# $D(-1)$-triples of the form $\{1, b, c\}$ in the ring $\mathbb{Z}[\sqrt{-t}], t > 0$

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**Abstract:** In this paper, we study $D(-1)$-triples of the form $\{1, b, c\}$ in the ring $\mathbb{Z}[\sqrt{-t}]$, $t > 0$, for positive integer $b$ such that $b$ is a prime, twice prime and twice prime squared. We prove that in those cases $c$ has to be an integer. In cases of $b = 26, 37$ or $50$ we prove that $D(-1)$-triples of the form $\{1, b, c\}$ cannot be extended to a $D(-1)$-quadruple in the ring $\mathbb{Z}[\sqrt{-t}], t > 0$, except in cases $t \in \{1, 4, 9, 25, 36, 49\}$. For those exceptional cases of $t$ we show that there exist infinitely many $D(-1)$-quadruples of the form $\{1, b, -c, d\}$, $c, d > 0$ in $\mathbb{Z}[\sqrt{-t}]$.

**Keywords:** Diophantine quadruples, quadratic field, simultaneous Pellian equations, linear form in logarithms

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## 1. Introduction

Let $R$ be a commutative ring and $z \in R$. A set $\{a_1, a_2, \dots, a_m\}$ in $R$ such that $a_i \neq 0$, $i = 1, \dots, m, a_i \neq a_j$ and $a_i a_j + z$ is a square in $R$ for all $1 \leq i < j \leq m$ is called *a Diophantine m-tuple with the property* $D(z)$, or simply *a* $D(z)$*-m-tuple* in the ring $R$. Diophantus of Alexandria was the first to look for such sets and it was in the case $z = 1$. He found the set of four rationals $\{1/16, 33/16, 17/4, 105/16\}$ with the property $D(1)$. However, Fermat found a first Diophantine quadruple with the property $D(1)$, the set $\{1, 3, 8, 120\}$ (see [6]).

Well studied is the case $z = -1$. Let us briefly review the history and some of the key results on this subject in integers and later in certain imaginary quadratic fields.

It is proved that some infinite families of $D(-1)$-triples cannot be extended to a $D(-1)$-quadruple. The non-extendibility of $\{1, b, c\}$ was confirmed for $b = 2$ by Dujella in [8], for $b = 5$ partially by Abu Muriefah and Al Rashed in [2], and completely by Filipin in [16]. The statement was also proved for $b = 10$ by Filipin in [16], and for $b = 17, 26, 37, 50$ by Fujita in [20]. While it is conjectured that $D(-1)$-quadruples do not exist in integers (see [9]), it is known that no $D(-1)$-quintuple exists and that if $\{a, b, c, d\}$ is a $D(-1)$-quadruple with $a < b < c < d$, then $a = 1$ (see [12]). Dujella, Filipin and Fuchs in [11] proved that there are at most finitely many $D(-1)$-quadruples, by giving an upper bound of $10^{903}$ for the number of $D(-1)$-quadruples. This bound was improved to $10^{356}$ by Filipin and Fujita in [17], and in [5] by Bonciocat, Cipu and Mignotte to $4 \cdot 10^{70}$. Very recently Elsholtz, Filipin and Fujita improved the result, i.e. the number of $D(-1)$-quadruples is less then $5 \cdot 10^{60}$ (see [15]).

Couple of authors tried to obtain some results about the existence of Diophantine quadruples over the imaginary quadratic fields. One important difference between real and complex quadratic fields is that in the real case there exist infinitely many units. The methods for the construction of Diophantine quadruples usually use elements with small norm, which makes a complex case harder to handle. Anyway, Dujella in [7] presented results on this subject in Gaussian integers. He proved that there does not exist a $D(a+bi)$-quadruple in $\mathbb{Z}[i]$ if $b$ is odd or $a \equiv b \equiv 2 \pmod 4$, i.e. if $a+bi$ is not representable as a difference of the squares of two elements in $\mathbb{Z}[i]$, and in contrary if $a+bi$ is not of such form and $a+bi \notin \{\pm 2, \pm 1 \pm 2i, \pm 4i\}$, then $D(a+bi)$-quadruple exists. In [18], Franušić also gave some results on Diophantine quadruples in Gaussian integers, and in [19] with Kreso showed that the Diophantine pair $\{1, 3\}$ cannot be extended to a Diophantine quintuple in the ring $\mathbb{Z}[\sqrt{-2}]$.

The problem of existence of $D(z)$-quadruples is almost completely solved in the ring $\mathbb{Z}[\sqrt{-2}]$. Several authors contributed to the characterization of elements $z$ in $\mathbb{Z}[\sqrt{-2}]$ for which a Diophantine quadruple with the property $D(z)$ exists (see [1, 14, 25]). In [26], author studied the existence of $D(-1)$-quadruples of the form $\{1,b,c,d\}, b\in\{2,5,10,17\}$, in the ring $\mathbb{Z}[\sqrt{-t}], t>0$. He proved that $D(-1)$-triples of the form $\{1,b,c\}$, for $b=2,5,10,17$, cannot be extended to a $D(-1)$-quadruple in the ring $\mathbb{Z}[\sqrt{-t}\,]$, except in cases $t\in\{1,4,9,16\}$.

