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# Scaling properties of the empirical structure function of linear fractional stable motion and estimation of its parameters

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**Abstract:** Linear fractional stable motion is an example of a self-similar stationary increments stochastic process exhibiting both long-range dependence and heavy-tails. In this paper we propose methods that are able to estimate simultaneously the self-similarity parameter and the tail parameter. These methods are based on the asymptotic behavior of the so-called “empirical structure function”, a statistic which resembles a sample moment of the process. We show and use the fact that the rate of growth of the empirical structure function is determined by the Hurst parameter and the tail index. We test the methods on simulated data and apply them to network traffic and solar flares data.

**Keywords:** Hurst parameter, tail index, self-similarity, stable index.

## 1 Introduction

Empirical time series which appear in many applications display both the “Joseph” and “Noah” effects, as coined by Mandelbrot (see e.g. Mandelbrot (1997)). While the Joseph effect refers to long-range dependence of the increments, the Noah effect refers to their high variability as expressed by power law tails of the marginal distributions. Fractional Brownian motion is an example of a process exhibiting only the Joseph effect: its increments are long-range dependent but with normal marginal distribution. On the other hand, the  $\alpha$ -stable Lévy process with  $0 < \alpha < 2$  exhibits only the Noah effect: it has independent but heavy-tailed increments with tail index equal to  $\alpha$ .

An example of a stochastic process which exhibits both effects is the linear fractional stable motion (LFSM). LFSM can be defined through the stochastic integral

$$X(t) = \frac{1}{C_{H,\alpha}} \int_{\mathbb{R}} \left( (t-u)_+^{H-1/\alpha} - (-u)_+^{H-1/\alpha} \right) M(du), \quad t \in \mathbb{R}, \quad (1)$$

where  $\alpha \in (0, 2)$ ,  $0 < H < 1$ ,  $(x)_+ = \max(x, 0)$  and where  $M$  is a random noise. More specifically,  $M$  is an  $\alpha$ -stable random measure on  $\mathbb{R}$  with Lebesgue control measure  $\lambda$  and skewness  $\beta$ . This means first that  $M$  is an independently scattered random measure: if  $A_1$  and  $A_2$  are disjoint sets, then  $M(A_1)$  and  $M(A_2)$  are independent random variables. Secondly, for all sets  $A$  such that  $\lambda(A) < \infty$ ,  $M(A)$  has an  $\alpha$ -stable distribution with scale parameter  $\lambda(A)^{1/\alpha}$  and skewness parameter  $\beta$ , i.e.  $M(A) \sim S_\alpha(\lambda(A)^{1/\alpha}, \beta, 0)$ . If  $\alpha = 1$  we

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assume  $\beta = 0$ , but for other values of  $\alpha$ ,  $M$  is allowed to be skewed. In general, a random variable  $X$  has an  $\alpha$ -stable distribution with index of stability  $\alpha \in (0, 2)$ , scale parameter  $\sigma \in (0, \infty)$ , skewness parameter  $\beta \in [-1, 1]$  and shift parameter  $\mu \in \mathbb{R}$ , denoted by  $X \sim S_\alpha(\sigma, \beta, \mu)$  if its characteristic function has the following form

$$E \exp\{i\zeta X\} = \begin{cases} \exp\left\{-\sigma^\alpha |\zeta|^\alpha \left(1 - i\beta \operatorname{sign}(\zeta) \tan \frac{\alpha\pi}{2} + i\zeta\mu\right)\right\}, & \text{if } \alpha \neq 1, \\ \exp\left\{-\sigma |\zeta| \left(1 - i\beta \frac{2}{\pi} \operatorname{sign}(\zeta) \ln |\zeta| + i\zeta\mu\right)\right\}, & \text{if } \alpha = 1, \end{cases} \quad \zeta \in \mathbb{R}.$$

If the constant  $C_{H,\alpha}$  in the representation (1) is chosen such that the scaling parameter of  $X(1)$  equals 1, i.e.

$$C_{H,\alpha} = \left( \int_{\mathbb{R}} |(1-u)_+^{H-1/\alpha} - (-u)_+^{H-1/\alpha}|^\alpha du \right)^{1/\alpha}.$$

then the process is called standard LFSM. The stationary sequence  $Y_i = X(i) - X(i-1)$ ,  $i \in \mathbb{N}$  is referred to as the *linear fractional stable noise*. The process LFSM  $\{X(t)\}$  is  $H$ -self-similar with stationary increments, that is for every  $a > 0$ , process  $\{X(at), t \in \mathbb{R}\}$  has the same finite dimensional distributions as  $\{a^H X(t), t \in \mathbb{R}\}$  (see (Samorodnitsky & Taqqu 1994, Proposition 7.4.2)). Setting  $\alpha = 2$  in (1) reduces the LFSM to the fractional Brownian motion. By analogy to this process, the case  $H > 1/\alpha$  is referred to as a long-range dependence and the case  $H < 1/\alpha$  as negative dependence. Since the second moment and thus the correlation function of  $X(t)$  is infinite, there are other ways to get a feeling for the dependence structure of the process. This is usually done using codifferences (Samorodnitsky & Taqqu (1994)) but see also a more recent approach described in Magdziarz (2009).

For each  $t$ ,  $X(t)$  has a strictly stable distribution with stable index  $\alpha$ . The parameter  $\alpha$  governs the tail behavior of the marginal distributions in the sense that for each  $t$ ,  $X(t)$  is heavy-tailed with tail index  $\alpha$ , i.e.  $P(|X(t)| > x) = L(x)x^{-\alpha}$  where  $L$  is a slowly varying function, that is,  $L(ax)/L(x) \rightarrow 1$  as  $|x| \rightarrow \infty$ , for every  $a > 0$ . This implies, in particular, that  $E|X(t)|^q = \infty$  for  $q \geq \alpha$ . See Samorodnitsky & Taqqu (1994) for more details.

Since LFSM combines both heavy-tails and long-range dependence it provides a rich modeling potential (see e.g. Willinger et al. (1998)). It is therefore important to have methods of estimating the parameters  $\alpha$  and  $H$ . Standard estimators of the Hurst exponent  $H$  usually assume that the underlying process has finite variance and this makes them inappropriate for the case of LFSM. Also, estimators of the tail index are known to behave well mostly on independent or weakly dependent samples (see e.g. Embrechts et al. (1997)). It is therefore necessary to construct an estimator of both parameters that takes into account the special structure of the LFSM.

A wavelet based estimator of the parameter  $H$  for the LFSM has been proposed in Stoev & Taqqu (2003) (see also Phipps et al. (2007) and Stoev & Taqqu (2005)). The authors define two estimators both of which are shown to be strongly consistent and asymptotically normal under some conditions. These estimators do not require the knowledge of  $\alpha$ . In Ayache & Hamonier (2012), a wavelet-based estimator of  $\alpha$  has been defined. However this method requires one to know the  $H$  value first. For the more general

model of linear multifractional stable motion, estimators of  $\alpha$  and the Hurst functional parameter  $H(\cdot)$  have also been developed (see [Ayache & Hamonier \(2013\)](#)).

