that σ is a subrepresentation of $\delta(\Delta_1)\times\cdots\times\delta(\Delta_{k-2})\rtimes\delta(\Delta_{k-1},\Delta_k;\sigma_{cusp})$. Thus, it is enough to prove that $\delta(\Delta_{k-1},\Delta_k;\sigma_{cusp})$ is not strongly positive, i.e., to prove $k \neq 2$.

Suppose, on the contrary, $k = 2$. Then we have an embedding $\sigma \hookrightarrow \delta([\nu^{a_1}\rho,\nu^{b_1}\rho])\times \delta([\nu^{\frac{1}{2}}\rho,\nu^{b_2}\rho]) \rtimes \sigma_{cusp}$. If $a_1 > \frac{1}{2}$ we get $b_1 < b_2$ and in the same way as in the proof of Theorem 4.4 from [17] we obtain a contradiction with the strong positivity of σ.

It remains to consider the case $a_1 = \frac{1}{2}$.

We will first show that there are no strongly positive irreducible subquotients of $\nu^{\frac{1}{2}}\rho \rtimes \delta([\nu^{\frac{1}{2}}\rho,\nu^{b_2}\rho];\sigma_{cusp})$, using induction over $b_2 - \frac{1}{2}$.

First, it can be seen in the same way as in discussion preceding Proposition 3.12 of [9] that the representation $\nu^{\frac{1}{2}}\rho \rtimes \delta(\nu^{\frac{1}{2}}\rho;\sigma_{cusp})$ does not have a strongly positive irreducible subquotient (it contains two irreducible subquotients, the Langlands quotient and a tempered representation).

Further, the representation $\pi = \nu^{\frac{1}{2}}\rho \rtimes \delta([\nu^{\frac{1}{2}}\rho,\nu^{\frac{3}{2}}\rho];\sigma_{cusp})$ does not have a strongly positive irreducible subquotient since we have:

$$\begin{aligned} r_{GL}(\pi) &= \nu^{\frac{1}{2}}\rho\times\delta([\nu^{\frac{1}{2}}\rho,\nu^{\frac{3}{2}}\rho])\otimes\sigma_{cusp} + \nu^{-\frac{1}{2}}\rho\times\delta([\nu^{\frac{1}{2}}\rho,\nu^{\frac{3}{2}}\rho])\otimes\sigma_{cusp} \\ r_{(l)}(\pi) &= \nu^{\frac{1}{2}}\rho\otimes\delta([\nu^{\frac{1}{2}}\rho,\nu^{\frac{3}{2}}\rho];\sigma_{cusp}) + \nu^{-\frac{1}{2}}\rho\otimes\delta([\nu^{\frac{1}{2}}\rho,\nu^{\frac{3}{2}}\rho];\sigma_{cusp}) + \\ &\quad \nu^{\frac{3}{2}}\rho\otimes\nu^{\frac{1}{2}}\rtimes\delta(\nu^{\frac{1}{2}}\rho;\sigma_{cusp}). \end{aligned}$$

If σ' is some strongly positive irreducible subquotient of π, then obviously $r_{GL}(\sigma') \geq \nu^{\frac{1}{2}}\rho\times\delta([\nu^{\frac{1}{2}}\rho,\nu^{\frac{3}{2}}\rho])\otimes\sigma_{cusp}$, but this implies that $r_{(l)}(\sigma')$ contains some irreducible subquotient of $\nu^{\frac{3}{2}}\rho\otimes\nu^{\frac{1}{2}}\rtimes\delta(\nu^{\frac{1}{2}}\rho;\sigma_{cusp})$, contradicting strong positivity of σ'.