The aim of the present paper is to go further, i.e. to obtain some general results about such $D(-1)$-triples and to extend results from [26], using the results presented in [20]. Our main result can be expressed as

**Theorem 1.1** *Let $t>1$.*

(*i*) *If $b\in\{26,50\}$ and $t\neq b-1$, then there does not exist a $D(-1)$-quadruple of the form $\{1,b,c,d\}$ in $\mathbb{Z}[\sqrt{-t}]$.*

(*ii*) *If $t\notin\{4,9,36\}$, then there does not exist a $D(-1)$-quadruple of the form $\{1,37,c,d\}$ in $\mathbb{Z}[\sqrt{-t}]$.*

By [7, Theorem 3], in the case $t=1$ for such $b$'s there exist infinitely many $D(-1)$-quadruples of the form $\{1,b,c,d\}$ in Gaussian integers. But for such $t$ and other exceptions of statements (i) and (ii) of Theorem 1.1, we also prove that in $\mathbb{Z}[\sqrt{-t}]$ exist infinitely many $D(-1)$-quadruples of the form $\{1,b,-c,d\}$, $c,d>0$.

The strategy of proof of the above theorem follows the similar lines as the proof of the results on extendibility of $D(-1)$-triples, presented in [26], and some other Diohphantine problems that can be transformed into systems of parametric Pellian equations (see e.g. [22]). Firstly we prove more general result, i.e. if $\{1,b,c\}$ is a $D(-1)$-triple in $\mathbb{Z}[\sqrt{-t}], t>0$ and $b$ is a prime, twice prime and twice prime squared, then $c$ is an integer. As a consequence of that result, in cases $b=26,37,50$ we show that for $t\notin\{1,4,9,25,36,49\}$ there does not exist a subset of $\mathbb{Z}[\sqrt{-t}]$ of the form $\{1,b,c,d\}$ with the property that the product of any two of its distinct elements diminished by 1 is a square of an element in $\mathbb{Z}[\sqrt{-t}]$. In the case $b=37$, $t=3$, we use the standard methods when considering extension of $D(n)$-triple. We reduce our problem to a system of simultaneous Pellian equations, which leads to the consideration of intersections of two binary recurrence sequences. Using the congruence method together with Bennett's theorem ([4, Theorem 3.2]) on simultaneous approximation of the square roots of algebraic numbers which are close to 1 we obtain an upper bound of extension element. And at the end, we use Baker's theory of linear forms in logarithms of algebraic numbers (in fact we apply Baker-Wüstholz result [3, Theorem]) to compute an upper bound for the indices of the recurring sequences, and then apply the reduction method of Dujella and Pethő ([13, Lemma 5a]), based on that of Baker and Davenport. Even the methods here are standard, there is more technical work to be done than usually in such kinds of problems.

If we summarize results from [26] and those from Theorem 1.1, we could say that in the ring $\mathbb{Z}[\sqrt{-t}]$ we have results about the extensibility of such $D(-1)$-triples, analog to results in integers presented in [20].

## 2. $D(-1)$-quadruples of the form $\{1,b,c,d\}, b\in\{26,50\}$

In this section, firstly we prove a general result on $D(-1)$-triples of the form $\{1,b,c\}$ in the ring $\mathbb{Z}[\sqrt{-t}], t>0$, depending on the form of $b$. We apply the obtained result to characterize all $D(-1)$-quadruples for $b\in\{26,50\}$ (if they exist).

For the convenience of the reader, at the beginning we will state one useful test based on the properties of Pellian equations.

**Lemma 2.1 ([21, Criterion 1])** *Let $U>1$, $V$ be positive integers such that $(U,V)=1$ and $D=UV$ is not a square of a natural number. Moreover let $\langle u_0,v_0\rangle$ denote the least positive* *integer solution of Pell equation* $u^2 - Dv^2 = 1$. *Then the equation*

$$Ux^2 - Vy^2 = 1$$

*has a solution in positive integers* $x, y$ *if and only if* $2U|u_0 + 1$ *and* $2V|u_0 - 1$.

Now we will prove the following result:

**Theorem 2.2** *Let* $t > 0$ *and* $\{1, b, c\}$ *be* $D(-1)$*-triple in the ring* $\mathbb{Z}[\sqrt{-t}]$.

(*i*) *If* $b$ *is a prime, then* $c \in \mathbb{Z}$.

(*ii*) *If* $b = 2b_1$, *where* $b_1$ *is a prime, then* $c \in \mathbb{Z}$.

(*iii*) *If* $b = 2b_2^2$, *where* $b_2$ *is a prime, then* $c \in \mathbb{Z}$.