In this paper we develop methods that are able to simultaneously estimate both parameters  $\alpha$  and  $H$ . The methods are based on the asymptotic behavior of the so-called “empirical structure function” which is essentially a  $q$ -th order sample moment of  $|X(t)|$ . Asymptotic properties of the empirical structure function have been investigated before in the context of multifractal theory ([Heyde & Sly \(2008\)](#)). Our method is based on the fact that the rate of growth of the empirical structure function depends on the parameters  $H$  and  $\alpha$ . We derive a limit in probability which identifies the relation between the two parameters. This is then used to define the so-called “scaling function” which provides an alternative method of estimation. We test the methods on simulated data in order to compare their performance.

## 2 Empirical structure function and the scaling function

Suppose  $\{X(t)\}$  is LFSM that is sampled in a regularly spaced time instants,  $X(\delta)$ ,  $X(2\delta)$ ,  $\dots$ ,  $X(n\delta)$ . For simplicity of notation, we assume  $\delta = 1$ , so we have a sample  $X_1, \dots, X_n$ . Based on a sample, the empirical structure function can be defined as follows:

$$S_q(n, t) = \frac{1}{[n/t]} \sum_{i=1}^{[n/t]} |X_{i[t]} - X_{(i-1)[t]}|^q, \quad (2)$$

where  $q \in \mathbb{R}$  and  $1 \leq t \leq n$ . We thus partition the data into consecutive blocks of length  $[t]$ , sum each block and take the power  $q$  of the absolute value of the sum. Finally, we average over all  $[n/t]$  blocks. For this reason, (2) is sometimes called “partition function”. Notice that for  $t = 1$  one gets the usual empirical  $q$ -th absolute moment. Because of the stationary increments, it is natural to use the empirical structure function as an estimator of the  $q$ -th absolute moment of  $X(t)$ .

Asymptotic properties of  $S_q(n, t)$  have been considered in the context of multifractality detection ([Grahovac et al. \(2014\)](#), [Grahovac & Leonenko \(2014\)](#), [Heyde \(2009\)](#)), and in particular for LFSM in [Heyde & Sly \(2008\)](#). We go over the methodology from [Grahovac et al. \(2014\)](#) to establish the asymptotic properties. Instead of keeping  $t$  fixed, we take it to be of the form  $t = n^s$  for some  $s \in (0, 1)$ , which allows the blocks to grow as the sample size increases

$$S_q(n, n^s) = \frac{1}{n^{1-s}} \sum_{i=1}^{n^{1-s}} |X_{in^s} - X_{(i-1)n^s}|^q.$$

Since  $s > 0$ ,  $S_q(n, n^s)$  will diverge as  $n \rightarrow \infty$ . We are interested in the rate of divergence of this statistic measured as a power of  $n$ . We can get this by considering the limiting behavior of  $\ln S_q(n, n^s) / \ln n$ . One can think of the limiting value as the value of the smallest power of  $n$  needed to normalize the empirical structure function in such a way that it will converge to some random variable not identically equal to zero.

In our analysis we will also include a range of negative  $q$  values. Although this may seem unusual, finite negative order moments provide additional information on the value of the Hurst parameter. In particular, for  $q \in (-1, 0)$ , stable-distributed random variables have finite  $q$ -th absolute moment since their probability density function is bounded (see e.g. [Zolotarev \(1986\)](#)).

The main argument in establishing asymptotic properties of the empirical structure function is based on the following lemma. A similar result has been proved in [Heyde & Sly \(2008\)](#), yet we prove it here with much simpler arguments.

**Lemma 1.** *Suppose  $X_i, i \in \mathbb{N}$  is a discretely observed sample from LFSM  $\{X(t)\}$  such that  $X(i) = X_i$ . Then for  $q \geq \alpha$ ,*

$$\frac{\ln(\sum_{i=1}^n |X_i - X_{i-1}|^q)}{\ln n} \xrightarrow{P} \frac{q}{\alpha},$$

as  $n \rightarrow \infty$ , where  $\xrightarrow{P}$  stands for convergence in probability.

*Proof.* Let  $\varepsilon > 0$ . Denote by  $Y_i = X_i - X_{i-1}$  and notice this is a stationary sequence. Suppose  $\delta < \varepsilon/(q - \alpha)$  and define

$$Z_{i,n} = Y_i \mathbf{1}(|Y_i| \leq n^{\frac{1}{\alpha} + \delta}), \quad i = 1, \dots, n, \quad n \in \mathbb{N}.$$

It follows from Karamata's theorem ([Embrechts et al. \(1997\)](#)) that for arbitrary  $r > \alpha$

$$\begin{aligned} E|Z_{i,n}|^r &= \int_0^\infty P(|Z_{i,n}|^r > x) dx = \int_0^{n^{r(\frac{1}{\alpha} + \delta)}} P(|Y_i|^r > x) dx \\ &= \int_0^{n^{r(\frac{1}{\alpha} + \delta)}} L(x^{\frac{1}{r}}) x^{-\frac{\alpha}{r}} dx \leq C_1 L(n^{\frac{1}{\alpha} + \delta}) n^{r(\frac{1}{\alpha} + \delta)(-\frac{\alpha}{r} + 1)} \\ &= C_1 L(n^{\frac{1}{\alpha} + \delta}) n^{\frac{r}{\alpha} - 1 + \delta(r - \alpha)} \end{aligned} \tag{3}$$

Next, notice that

$$P\left(\max_{i=1, \dots, n} |Y_i| > n^{\frac{1}{\alpha} + \delta}\right) \leq \sum_{i=1}^n P(|Y_i| > n^{\frac{1}{\alpha} + \delta}) \leq C_2 n \frac{L(n^{\frac{1}{\alpha} + \delta})}{(n^{\frac{1}{\alpha} + \delta})^\alpha} \leq C_2 \frac{L(n^{\frac{1}{\alpha} + \delta})}{n^{\alpha\delta}}.$$

Now by partitioning on the event

$$\{Y_i = Z_{i,n}, i = 1, \dots, n\} = \{Y_i \leq n^{\frac{1}{\alpha} + \delta}, i = 1, \dots, n\} = \left\{ \max_{i=1, \dots, n} |Y_i| \leq n^{\frac{1}{\alpha} + \delta} \right\},$$

using Markov's inequality and (3) we have

$$\begin{aligned}
P\left(\frac{\ln(\sum_{i=1}^n |Y_i|^q)}{\ln n} > \frac{q}{\alpha} + \varepsilon\right) &= P\left(\sum_{i=1}^n |Y_i|^q > n^{\frac{q}{\alpha} + \varepsilon}\right) \\
&\leq P\left(\sum_{i=1}^n |Y_i|^q > n^{\frac{q}{\alpha} + \varepsilon}, \max_{i=1, \dots, n} |Y_i| \leq n^{\frac{1}{\alpha} + \delta}\right) + P\left(\max_{i=1, \dots, n} |Y_i| > n^{\frac{1}{\alpha} + \delta}\right) \\
&\leq P\left(\sum_{i=1}^n |Z_{i,n}|^q > n^{\frac{q}{\alpha} + \varepsilon}\right) + P\left(\max_{i=1, \dots, n} |Y_i| > n^{\frac{1}{\alpha} + \delta}\right) \leq \frac{nE|Z_{i,n}|^q}{n^{\frac{q}{\alpha} + \varepsilon}} + C_2 \frac{L(n^{\frac{1}{\alpha} + \delta})}{n^{\alpha\delta}} \\
&\leq \frac{n}{n^{\frac{q}{\alpha} + \varepsilon}} C_1 L(n^{\frac{1}{\alpha} + \delta}) n^{\frac{q}{\alpha} - 1 + \delta(q - \alpha)} + C_2 \frac{L(n^{\frac{1}{\alpha} + \delta})}{n^{\alpha\delta}} \\
&\leq C_1 L(n^{\frac{1}{\alpha} + \delta}) n^{\delta(q - \alpha) - \varepsilon} + C_2 \frac{L(n^{\frac{1}{\alpha} + \delta})}{n^{\alpha\delta}} \rightarrow 0,
\end{aligned}$$

as  $n \rightarrow \infty$ , since  $\delta(q - \alpha) - \varepsilon < 0$  and  $L(x)$  is slowly varying, thus bounded by any positive power of  $x$ .