Let us now suppose that the representation $\nu^{\frac{1}{2}}\rho \rtimes \delta([\nu^{\frac{1}{2}}\rho,\nu^{m}\rho];\sigma_{cusp})$ does not have a strongly positive irreducible subquotient for $m < b_k$. We study the induced representation $\pi = \nu^{\frac{1}{2}}\rho \rtimes \delta([\nu^{\frac{1}{2}}\rho,\nu^{b_k}\rho];\sigma_{cusp})$. Similarly as in the previously considered case we have:

$$\begin{aligned} r_{GL}(\pi) &= \nu^{\frac{1}{2}}\rho\times\delta([\nu^{\frac{1}{2}}\rho,\nu^{b_k}\rho])\otimes\sigma_{cusp} + \nu^{-\frac{1}{2}}\rho\times\delta([\nu^{\frac{1}{2}}\rho,\nu^{b_k}\rho])\otimes\sigma_{cusp} \\ r_{(l)}(\pi) &= \nu^{\frac{1}{2}}\rho\otimes\delta([\nu^{\frac{1}{2}}\rho,\nu^{b_k}\rho];\sigma_{cusp}) + \nu^{-\frac{1}{2}}\rho\otimes\delta([\nu^{\frac{1}{2}}\rho,\nu^{b_k}\rho];\sigma_{cusp}) + \\ &\quad \nu^{b_k}\rho\otimes\nu^{\frac{1}{2}}\rtimes\delta([\nu^{\frac{1}{2}}\rho,\nu^{b_k-1}\rho];\sigma_{cusp}). \end{aligned}$$

Using the inductive assumption, in completely same way as in the case $b_k = \frac{3}{2}$ we deduce that π does not contain a strongly positive irreducible subquotient.

Now, suppose that strongly positive discrete series σ is a subrepresentation of $\delta([\nu^{\frac{1}{2}}\rho,\nu^{b_1}\rho]) \rtimes \delta([\nu^{\frac{1}{2}}\rho,\nu^{b_2}\rho];\sigma_{cusp})$, where $b_1 \leq b_2$. We have

$$\sigma \hookrightarrow \delta([\nu^{\frac{3}{2}}\rho,\nu^{b_1}\rho])\times\nu^{\frac{1}{2}}\rho\rtimes\delta([\nu^{\frac{1}{2}}\rho,\nu^{b_2}\rho];\sigma_{cusp}).$$

There is some irreducible representation π such that $\sigma \hookrightarrow \delta([\nu^{\frac{3}{2}}\rho,\nu^{b_1}\rho]) \rtimes \pi$. Since σ is strongly positive, π also has to be strongly positive. Also, Frobenius reciprocity gives $\mu^*(\sigma) \geq \delta([\nu^{\frac{3}{2}}\rho,\nu^{b_1}\rho])\otimes\pi$. Using the structural formula for μ^*, we get $\pi \leq \nu^{\frac{1}{2}}\rho\times\delta([\nu^{\frac{3}{2}}\rho,\nu^{b_2}\rho])\times\nu^{\frac{1}{2}}\rho\rtimes\sigma_{cusp}$. Since π is strongly positive, looking at Jacquet modules with respect to Siegel parabolic subgroup first we obtain $\pi \leq$

$\nu^{\frac{1}{2}}\rho \times \delta([\nu^{\frac{3}{2}}\rho, \nu^{b_2}\rho]) \rtimes \delta(\nu^{\frac{1}{2}}\rho; \sigma_{cusp})$. Since $\delta([\nu^{\frac{1}{2}}\rho, \nu^{b_2}\rho]; \sigma_{cusp})$ is the only irreducible subquotient of $\delta([\nu^{\frac{3}{2}}\rho, \nu^{b_2}\rho]) \rtimes \delta(\nu^{\frac{1}{2}}\rho; \sigma_{cusp})$ whose Jacquet module with respect to Siegel parabolic subgroup contains only representations of the form $\pi' \otimes \sigma_{cusp}$ with no $\nu^x\rho$, $x \leq 0$, appearing in the cuspidal support of π', it follows that π is a subquotient of $\nu^{\frac{1}{2}}\rho \rtimes \delta([\nu^{\frac{1}{2}}\rho, \nu^{b_2}\rho]; \sigma_{cusp})$, a contradiction.

Therefore, $k = 1$ and the proof is complete. □

Department of Mathematics, University of Osijek, Osijek, Croatia
E-mail address: imatic@mathos.hr

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Department of Mathematics, University of Iowa, Iowa city, IA
E-mail address: yeansu-kim@uiowa.edu