**Proof**: Let us suppose that $t > 0$ and $\{1, b, c\}$ is a $D(-1)$-triple in $\mathbb{Z}[\sqrt{-t}]$. Then there exist integers $c_1, d_1, x, y, u, v$ such that $c = c_1 + d_1\sqrt{-t}$ and

$$\begin{aligned} c - 1 &= (x + y\sqrt{-t})^2, \\ bc - 1 &= (u + v\sqrt{-t})^2. \end{aligned}$$

We obtain

$$\begin{aligned} c_1 &= x^2 - ty^2 + 1, \quad & d_1 &= 2xy, \\ bc_1 &= u^2 - tv^2 + 1, \quad & bd_1 &= 2uv. \end{aligned}$$

Therefore, we have to consider the system of equations

$$b(x^2 - ty^2 + 1) = u^2 - tv^2 + 1, \qquad (2.1)$$

$$bxy = uv, \qquad (2.2)$$

with the condition that $xy \neq 0$.

(i) If $b$ is a prime, then the equation (2.2) implies $v = bw$ or $u = bw$, where $w \in \mathbb{Z}$.

(I) Let us suppose that $v = bw, w \in \mathbb{Z}$. Then (2.2) implies $xy = uw$, with general solution $x = pq, y = rs, u = pr, w = qs, p, q, r, s \in \mathbb{Z}$. Inserting this in the equation (2.1) we get

$$t = \frac{1 - b - bp^2q^2 + p^2r^2}{b(bq^2 - r^2)s^2} = \frac{b - 1 - p^2h}{bs^2h},$$

where $h = r^2 - bq^2$. If $h < 0$, then clearly $t < 0$. If $h > 0$, then $b - 1 - p^2h < b$ and we obtain $t < 1/(s^2h)$, so $t \leq 0$. The case $h = 0$ is not possible for $xy \neq 0$, since $\sqrt{b}$ is irrational.

(II) Now we suppose that $u = bw, w \in \mathbb{Z}$. Then from (2.2) we obtain $xy = vw$, so there exist integers $p, q, r, s$ such that $x = pq, y = rs, v = pr, w = qs$, where $p, q, r, s \in \mathbb{Z}$. Inserting this into (2.1) we obtain

$$t = \frac{b - 1 - bq^2h}{r^2h},$$

where $h = bs^2 - p^2$. If $h < 0$, then clearly $t < 0$. If $h > 0$, then $b - 1 - bq^2h < 0$, so $t < 0$. Since $\sqrt{b}$ is irrational, the case $h = 0$ is not possible for $xy \neq 0$.

Since the assumption is $t > 0$, in cases (I) and (II) there is no solution of the system of equations (2.1) and (2.2). Hence $y = 0$ or $x = 0$ or $x = y = 0$.

a) If $y = 0$, then $d_1 = 0$, which implies $c = c_1 = x^2 + 1$, i.e. $c \in \mathbb{N}$.

b) If we suppose that $x = 0$, then we will obtain $c = c_1 = 1 - ty^2 < 0$. Now we have to investigate when it is possible. From (2.2) we conclude that $u = 0$ or $v = 0$ or $u = v = 0$.

If $u=0$, then from (2.1) we obtain $v^2-by^2=(1-b)/t$.
If $v=0$, then from (2.1) we obtain $u^2+bty^2=b-1$. Since $u^2+bty^2>b-1$ for $y\neq 0$, it follows $y=0$, so $c=1$, which is a contradiction with $(1,b,c)$ is a $D(-1)$-triple.
If $u=v=0$, we obtain $bc_1=1$, which is not possible.

Therefore, the case $x=0$ is possible only if $t|b-1$ and the equation

$$v^2-by^2=\frac{1-b}{t}$$

has a solution.

c) If $x=y=0$, then $c=1$. That is not possible, too.

(ii) Let us suppose that $b=2b_1$, where $b_1$ is a prime. Since $2b_1-1$ is a square, we obtain that $b_1$ is of the special form $b_1=2k_1^2+2k_1+1$, $k_1\in\mathbb{Z}$. The equation (2.2) implies $v=2b_1w$ or $u=2b_1w$ or $u=2k, v=b_1l$ or $u=b_1l, v=2k$, for $k,l,w\in\mathbb{Z}$.

(I) If $v=2b_1w, w\in\mathbb{Z}$, from (2.2) we obtain $xy=uw$, so there exist integers $p,q,r,s$ such that $x=pq, y=rs, u=pr, w=qs$. Inserting this in the equation (2.1) we get

$$t=\frac{2b_1-1-p^2h}{2b_1s^2h},$$

where $h=r^2-2b_1q^2$. If $h<0$, then clearly $t<0$. If $h>0$, then $2b_1-1-p^2h<2b_1$ and $t<1/(s^2h)$. Therefore, $t\leq 0$. The case $h=0$ is not possible for $xy\neq 0$, since $\sqrt{2b_1}$ is irrational.

(II) Let us suppose that $u=2b_1w, w\in\mathbb{Z}$. Then from (2.2) follows $xy=vw$, with general solution $x=pq, y=rs, v=pr, w=qs$, where $p,q,r,s\in\mathbb{Z}$. Equation (2.1) now implies

$$t=\frac{2b_1-1-2b_1q^2h}{r^2h},$$

where $h=2b_1s^2-p^2$. If $h<0$, then clearly $t<0$. If $h>0$, then $2b_1-1-2b_1q^2h<0$, so $t<0$. Since $\sqrt{2b_1}$ is irrational, the case $h=0$ is not possible for $xy\neq 0$.