For the reverse inequality notice that since  $Y_i$  is a stationary strictly  $\alpha$ -stable sequence corresponding to a dissipative flow it follows by Theorem 4.8 in Samorodnitsky (2004) that  $\max_{i=1, \dots, n} |Y_i|/n^{1/\alpha}$  converges in distribution to some positive random variable. So for any  $\delta > 0$ ,

$$P\left(\max_{i=1, \dots, n} |Y_i| < n^{\frac{1}{\alpha} - \delta}\right) \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Now it follows that

$$\begin{aligned}
P\left(\frac{\ln(\sum_{i=1}^n |Y_i|^q)}{\ln n} < \frac{q}{\alpha} - \varepsilon\right) &= P\left(\sum_{i=1}^n |Y_i|^q < n^{\frac{q}{\alpha} - \varepsilon}\right) \\
&\leq P\left(\max_{i=1, \dots, n} |Y_i| < n^{\frac{1}{\alpha} - \frac{\varepsilon}{q}}\right) \rightarrow 0,
\end{aligned}$$

as  $n \rightarrow \infty$  which proves the statement.  $\square$

**Theorem 1.** Suppose  $X_i, i \in \mathbb{N}$  is a discretely observed sample from LFSM  $\{X(t)\}$  such that  $X(i) = X_i$ . Then for  $q > -1$  and every  $s \in [0, 1]$

$$\frac{\ln S_q(n, n^s)}{\ln n} \xrightarrow{P} R_{H, \alpha}(q, s) := \begin{cases} sqH, & \text{if } q < \alpha, \\ s\left(1 + qH - \frac{q}{\alpha}\right) + \frac{q}{\alpha} - 1, & \text{if } q \geq \alpha, \end{cases} \quad (4)$$

as  $n \rightarrow \infty$ .

*Proof.* By  $H$ -self-similarity of LFSM it follows that

$$S_q(n, n^s) \stackrel{d}{=} \frac{n^{sqH}}{n^{1-s}} \sum_{i=1}^{n^{1-s}} |X_i - X_{i-1}|^q. \quad (5)$$

The sequence  $X_i - X_{i-1}, i \in \mathbb{N}$  is a stationary linear fractional stable noise and thus it is a stable mixed moving average which is known to be ergodic (see Cambanis et al. (1987),

Pipiras & Taqqu (2002), Surgailis et al. (1993)). For  $q \in (-1, \alpha)$ ,  $E|X_i - X_{i-1}|^q < \infty$ , so it follows by the ergodic theorem that

$$\frac{S_q(n, n^s)}{n^{sqH}} \rightarrow E|X(1)|^q \text{ a.s.}$$

In particular,

$$\ln n \left( \frac{\ln S_q(n, n^s)}{\ln n} - sqH \right) = \ln S_q(n, n^s) - \ln n^{sqH} \rightarrow \ln E|X(1)|^q \text{ a.s.}$$

which implies the statement of the theorem for  $q < \alpha$ .

Now we consider the case  $q \geq \alpha$ . We have by (5) that

$$\begin{aligned} & \frac{\ln S_q(n, n^s)}{\ln n} - s \left( 1 + qH - \frac{q}{\alpha} \right) - \left( \frac{q}{\alpha} - 1 \right) \\ &= \frac{(sqH - 1 + s) \ln n + \ln \left( \sum_{i=1}^{n^{1-s}} |X_i - X_{i-1}|^q \right)}{\ln n} - s \left( 1 + qH - \frac{q}{\alpha} \right) - \left( \frac{q}{\alpha} - 1 \right) \\ &= \frac{\ln \left( \sum_{i=1}^{n^{1-s}} |X_i - X_{i-1}|^q \right)}{\ln n} - \frac{q}{\alpha} (1 - s) \\ &= \frac{\ln \left( \sum_{i=1}^{n^{1-s}} |X_i - X_{i-1}|^q \right)}{\ln n^{1-s}} (1 - s) - \frac{q}{\alpha} (1 - s). \end{aligned}$$

Since by Lemma 1

$$\frac{\ln \left( \sum_{i=1}^{n^{1-s}} |X_i - X_{i-1}|^q \right)}{\ln n^{1-s}} \xrightarrow{P} \frac{q}{\alpha}, \quad \text{as } n \rightarrow \infty,$$

it follows that

$$\begin{aligned} & P \left( \left| \frac{\ln S_q(n, n^s)}{\ln n} - s \left( 1 + qH - \frac{q}{\alpha} \right) - \left( \frac{q}{\alpha} - 1 \right) \right| > \varepsilon \right) \\ &= P \left( \left| \frac{\ln \left( \sum_{i=1}^{n^{1-s}} |X_i - X_{i-1}|^q \right)}{\ln n^{1-s}} - \frac{q}{\alpha} \right| (1 - s) > \varepsilon \right) \rightarrow 0, \end{aligned}$$

as  $n \rightarrow \infty$  and this proves the second case.  $\square$

As it is clear from the theorem, the rate of growth of the empirical structure function is heavily influenced by two parameters, namely  $H$  and  $\alpha$ . This motivates the idea of estimating these parameters from the structure function. We describe the method in the next section, but first we establish results that will be the basis for another method of estimation.

The second estimation method we propose relies on ideas presented in Grahovac et al. (2014) and Grahovac & Leonenko (2014) where similar results were used to establish an estimation method for the tail index of weakly dependent samples. It is clear from (4)

that  $\ln S_q(n, n^s)/\ln n$  should behave approximately linearly in  $s$ . It thus makes sense to focus on the slope of the simple linear regression of  $\ln S_q(n, n^s)/\ln n$  on  $s$ .

Fixing  $q$  and taking  $0 \leq s_1 < \dots < s_N \leq 1$  we can form a set of  $N$  points

$$(s_i, \ln S_q(n, n^{s_i})/\ln n), \quad i = 1, \dots, N.$$

Using the well known formula for the slope of the linear regression line, we define the empirical scaling function (slope) based on these points to be

$$\hat{\tau}_{N,n}(q) = \frac{\sum_{i=1}^N s_i \frac{\ln S_q(n, n^{s_i})}{\ln n} - \frac{1}{N} \sum_{i=1}^N s_i \sum_{j=1}^N \frac{\ln S_q(n, n^{s_j})}{\ln n}}{\sum_{i=1}^N (s_i)^2 - \frac{1}{N} \left( \sum_{i=1}^N s_i \right)^2}. \quad (6)$$

The next theorem establishes the asymptotic properties of  $\hat{\tau}_{N,n}$ .

**Theorem 2.** *Suppose that the assumptions of Theorem 1 hold and fix  $s_1, \dots, s_N$  in (6). Then, for every  $q > -1$ ,*

$$\hat{\tau}_{N,n}(q) \xrightarrow{P} \tau_{H,\alpha}^\infty(q) := \begin{cases} Hq, & \text{if } -1 < q < \alpha, \\ \left(H - \frac{1}{\alpha}\right)q + 1, & \text{if } q \geq \alpha, \end{cases} \quad (7)$$

as  $n \rightarrow \infty$ .

*Proof.* Fix  $q > -1$  and let  $\varepsilon > 0$ ,  $\delta > 0$  and

$$C = \sum_{i=1}^N (s_i)^2 - \frac{1}{N} \left( \sum_{i=1}^N s_i \right)^2 > 0.$$

By Theorem 1, for each  $i = 1, \dots, N$  there exist  $n_i^{(1)}$  such that

$$P \left( \left| \frac{\ln S_q(n, n^{s_i})}{\ln n} - R_{H,\alpha}(q, s_i) \right| > \frac{\varepsilon C}{2s_i N} \right) < \frac{\delta}{2N}, \quad n \geq n_i^{(1)}.$$

It follows then that for  $n \geq n_{max}^{(1)} := \max\{n_1^{(1)}, \dots, n_N^{(1)}\}$

$$\begin{aligned} & P \left( \left| \sum_{i=1}^N s_i \frac{\ln S_q(n, n^{s_i})}{\ln n} - \sum_{i=1}^N s_i R_{H,\alpha}(q, s_i) \right| > \frac{\varepsilon C}{2} \right) \\ & \leq P \left( \sum_{i=1}^N s_i \left| \frac{\ln S_q(n, n^{s_i})}{\ln n} - R_{H,\alpha}(q, s_i) \right| > \frac{\varepsilon C}{2} \right) \\ & \leq \sum_{i=1}^N P \left( \left| \frac{\ln S_q(n, n^{s_i})}{\ln n} - R_{H,\alpha}(q, s_i) \right| > \frac{\varepsilon C}{2s_i N} \right) < \frac{\delta}{2}. \end{aligned}$$

Similarly, for each  $i = 1, \dots, N$  there exist  $n_i^{(2)}$  such that

$$P \left( \left| \frac{\ln S_q(n, n^{s_i})}{\ln n} - R_{H,\alpha}(q, s_i) \right| > \frac{\varepsilon C}{2 \left( \sum_{i=1}^N s_i \right)} \right) < \frac{\delta}{2N}, \quad n \geq n_i^{(2)},$$



and for  $n \geq n_{max}^{(2)} := \max\{n_1^{(2)}, \dots, n_N^{(2)}\}$

$$\begin{aligned} & P \left( \left| \frac{1}{N} \sum_{i=1}^N s_i \sum_{j=1}^N \frac{\ln S_q(n, n^{s_j})}{\ln n} - \frac{1}{N} \sum_{i=1}^N s_i \sum_{j=1}^N R_{H,\alpha}(q, s_j) \right| > \frac{\varepsilon C}{2} \right) \\ & \leq P \left( \sum_{j=1}^N \left| \frac{\ln S_q(n, n^{s_j})}{\ln n} - R_{H,\alpha}(q, s_j) \right| > \frac{N\varepsilon C}{2 \left( \sum_{i=1}^N s_i \right)} \right) \\ & \leq \sum_{j=1}^N P \left( \left| \frac{\ln S_q(n, n^{s_j})}{\ln n} - R_{H,\alpha}(q, s_j) \right| > \frac{\varepsilon C}{2 \left( \sum_{i=1}^N s_i \right)} \right) < \frac{\delta}{2}. \end{aligned}$$

Finally then, for  $n \geq \max\{n_{max}^{(1)}, n_{max}^{(2)}\}$  it follows that

$$\begin{aligned} & P \left( \left| \hat{\tau}_{N,n}(q) - \frac{\sum_{i=1}^N s_i R_{H,\alpha}(q, s_i) - \frac{1}{N} \sum_{i=1}^N s_i \sum_{j=1}^N R_{H,\alpha}(q, s_j)}{\sum_{i=1}^N (s_i)^2 - \frac{1}{N} \left( \sum_{i=1}^N s_i \right)^2} \right| > \varepsilon \right) \\ & \leq P \left( \left| \sum_{i=1}^N s_i \frac{\ln S_q(n, n^{s_i})}{\ln n} - \sum_{i=1}^N s_i R_{H,\alpha}(q, s_i) \right| \right. \\ & \quad \left. + \left| \frac{1}{N} \sum_{i=1}^N s_i \sum_{j=1}^N \frac{\ln S_q(n, n^{s_j})}{\ln n} - \frac{1}{N} \sum_{i=1}^N s_i \sum_{j=1}^N R_{H,\alpha}(q, s_j) \right| > \varepsilon C \right) \\ & \leq P \left( \left| \sum_{i=1}^N s_i \frac{\ln S_q(n, n^{s_i})}{\ln n} - \sum_{i=1}^N s_i R_{H,\alpha}(q, s_i) \right| > \frac{\varepsilon C}{2} \right) \\ & \quad + P \left( \left| \frac{1}{N} \sum_{i=1}^N s_i \sum_{j=1}^N \frac{\ln S_q(n, n^{s_j})}{\ln n} - \frac{1}{N} \sum_{i=1}^N s_i \sum_{j=1}^N R_{H,\alpha}(q, s_j) \right| > \frac{\varepsilon C}{2} \right) < \delta, \end{aligned}$$

and thus

$$\hat{\tau}_{N,n}(q) \xrightarrow{P} \frac{\sum_{i=1}^N s_i R_{H,\alpha}(q, s_i) - \frac{1}{N} \sum_{i=1}^N s_i \sum_{j=1}^N R_{H,\alpha}(q, s_j)}{\sum_{i=1}^N (s_i)^2 - \frac{1}{N} \left( \sum_{i=1}^N s_i \right)^2}.$$

It remains to show that the right hand side is exactly  $\tau_{H,\alpha}^\infty(q)$  from (7). Indeed, when  $q < \alpha$  we have

$$\frac{\frac{1}{N} \sum_{i=1}^N q H s_i^2 - \frac{1}{N^2} \sum_{i=1}^N s_i \sum_{j=1}^N q H s_j}{\frac{1}{N} \sum_{i=1}^N (s_i)^2 - \frac{1}{N^2} \left( \sum_{i=1}^N s_i \right)^2} = Hq$$