(III) If we set $u=2k, v=b_1l, k,l\in\mathbb{Z}$ in (2.2), we get $xy=kl$. Therefore exist $p,q,r,s\in\mathbb{Z}$ such that $x=pq, y=rs, k=pr, l=qs$. Now from (2.1) we obtain

$$t=\frac{2b_1-1-2p^2h}{b_1s^2h},$$

where $h=2r^2-b_1q^2$.

If $h<0$, then clearly $t<0$.

If $h>0$, then $2b_1-1-2p^2h<2b_1$, so we obtain $t<2/(s^2h)$. If $h>1$ or $s>1$, then $t\leq 0$, so we have to study the case $h=s=1$ with $t=1$. We obtain the equation $2p^2=b_1-1$. Since $b_1$ is a prime and of the form $b_1=2k_1^2+2k_1+1, k_1\in\mathbb{Z}$, it follows the equation $p^2=k_1(k_1+1)$ with integer solutions $p=0, k_1=-1,0$. That implies $b_1=1$, which is not possible since $b_1$ is a prime.

Since $\sqrt{2b_1}$ is irrational, the case $h=0$ is not possible for $xy\neq 0$.

(IV) Inserting $u=b_1l, v=2k, k,l\in\mathbb{Z}$ in (2.2), we obtain $xy=kl$, so there exist integers $p,q,r,s$ such that $x=pq, y=rs, k=pr, l=qs$. Equation (2.1) now implies

$$t=\frac{2b_1-1-b_1q^2h}{2r^2h},$$

where $h=b_1s^2-2p^2$.

If $h<0$, then clearly $t<0$.

If $h > 1$, then $2b_1 - 1 - b_1q^2h < 0$, so $t < 0$. Therefore, we have to study the case $h = 1$, which means to find all integer solutions of the equation

$$b_1s^2 - 2p^2 = 1, \tag{2.3}$$

with prime $b_1$ of the form $b_1 = 2k_1^2 + 2k_1 + 1, k_1 \in \mathbb{Z}$. In the notation of Lemma 2.1, we have $U = b_1, V = 2, D = 2b_1, u_0 = 8k_1^2 + 8k_1 + 3, v_0 = 4k_1 + 2$ and conclude that $4 \not| 8k_1^2 + 8k_1 + 2$. Therefore the equation (2.3) has no integer solutions.

The case $h = 0$ is not possible for $xy \neq 0$, since $\sqrt{2b_1}$ is irrational.

Since the assumption is $t > 0$, in cases (I)–(IV) there is no solution of the system of equations (2.1) and (2.2). Therefore $y = 0$ or $x = 0$ or $x = y = 0$. By the similar argumentation as in (i), the first possibility gives $c \in \mathbb{N}$, while the second is possible only if $t | 2b_1 - 1$ and the equation $v^2 - 2b_1y^2 = (1 - 2b_1)/t$ has a solution.

(iii) If we set $b = 2b_2^2$, where $b_2$ is a prime, then the equation (2.2) implies $v = 2b_2^2w$ or $u = 2b_2^2w$ or $u = 2k, v = b_2^2l$ or $u = b_2^2l, v = 2k$, where $k, l, w \in \mathbb{Z}$. Cases $u = 2b_2k, v = b_2l$ and $u = b_2k, v = 2b_2l$, for $k, l \in \mathbb{Z}$ are not possible, since the condition $b_2^2 | u^2 - tv^2 + 1$ is not satisfied. Similarly as in (ii), for such $b_2$, $b_2^2$ is of the special form $b_2^2 = 2k_1^2 + 2k_1 + 1, k_1 \in \mathbb{Z}$, and for each case of $u$ and $v$ the proof that the system of equations (2.1) and (2.2) has no solution follows the same lines as the proof of the case (ii), so it will be omitted. By the same argumentation, we conclude that the possibility $y = 0$ gives $c \in \mathbb{N}$, while $x = 0$ is possible only if $t | 2b_2^2 - 1$ and the equation $v^2 - 2b_2^2y^2 = (1 - 2b_2^2)/t$ has a solution.

□

Now, we have the following consequence of Theorem 2.2, which will be very useful in consideration of the existence of $D(-1)$-quadruples of the form $\{1, b, c, d\}$ in $\mathbb{Z}[\sqrt{-t}]$, for $b \in \{26, 50\}$.

**Corollary 2.3** *Let $b \in \{26, 50\}$. If $t > 1, t \neq b - 1$ and $\{1, b, c\}$ is a $D(-1)$-triple in $\mathbb{Z}[\sqrt{-t}]$, then $c \in \mathbb{N}$. If $t \in \{1, b - 1\}$, then $c \in \mathbb{Z}$.*

**Proof:** It follows from the proof of Theorem 2.2 (ii) and (iii). For every $t > 0$, we have that $c \in \mathbb{Z}$.

If $b_1 = 13$, then the condition $t | 2b_1 - 1$ is satisfied for $t \in \{1, 5, 25\}$. The equation $v^2 - 2b_1y^2 = (1 - 2b_1)/t$ is solvable for $t \in \{1, 25\}$. If $t = 5$, then we obtain the equation $v^2 - 26y^2 = -5$, which is not solvable modulo 8.