and if  $q \geq \alpha$

$$\begin{aligned}
& \frac{\frac{1}{N} \sum_{i=1}^N s_i \left( (1 + qH - \frac{q}{\alpha}) s_i + \frac{q}{\alpha} - 1 \right) - \frac{1}{N^2} \sum_{i=1}^N s_i \sum_{j=1}^N \left( (1 + qH - \frac{q}{\alpha}) s_j + \frac{q}{\alpha} - 1 \right)}{\frac{1}{N} \sum_{i=1}^N (s_i)^2 - \frac{1}{N^2} \left( \sum_{i=1}^N s_i \right)^2} \\
&= \frac{\left( 1 + qH - \frac{q}{\alpha} \right) \frac{1}{N} \sum_{i=1}^N s_i^2 + \left( \frac{q}{\alpha} - 1 \right) \frac{1}{N} \sum_{i=1}^N s_i}{\frac{1}{N} \sum_{i=1}^N (s_i)^2 - \frac{1}{N^2} \left( \sum_{i=1}^N s_i \right)^2} \\
&\quad - \frac{\left( 1 + qH - \frac{q}{\alpha} \right) \frac{1}{N^2} \left( \sum_{i=1}^N s_i \right)^2 + \left( \frac{q}{\alpha} - 1 \right) \frac{1}{N} \sum_{i=1}^N s_i}{\frac{1}{N} \sum_{i=1}^N (s_i)^2 - \frac{1}{N^2} \left( \sum_{i=1}^N s_i \right)^2} \\
&= \left( H - \frac{1}{\alpha} \right) q + 1 + \frac{\left( \frac{q}{\alpha} - 1 \right) \frac{1}{N} \sum_{i=1}^N s_i - \left( \frac{q}{\alpha} - 1 \right) \frac{1}{N} \sum_{i=1}^N s_i}{\frac{1}{N} \sum_{i=1}^N (s_i)^2 - \frac{1}{N^2} \left( \sum_{i=1}^N s_i \right)^2} = \left( H - \frac{1}{\alpha} \right) q + 1.
\end{aligned}$$

□

### 3 Estimation methods

Based on the results of the preceding section we now specify the estimation methods for the parameters  $H$  and  $\alpha$ .

**Method M1** is based on the results of Theorem 1. By choosing points  $0 \leq s_1 < \dots < s_N \leq 1$  and  $q_j \in (-1, q_{max})$ ,  $j = 1, \dots, M$ , based on the sample of length  $n$  we can calculate

$$\left\{ \frac{\ln S_{q_j}(n, n^{s_i})}{\ln n} : i = 1, \dots, N, j = 1, \dots, M \right\}. \quad (8)$$

In simulations and examples below we choose  $q_{max} = 4$  to cover the critical range. As  $\ln S_{q_j}(n, n^{s_i}) / \ln n$  is expected to behave as  $R_{H,\alpha}(q_j, s_i)$  defined in (4), we define an estimator for  $(\alpha, H)$  by minimizing the difference between the two in the sense of the ordinary least squares, i.e.

$$(\hat{H}_1, \hat{\alpha}_1) = \arg \min_{(H,\alpha) \in (0,1) \times (0,2)} \sum_{i=1}^N \sum_{j=1}^M \left( \frac{\ln S_{q_j}(n, n^{s_i})}{\ln n} - R_{H,\alpha}(q_j, s_i) \right)^2. \quad (9)$$

Although method M1 follows naturally from Theorem 1, it has a disadvantage of being sensitive to the scale parameter of the data. Indeed, any scale parameter  $c$  would scale the empirical structure function by a factor  $|c|^q$ . On finite samples, this produces an additional term  $\ln |c|^q / \ln n$  in Equation (9) that needs to be compensated with the intercept of  $R_{H,\alpha}$ . To avoid this issue, we specify two other methods which are based only on the slope of  $R_{H,\alpha}$  and are not affected by a scale parameter.

**Method M2** is based on the empirical scaling function (6) and Theorem 2. In contrast to M1, we proceed here in two steps. First, based on the data set (8) for each  $q_j$ ,  $j = 1, \dots, M$  we compute the empirical scaling function  $\hat{\tau}_{N,n}(q_j)$  as defined in (6). Since for

large samples this converges to  $\tau_{H,\alpha}^\infty(q_j)$  for each  $j$ , we define estimators based on the scaling function

$$(\hat{H}_2, \hat{\alpha}_2) = \arg \min_{(H,\alpha) \in (0,1) \times (0,2)} \sum_{j=1}^M \left( \hat{\tau}_{N,n}(q_j) - \tau_{H,\alpha}^\infty(q_j) \right)^2. \quad (10)$$

The shape of the asymptotic scaling function  $\tau_{H,\alpha}^\infty$  in (7) is shown in the Figure 1 for a range of values of  $\alpha$  and  $H$ . Figure 1a shows the long-range dependent case. As indicated in (7), the scaling function is bilinear in  $q$  with the first part having the slope  $H$ . A break occurs at  $\alpha$  and the plot is linear again but now with the slope  $H - 1/\alpha$ . In the negative dependence case  $H < 1/\alpha$  (Figure 1b), the second part has a negative slope.

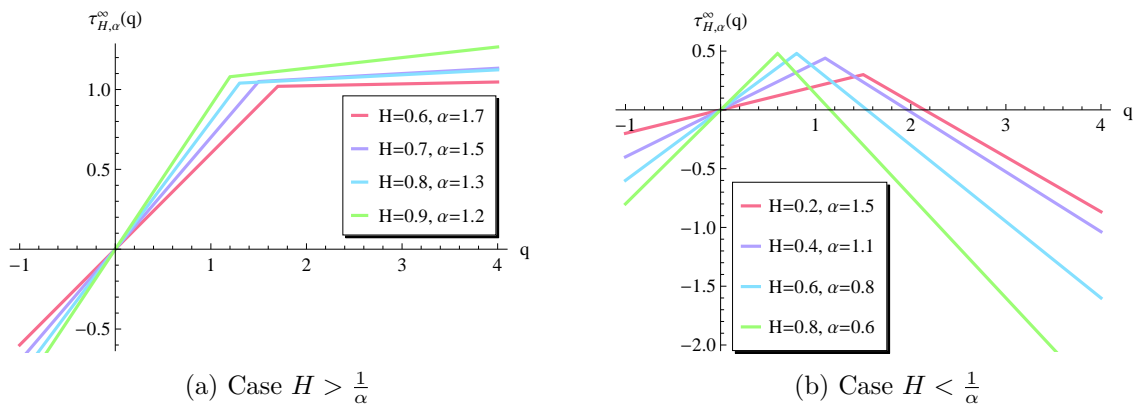


Figure 1: Asymptotic scaling function  $\tau_{H,\alpha}^\infty$

Figure 2 shows the estimated scaling functions (dashed) for some of the  $H$  and  $\alpha$  values presented in Figure 1. The estimation is based on one sample path realization of length 15784 (explained later). Here, and in every other example,  $s_i, i = 1, \dots, N$  in (6) are chosen equidistantly in the interval  $[0.1, 0.9]$  with  $N = 23$ . Scaling function is estimated at points  $q_j$  chosen equidistantly in the interval  $[-1, 4]$  with step 0.1. On each plot in Figure 2 the corresponding true scaling function is shown by a solid line. Although the break is not sharp, one can notice the bilinear shape.

The slope of the first part ( $q < \alpha$ ) of the estimated scaling function corresponds to  $H$ , the breakpoint corresponds to  $\alpha$  and the slope of the second part ( $q > \alpha$ ) contains information about both parameters  $H$  and  $\alpha$ . In some examples, as well as in those in Figure 2, the slope of the second part does not give the value  $H - 1/\alpha$  very precisely, although the first part and the breakpoint behave as expected from (7). The second part corresponds to the rate of growth under infinite moments, which makes it a sensitive quantity to measure. Moreover it depends on both parameters  $H$  and  $\alpha$ . This can affect the estimation even when there is an obvious bilinear shape. For this reason we provide alternative estimation method which uses only the information from the first part of the scaling function and the breakpoint.

**Method M3** fits the following general continuous bilinear function to the empirical

scaling function

$$\varsigma(q) = \begin{cases} aq, & \text{if } -1 < q \leq b, \\ cq + b(a - c), & \text{if } q > b. \end{cases} \quad (11)$$

Here we are interested only in parameter  $a$  which corresponds to  $H$  and  $b$  which corresponds to  $\alpha$ . The estimators in method M3 are now defined as

$$(\hat{H}_3, \hat{\alpha}_3, \hat{c}) = \arg \min_{(a,b,c) \in (0,1) \times (0,2) \times \mathbb{R}} \sum_{j=1}^M (\hat{\tau}_{N,n}(q_j) - \varsigma(q_j))^2. \quad (12)$$

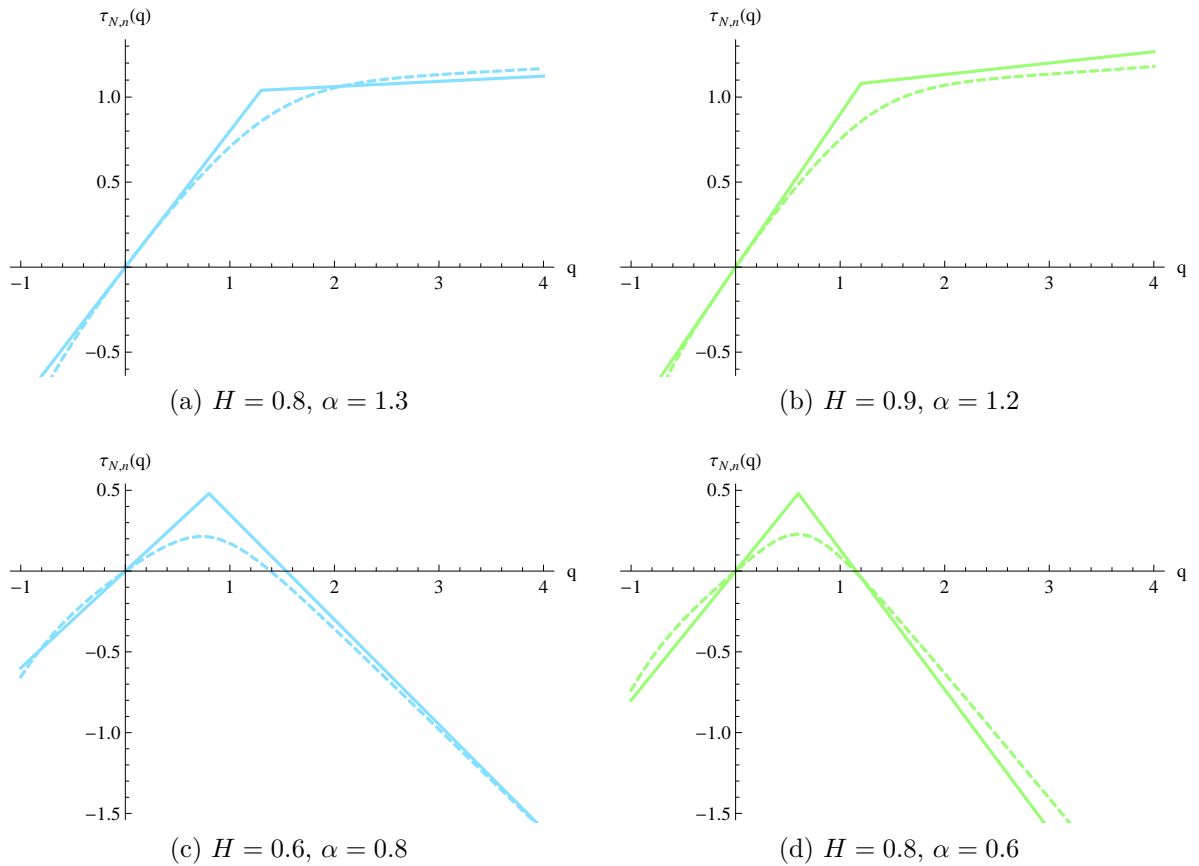


Figure 2: Estimated scaling functions (dashed) with the corresponding  $\tau_{H,\alpha}^\infty$

## 4 Simulation study

We use simulation to test the bias and variability of the estimators. We also compare the methods to see which one provides the best results.

In order to simulate paths of LFSM we have used FFT (fast Fourier transform) based algorithm described in [Stoev & Taqqu \(2004\)](#). All generated sample paths are of length 15784 and additional parameters of the generator are chosen to be  $m = 128$  and  $M = 600$ .

This makes  $m(M + 15784)$  to be a power of 2 and the algorithm uses FFT (see [Stoev & Taqqu \(2004\)](#) for more details). In all cases we use symmetric  $\alpha$ -stable LFSM and the scale parameter of  $X(1)$  is set to 1.

Simulations were conducted as follows. We chose for  $\alpha$  values 0.3, 0.7, 1, 1.3, 1.7 and for  $H$  values 0.2, 0.4, 0.6, 0.8 which makes a total of 20 cases. For each case, 100 sample paths of length 15784 have been simulated. For each sample we compute the estimates  $(\hat{H}_1, \hat{\alpha}_1)$ ,  $(\hat{\alpha}_2, \hat{H}_2)$  and  $(\hat{\alpha}_3, \hat{H}_3)$  corresponding to each of the methods. The mean bias and root mean square error (RMSE) of each estimator have been computed for each case and the results are shown in [Table 1](#) and [Table 2](#). We compare the methods by indicating the better values in bold.