If $b_2 = 5$, the condition $t | 2b_2^2 - 1$ is satisfied for $t \in \{1, 7, 49\}$. The equation $v^2 - 2b_2^2y^2 = (1 - 2b_2^2)/t$ is solvable for $t \in \{1, 49\}$. In case of $t = 7$, the equation $v^2 - 50y^2 = -7$ is not solvable modulo 5.

Therefore, if $t \neq 1, 25$, resp. $t \neq 1, 49$, we have that $c \in \mathbb{N}$.

□

**Example 2.4** *The sets $\{1, 26, 37\}, \{1, 26, -24\}$ are $D(-1)$-triples in $\mathbb{Z}[\sqrt{-t}]$ for $t \in \{1, 25\}$, while the sets $\{1, 50, 65\}, \{1, 50, -48\}$ are $D(-1)$-triples in $\mathbb{Z}[\sqrt{-t}]$ for $t \in \{1, 49\}$.*

We are ready to formulate the next proposition:

**Proposition 2.5** *Let $b \in \{26, 50\}$. If $t > 1, t \neq b - 1$, then there does not exist a $D(-1)$-quadruple of the form $\{1, b, c, d\}$ in $\mathbb{Z}[\sqrt{-t}]$.*

We will omit the proof of Proposition 2.5, since it follows the similar steps as the proof of [26, Proposition 2.4]. In this case of $b$, in the last step we obtain contradiction with [20, Theorem 1.3].

**Proposition 2.6** *Let $b \in \{26, 50\}$. If $t \in \{1, b-1\}$, then there exist infinitely many $D(-1)$-quadruples of the form $\{1, b, -c, d\}$, $c, d > 0$ in $\mathbb{Z}[\sqrt{-t}]$.*

Since $\mathbb{Z}[5i]$ and $\mathbb{Z}[7i]$ are subrings of the ring $\mathbb{Z}[i]$, it suffices to prove the statement of Proposition 2.6 for $t = b - 1$. By using [10, Lemma 3] the proof is the same as the proof of [26, Proposition 2.6], so it will be omitted.

# 3. $D(-1)$-quadruples of the form $\{1, 37, c, d\}$

In this section we consider the problem of existence of $D(-1)$-quadruples of the form $\{1, 37, c, d\}$ in the ring $\mathbb{Z}[\sqrt{-t}]$, for positive integer $t$. In view of $t$, we will separate the problem on two cases.

## 3.1. The case $t \geq 1, t \neq 3$

From Theorem 2.2 (i), for $b = 37$ we have the following result:

**Corollary 3.1** *If $t > 1, t \notin \{3, 4, 9, 36\}$ and $\{1, 37, c\}$ is a $D(-1)$-triple in $\mathbb{Z}[\sqrt{-t}]$, then $c \in \mathbb{N}$. If $t \in \{1, 3, 4, 9, 36\}$, then $c \in \mathbb{Z}$.*

**Proof:** It follows from the proof of Theorem 2.2 (i). For every $t > 0$, we have that $c \in \mathbb{Z}$.

If $b = 37$, then the condition $t|b-1$ is satisfied for $t \in \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$. The equation $v^2 - by^2 = (1-b)/t$ is solvable for all $t \in \{1, 3, 4, 9, 36\}$.

If $t = 2, 6, 18$, then we obtain equations $v^2 - 37y^2 = -18$, $v^2 - 37y^2 = -6$ and $v^2 - 37y^2 = -2$, which are not solvable modulo 4.

In case of $t = 12$ we obtain the equation

$$v^2 - 37y^2 = -3. \tag{3.1}$$

By using [23, Theorem 108 a] every fundamental solution $v^\star + y^\star\sqrt{37}$ of (3.1) has to satisfy $|v^\star| \leq 10$ and $0 \leq y^\star \leq 1$. If we check all possibilities we conclude that the equation (3.1) has no integer solutions.

Therefore, if $t \notin \{1, 3, 4, 9, 36\}$, we have that $c \in \mathbb{N}$.

□

**Example 3.2** *For $t \in \{1, 4, 9, 36\}$ the sets $\{1, 37, 50\}, \{1, 37, -35\}$ are $D(-1)$-triples in $\mathbb{Z}[\sqrt{-t}]$. The sets $\{1, 37, 50\}, \{1, 37, -2\}$ are $D(-1)$-triples in $\mathbb{Z}[\sqrt{-3}]$.*

Now we can formulate the next proposition:

**Proposition 3.3** *If $t > 1, t \notin \{3, 4, 9, 36\}$, then there does not exist a $D(-1)$-quadruple of the form $\{1, 37, c, d\}$ in $\mathbb{Z}[\sqrt{-t}]$.*

The proof of Proposition 3.3 follows easily from Corollary 3.1 and [20, Theorem 1.3], similarly to the proof of Proposition 2.5.

**Proposition 3.4** *If $t \in \{1, 4, 9, 36\}$, then there exist infinitely many $D(-1)$-quadruples of the form $\{1, 37, -c, d\}$, $c, d > 0$ in $\mathbb{Z}[\sqrt{-t}]$.*

We will omit the proof of Proposition 3.4 too, since it follows the same steps as the proof of [26, Proposition 2.6], and it is enough to consider $t = 36$.

### 3.2. The case $t = 3$

If we suppose that $\{1, 37, c, d\}$ is a $D(-1)$-quadruple in $\mathbb{Z}[\sqrt{-3}]$, then by Corollary 3.1, we conclude that $c, d \in \mathbb{Z}$. If there exist integers $x_1, y_1, u_1, v_1, w_1$ such that

$$c - 1 = x_1^2,\ d - 1 = y_1^2,\ 37c - 1 = u_1^2,\ 37d - 1 = v_1^2,\ cd - 1 = w_1^2,$$

we get a contradiction with [20, Theorem 1.3], i.e. there does not exist $c, d \in \mathbb{Z}$ such that $\{1, 37, c, d\}$ is a $D(-1)$-quadruple in $\mathbb{Z}$. Therefore, if such quadruple exists, then at least one of $c-1, d-1, 37c-1, 37d-1, cd-1$ is equal to $-3w^2$, for some $w \in \mathbb{Z}$.

Let $\tilde{s}, \tilde{t}, x, y, z \in \mathbb{Z}$. Considering the positivity of $c$ and $d$, there are three possible combinations without contradiction:

(I) $c - 1 = -3\tilde{s}^2,\ 37c - 1 = -3\tilde{t}^2,\ d - 1 = -3x^2,\ 37d - 1 = -3y^2,\ cd - 1 = z^2,$

(II) $c - 1 = -3\tilde{s}^2,\ 37c - 1 = -3\tilde{t}^2,\ d - 1 = x^2,\ 37d - 1 = y^2,\ cd - 1 = -3z^2,$

(III) $c - 1 = \tilde{s}^2,\ 37c - 1 = \tilde{t}^2,\ d - 1 = -3x^2,\ 37d - 1 = -3y^2,\ cd - 1 = -3z^2.$

Thus we have to analyze each of above cases. It is obvious that combinations (II) and (III) determine the same $D(-1)$-quadruples, up on the order of their elements. Therefore it is sufficient to consider the existence of $D(-1)$-quadruples determined by (I) and (II).

#### 3.2.1. $D(-1)$-quadruples of the form $\{1, 37, c, d\}$, $c < 0$, $d < 0$

Concerning the case (I), we consider the existence of $D(-1)$-quadruples of the form $\{1, 37, c, d\}$, with $c < 0, d < 0$. From

$$\begin{aligned} c - 1 &= -3\tilde{s}^2, \\ 37c - 1 &= -3\tilde{t}^2 \end{aligned}$$

we obtain

$$\tilde{t}^2 - 37\tilde{s}^2 = -12. \tag{3.2}$$

By using [23, Theorem 108] all solutions in positive integers of Pellian equation (3.2) are given by two sequences

$$\tilde{t}_0 = 5, \tilde{s}_0 = 1, \tilde{t}_1 = 809, \tilde{s}_1 = 133, \tilde{t}_{n+2} = 146\tilde{t}_{n+1} - \tilde{t}_n, \tilde{s}_{n+2} = 146\tilde{s}_{n+1} - \tilde{s}_n, \tag{3.3}$$

$$\tilde{t}_0^{'} = -5, \tilde{s}_0^{'} = 1, \tilde{t}_1^{'} = 79, \tilde{s}_1^{'} = 13, \tilde{t}_{n+2}^{'} = 146\tilde{t}_{n+1}^{'} - \tilde{t}_n^{'}, \tilde{s}_{n+2}^{'} = 16\tilde{s}_{n+1}^{'} - \tilde{s}_n^{'}. \tag{3.4}$$

From

$$\begin{aligned} d - 1 &= -3x^2, \\ 37d - 1 &= -3y^2, \\ cd - 1 &= z^2 \end{aligned}$$

eliminating $d$, we obtain the system of simultaneous Pellian equations

$$z^2 + 3cx^2 = c - 1, \tag{3.5}$$

$$37z^2 + 3cy^2 = c - 37. \tag{3.6}$$

Now we formulate our result:

**Theorem 3.5** *Let $(\tilde{t_k}, \tilde{s_k}), k = 0, 1, 2, \ldots$ denote all integer solutions of the Pellian equation* (3.2) *given by* (3.3) *and* (3.4), *respectively. If $c = 1 - 3\tilde{s_k}^2$, then the system of simultaneous Pellian equations* (3.5) *and* (3.6) *has no integer solutions.*

**Proof**: By using $c = 1 - 3\tilde{s_k}^2$, from (3.5) and (3.6) we obtain equations

$$z^2 = 9x^2\tilde{s_k}^2 - 3x^2 - 3\tilde{s_k}^2, \tag{3.7}$$
$$37z^2 = 9y^2\tilde{s_k}^2 - 3y^2 - 3\tilde{s_k}^2 - 36. \tag{3.8}$$

By considering (3.8) we conclude that $3|z^2$, and therefore $9|z^2$. If we express (3.8) in the form

$$37z^2 - 9y^2\tilde{s_k}^2 + 36 = -3y^2 - 3\tilde{s_k}^2,$$

it follows that $9|-3y^2 - 3\tilde{s_k}^2$, i.e. $3|\tilde{s_k}^2 + y^2$. Therefore, $3|\tilde{s_k}$ and $3|y$.