Table 1: Bias of the estimators based on 100 sample paths

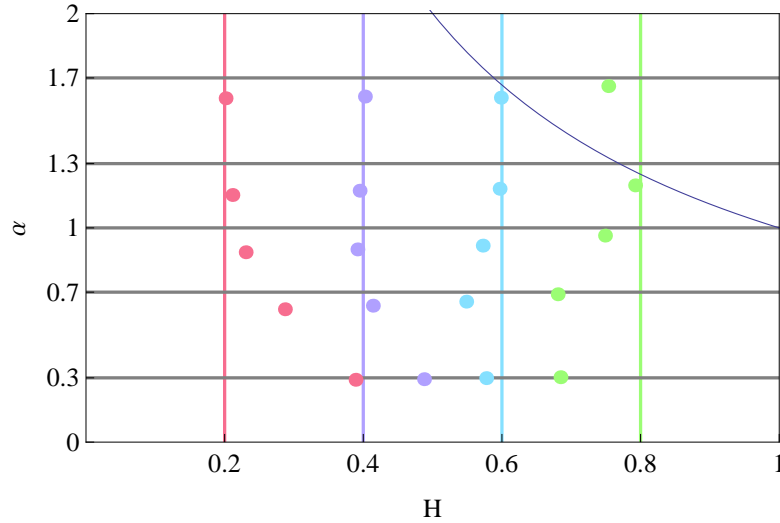
$H$	$\alpha$	$\hat{H}_1 - H$	$\hat{H}_2 - H$	$\hat{H}_3 - H$	$\hat{\alpha}_1 - \alpha$	$\hat{\alpha}_2 - \alpha$	$\hat{\alpha}_3 - \alpha$
0.2	0.3	<b>0.1076</b>	0.1898	0.1888	0.0199	0.0124	<b>-0.0089</b>
	0.7	<b>0.0349</b>	0.0472	0.0869	0.0325	<b>0.0317</b>	-0.0798
	1	0.0250	<b>-0.0162</b>	0.0304	<b>0.0396</b>	0.0958	-0.1137
	1.3	0.0227	-0.0377	<b>0.0112</b>	<b>0.0590</b>	0.1854	-0.1465
	1.7	0.0266	0.0048	<b>0.0014</b>	0.1640	0.2253	<b>-0.0950</b>
0.4	0.3	<b>-0.0129</b>	0.0877	0.0876	0.0243	0.0167	<b>-0.0062</b>
	0.7	-0.0208	-0.0208	<b>0.0138</b>	<b>0.0456</b>	0.0486	-0.0632
	1	<b>-0.0071</b>	-0.0590	-0.0085	<b>0.0543</b>	0.1222	-0.1005
	1.3	0.0071	-0.0533	<b>-0.0053</b>	<b>0.0707</b>	0.1951	-0.1265
	1.7	0.0176	0.0040	<b>0.0024</b>	0.1572	0.2045	<b>-0.0870</b>
0.6	0.3	-0.1210	-0.0240	<b>-0.0227</b>	0.0295	0.0242	<b>-0.0013</b>
	0.7	-0.0650	-0.0910	<b>-0.0515</b>	0.0627	0.0793	<b>-0.0442</b>
	1	<b>-0.0234</b>	-0.0855	-0.0277	<b>0.0663</b>	0.1423	-0.0828
	1.3	0.0064	-0.0432	<b>-0.0033</b>	<b>0.0579</b>	0.1546	-0.1174
	1.7	0.0139	-0.0022	<b>-0.0016</b>	0.1395	0.1899	<b>-0.0921</b>
0.8	0.3	-0.2230	-0.1159	<b>-0.1156</b>	0.0360	0.0305	<b>0.0030</b>
	0.7	<b>-0.1109</b>	-0.1507	-0.1195	0.0824	0.1097	<b>-0.0098</b>
	1	<b>-0.0309</b>	-0.0883	-0.0513	0.0622	0.1334	<b>-0.0360</b>
	1.3	<b>0.0072</b>	-0.0379	-0.0079	0.0276	0.1126	<b>-0.1016</b>
	1.7	<b>-0.0176</b>	-0.0356	-0.0466	0.1626	0.2261	<b>-0.0382</b>

Table 2: RMSE of the estimators based on 100 sample paths

$H$	$\alpha$	$\hat{H}_1$	$\hat{H}_2$	$\hat{H}_3$	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\alpha}_3$
0.2	0.3	0.2733	0.2537	<b>0.2516</b>	0.0416	0.0423	<b>0.0318</b>
	0.7	0.1301	<b>0.1155</b>	0.1333	<b>0.0866</b>	0.1124	0.1104
	1	0.0902	0.0899	<b>0.0838</b>	<b>0.1133</b>	0.2014	0.1513
	1.3	0.0652	0.0833	<b>0.0644</b>	<b>0.1461</b>	0.2902	0.1903
	1.7	0.0444	<b>0.0432</b>	0.0483	0.2238	0.2621	<b>0.2030</b>
0.4	0.3	0.2597	0.2000	<b>0.1975</b>	0.0454	0.0431	<b>0.0307</b>
	0.7	0.1438	0.1148	<b>0.1073</b>	0.0952	0.1128	<b>0.0940</b>
	1	0.0996	0.1167	<b>0.0811</b>	<b>0.1257</b>	0.2128	0.1385
	1.3	<b>0.0691</b>	0.0970	0.0731	<b>0.1607</b>	0.2940	0.1820
	1.7	<b>0.0418</b>	0.0432	0.0500	0.2266	0.2537	<b>0.1815</b>
0.6	0.3	0.2846	0.1965	<b>0.1935</b>	0.0503	0.0478	<b>0.0320</b>
	0.7	0.1654	0.1465	<b>0.1176</b>	0.1095	0.1258	<b>0.0828</b>
	1	0.1091	0.1292	<b>0.0963</b>	0.1390	0.2105	<b>0.1268</b>
	1.3	0.0731	0.0958	<b>0.0695</b>	0.1592	0.2577	<b>0.1517</b>
	1.7	<b>0.0413</b>	0.0469	0.0531	0.2206	0.2512	<b>0.1775</b>
0.8	0.3	0.3357	0.2264	<b>0.2261</b>	0.0566	0.0495	<b>0.0313</b>
	0.7	0.1881	0.1859	<b>0.1523</b>	0.1288	0.1446	<b>0.0612</b>
	1	0.1094	0.1350	<b>0.1003</b>	0.1415	0.1984	<b>0.0916</b>
	1.3	0.0746	0.0939	<b>0.0699</b>	0.1461	0.2255	<b>0.1363</b>
	1.7	<b>0.0446</b>	0.0563	0.0681	0.2305	0.2601	<b>0.1541</b>

When compared using the RMSE, it is clear from Table 2 that method M3 provides the best results for both parameters. Table 1 indicates that M1 and M3 provide smaller bias than M2, although the differences between the estimators are not substantial. Having in mind the sensitivity of M1 to the scale parameter, we can definitely recommend method M3 as the best one. The performance of method M3 is also shown in Figure 3. Mean estimates based on the 100 samples are shown as points in the  $(H, \alpha)$  plane and the gridlines show the true value of the parameters. For each value of  $H$  the corresponding points and gridline are shown with different colors. We also plot the function  $1/H$  to distinguish between long-range dependence ( $\alpha > 1/H$ ) and negative dependence ( $\alpha < 1/H$ ) case.

It can be seen from Figure 3 that the value of  $H$  does not seem to have a significant influence on the estimation of the tail index  $\alpha$ . However, the quality of the estimates of  $H$  worsens as  $\alpha$  takes smaller values. This can be explained from (4) since for small  $\alpha$ , the value of  $H$  has only a small impact on the shape of  $R_{H,\alpha}$ . One can see this also from the shape of the scaling function (7). The scaling function contains information on  $H$  in the slopes of the two parts of the broken line. When  $\alpha$  is small, the first linear part is short and the value  $1/\alpha$  dominates in the slope of the second part. This makes it hard to estimate  $H$  in this case.

Figure 3: Mean estimates of  $(\hat{H}_3, \hat{\alpha}_3)$ 

All three methods can estimate both parameters  $\alpha$  and  $H$  simultaneously. An estimator of the tail index based on the asymptotic behavior of the scaling function for *weakly dependent* sequences has been proposed in [Grahovac et al. \(2014\)](#). We find it interesting to also study how an estimation of the tail index would behave if the dependence structure were stronger and the parameter  $H$  measuring dependence is known. So we included in the simulation the behavior of the estimators M2 and M3 when  $H$  is known. We also did this for the estimators of  $H$  assuming  $\alpha$  is known. In these cases (10) and (12) reduce to the minimization of a univariate function.

When one of the parameters is known both methods M2 and M3 behave equally well. Here we present only mean estimates of method M2 (Figure 4). Mean estimates of  $\alpha$  when  $H$  is known are shown in Figure 4a. Figure 4b shows a similar plot for the estimated value of  $H$  assuming  $\alpha$  known. In the case of estimation of  $\alpha$  (Figure 4a), one sees that estimators based on the scaling function, like the one proposed in [Grahovac et al. \(2014\)](#), can perform well even under complicated dependence structure.