Now we set $z = 3k, \tilde{s_k} = 3l, y = 3m,\ k, l, m \in \mathbb{Z}$. Then the equation (3.8) is equivalent to

$$37k^2 + 4 = 81l^2m^2 - 3l^2 - 3m^2.$$

Since $37k^2 + 4 \equiv 1$ or $2 \pmod 3$, we get a contradiction. Therefore the equation (3.8) is not solvable.

□

Through the above analysis we conclude that there does not exist $D(-1)$-quadruple of the form $\{1, 37, c, d\}$ with $c < 0, d < 0$ in the ring $\mathbb{Z}[\sqrt{-3}]$, determined by (I).

### 3.2.2. $D(-1)$-quadruples of the form $\{1, 37, -c, d\}$, $c > 0$, $d > 0$

Now we are considering the case (II). We will consider the existence of $D(-1)$-quadruples of the form $\{1, 37, -c, d\}$ with $c > 0$, $d > 0$.

From

$$\begin{aligned} -c - 1 &= -3\tilde{s}^2, \\ -37c - 1 &= -3\tilde{t}^2 \end{aligned}$$

we obtain

$$\tilde{t}^2 - 37\tilde{s}^2 = -12.$$

If we denote with $(\tilde{t_k}, \tilde{s_k}), k = 0, 1, 2, \dots$ denote all positive solutions of the above Pellian equation given by (3.3) and (3.4), respectively, then there exists an integer $k$ such that

$$c = c_k = 3\tilde{s_k}^2 - 1. \tag{3.9}$$

From

$$\begin{aligned} d - 1 &= x^2, \\ 37d - 1 &= y^2, \\ -cd - 1 &= -3z^2 \end{aligned}$$

by elimination of $d$, we obtain the system of simultaneous Pellian equations

$$3z^2 - cx^2 = c + 1, \tag{3.10}$$
$$111z^2 - cy^2 = c + 37. \tag{3.11}$$

Now, we can formulate our main result:

**Theorem 3.6** *Let $k$ be a nonnegative integer and $c = c_k$ defined by* (3.9). *All solutions of the system of simultaneous Pellian equations* (3.10) *and* (3.11) *are given by* $(x, y, z) = (0, 6, \pm\sqrt{(c+1)/3})$.

The proof of Theorem 3.6 (roughly mentioned by steps at the end of the first section) will follow from the forthcoming analysis.

Since $z + x\sqrt{3c} = 2c + 1 + 2\tilde{s}\sqrt{3c}$ and $z + y\sqrt{111c} = 74c + 1 + 2\tilde{t}\sqrt{111c}$ are solutions of Pell equations

$$\begin{aligned} z^2 - 3cx^2 = 1, \\ z^2 - 111cy^2 = 1, \end{aligned} \tag{3.12}$$

if there exist solutions of equations (3.10) and (3.11), then there exist finite sets $\{z_0^{(i)} + x_0^{(i)}\sqrt{3c} : i = 1, \dots, i_0\}$ and $\{z_1^{(j)} + y_1^{(i)}\sqrt{111c} : j = 1, \dots, j_0\}$ of elements of $\mathbb{Z}[\sqrt{3c}]$ and $\mathbb{Z}[\sqrt{111c}]$ respectively, such that all solutions of (3.10) and (3.11) are given by

$$z\sqrt{3} + x\sqrt{c} = (z_0^{(i)}\sqrt{3} + x_0^{(i)}\sqrt{c})(2c + 1 + 2\tilde{s}\sqrt{3c})^m, \tag{3.13}$$
$$z\sqrt{111} + y\sqrt{c} = (z_1^{(j)}\sqrt{111} + y_1^{(j)}\sqrt{c})(74c + 1 + 2\tilde{t}\sqrt{111c})^n, \tag{3.14}$$

respectively.

From (3.13) we conclude that $z = v_m^{(i)}$ for some index $i$ and integer $m$, where

$$v_0^{(i)} = z_0^{(i)}, v_1^{(i)} = (2c + 1)z_0^{(i)} + 2c\tilde{s}x_0^{(i)}, v_{m+2}^{(i)} = (4c + 2)v_{m+1}^{(i)} - v_m^{(i)}, \tag{3.15}$$

and from (3.14) we conclude that $z = w_n^{(j)}$ for some index $j$ and integer $n$, where

$$w_0^{(j)} = z_1^{(j)}, w_1^{(j)} = (74c + 1)z_1^{(j)} + 2c\tilde{t}y_1^{(j)}, w_{n+2}^{(j)} = (148c + 2)w_{n+1}^{(j)} - w_n^{(j)}. \tag{3.16}$$

Thus we reformulated the system of equations (3.10) and (3.11) to finitely many Diophantine equations of the form

$$v_m^{(i)} = w_n^{(j)}.$$

If we choose representatives $z_0^{(i)}\sqrt{3} + x_0^{(i)}\sqrt{c}$ and $z_1^{(j)}\sqrt{111} + y_1^{(j)}\sqrt{c}$ such that $|z_0^{(i)}|$ and $|z_1^{(j)}|$ are minimal, then by using [23, Theorem 108] we obtain following estimates:

$$\begin{aligned} |z_0^{(i)}| &\leq c, \\ |z_1^{(j)}| &\leq c + 6. \end{aligned}$$

For the simplicity, from now on, the superscripts $(i)$ and $(j)$ will be omitted.