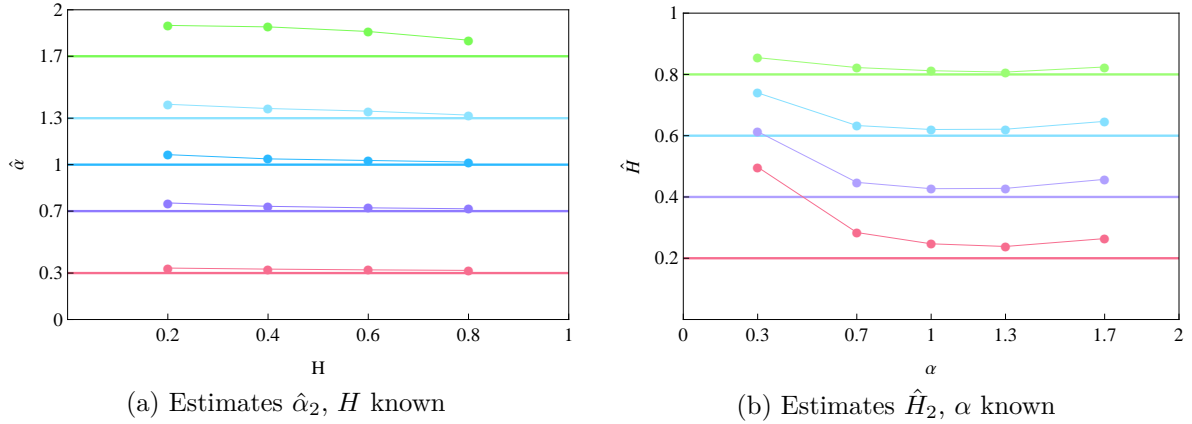


Figure 4: Mean estimates assuming one parameter is known

## 5 Real data applications

Empirical studies show that network traffic data can exhibit both self-similarity property and heavy tails (see e.g. [Leland et al. \(1994\)](#), [Willinger et al. \(1998\)](#)). Many models have been built explaining this behavior. In [Karasaridis & Hatzinakos \(2001\)](#), the authors propose to model network traffic as a linear transformation of the totally skewed linear fractional stable noise. Here we take one network traffic data set and assuming the data is a realization of this model we estimate the self-similarity and tail parameters.

The data we analyze is the Ethernet trace recorded at the Bellcore Morristown Research and Engineering facility (BC-Oct89Ext) (see [Leland et al. \(1994\)](#) and [Leland & Wilson \(1991\)](#) for more details). It contains packet arrival times (in seconds) and number of packets (in bytes). The original data has been modified by counting the packets in the blocks of 1 second. We express the time series as the number of packets per time unit and take only the first 25000 values, which is around 20% of all data (Figure 5a). The sample mean has been subtracted according to the model and the estimated scaling function (dotted) is shown in Figure 5b with the fitted bilinear function (11) (solid). The shape indeed resembles the one characteristic for the LFSM and estimation with method M3 yields values  $\hat{H} = 0.88$  and  $\hat{\alpha} = 1.33$ . The same data set has been analyzed in [Karasaridis & Hatzinakos \(2001\)](#). The authors report the estimated value 0.8 for the Hurst parameter and 1.63 for the tail index, which is in accordance with our analysis.



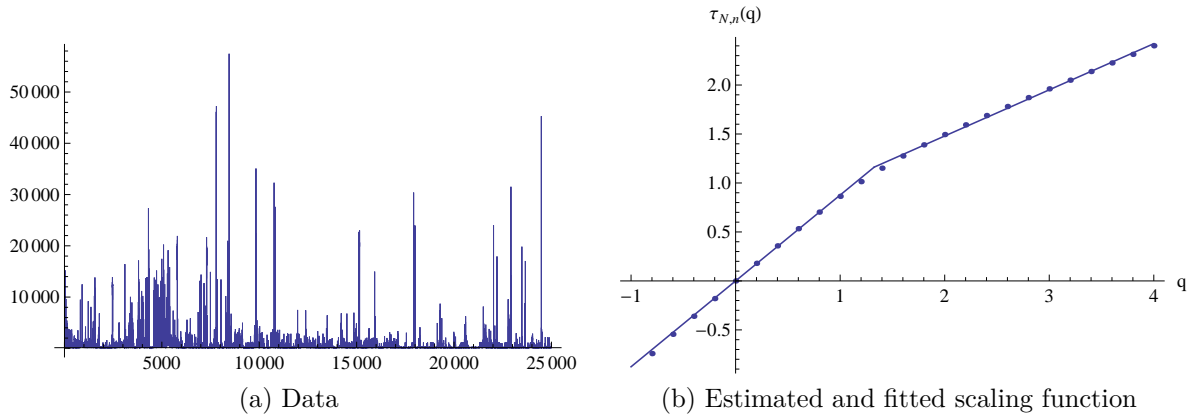


Figure 5: BC-Oct89Ext trace

For the second illustration we analyze the solar flare X-ray data observed by GOES satellite and publicly available at <http://www.ngdc.noaa.gov/stp/solar/solarflares.html>. This type of data is considered to exhibit both self-similarity and heavy tails and claimed to be modeled well with the LFSM (see Stanislavsky et al. (2009) and Weron et al. (2005)). Assuming the data is indeed a realization of the mean shifted linear fractional stable noise, we estimate the parameters  $H$  and  $\alpha$ . The data contains the information about the time of appearance and energy of the solar flares. We take the data in the period from August, 1999 to December, 2003, aggregate the maximum flux values on a daily basis and set the mean to 0, which provides 1405 data points. Figure 6a shows the plot of the data and Figure 6b the estimated scaling function with the fitted bilinear function (11). The estimated values of the parameters are  $\hat{H} = 0.75$  and  $\hat{\alpha} = 1.56$ .

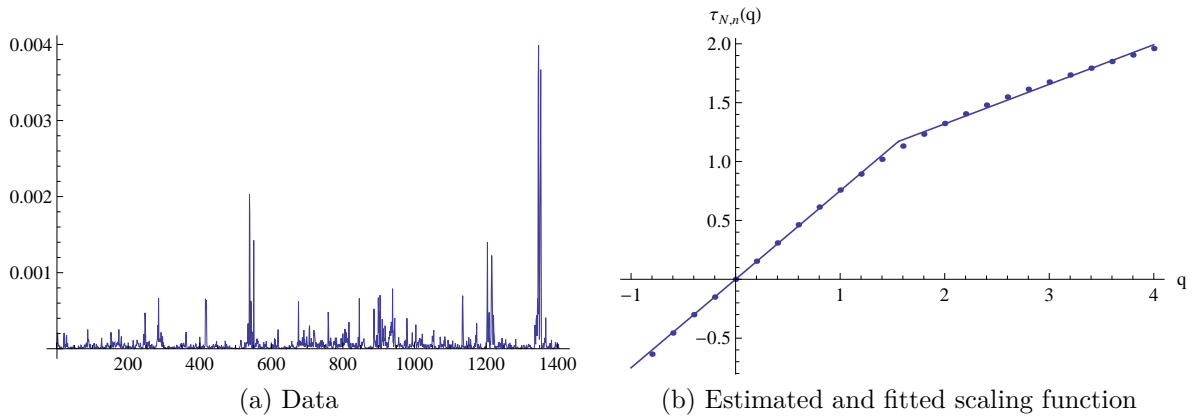


Figure 6: Solar flare X-ray data

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