From (3.15) and (3.16), it follows by induction that

$$\begin{aligned} v_m &\equiv z_0 \pmod{2c}, \\ w_n &\equiv z_1 \pmod{2c}. \end{aligned}$$

We conclude if there is a solution of the equation $v_m = w_n$ in integers $m$ and $n$, then we have

$$z_0 = z_1, \tag{3.17}$$
$$z_1 = z_0 - 2c, \quad z_0 > 0, \tag{3.18}$$
$$z_1 = z_0 + 2c, \quad z_0 < 0. \tag{3.19}$$

In observation of cases (3.18) and (3.19) we conclude that $z_0 = c - i, z_1 = -c - i$, and $z_0 = -c+i, z_1 = c+i$, where $i \in \{0, 1, 2, 3, 4, 5, 6\}$. Now from (3.10) we obtain $cx_0^2 = 3z_0^2 - c - 1$, so it follows that $c|3i^2 - 1$.

If $c=2$, including the possibility (3.17), for $i=1,3,5$ the condition $c|3i^2-1$ is satisfied, so we have some new possibilities for $z_0$ and $z_1$ determined by (3.18) and (3.19), i.e. $(z_0,z_1)=(1,-3),(z_0,z_1)=(-1,3)$. Inserting that into equations (3.10) and (3.11) we obtain that at least one equation has no corresponding integer solutions $x_0,y_1$. So in case $c=2$ we can omit possibilities (3.18) and (3.19).

We will now assume that $c>2$ is the minimal positive integer such that the solution of the system of equations (3.10) and (3.11) exists. Since $c=c_k=3\tilde{s_k}^2-1$ (resp. $c=c_k=3\tilde{s_k}'^2-1$), from (3.3) (resp. from (3.4)) it follows that the minimal positive integer such that the $D(-1)$-triple of the form $\{1,37,-c\}$ can be extended satisfies $c\geq 506$. In that case the condition $c|3i^2-1$ is not satisfied, so we have $z_0=z_1$. Keeping in mind that the case $z_0=z_1$ can also appear for $c=2$, in all further results that might be necessary for the reduction method, we will also include the case $c=2$ and use the reduction method in this case as well.

Let $d_0=(3z_0^2-1)/c$. Then we have

$$\begin{aligned} d_0-1 &= \frac{3z_0^2-c-1}{c}=x_0^2,\\ 37d_0-1 &= \frac{111z_0^2-c-37}{c}=y_1^2,\\ -cd_0-1 &= -3z_0^2, \end{aligned}$$

and

$$0<d_0\leq \frac{c^2+2c}{c}=c+2.$$

If $d_0=c+2$, then $(c+1)^2=3z_0^2$, so $c=-1$ and $z_0=0$. That is the contradiction with $c>2$.

If $d_0=c+1$, then $c^2+c+1=3z_0^2$. We conclude that $c\equiv 1 \pmod 3$ and that is the contradiction with (3.9).

If $d_0=c$, we obtain the equation $c^2-3z_0^2=-1$ which is not solvable modulo 3.

Therefore, $d_0<c$.

Let $d_0>1$. Now, we will consider the extensibility of $D(-1)$-triple $\{1,37,d\}, d=d_0$ to $D(-1)$-quadruple $\{1,37,d,c\}$. By Proposition 2.5, there does not exist $D(-1)$-quadruple of the form $\{1,26,d,c\}$, so we can assume that $d>37$.

From

$$\begin{aligned} d-1 &= \hat{s}^2,\\ 37d-1 &= \hat{t}^2, \end{aligned}$$

we obtain

$$\hat{t}^2-37\hat{s}^2=36. \tag{3.20}$$

Moreover, from

$$\begin{aligned} c-1 &= -3\hat{x}^2,\\ 37c-1 &= -3\hat{y}^2,\\ cd-1 &= -3\hat{z}^2 \end{aligned}$$

it follows

$$\begin{aligned} 3\hat{z}^2-3d\hat{x}^2 &= 1-d, &(3.21)\\ 111\hat{z}^2-3d\hat{y}^2 &= 37-d, &(3.22) \end{aligned}$$

which is equivalent to

$$\hat{z}^2 - d\hat{x}^2 = \frac{1-d}{3}, \tag{3.23}$$

$$37\hat{z}^2 - d\hat{y}^2 = \frac{37-d}{3}. \tag{3.24}$$

It is obvious that $\hat{z}+\hat{x}\sqrt{d} = 2d-1+2\hat{s}\sqrt{d}$ and $\hat{z}+\hat{y}\sqrt{37d} = 74d-1+2\hat{t}\sqrt{37d}$ are solutions of Pell equations

$$\begin{aligned} \hat{z}^2 - d\hat{x}^2 &= 1, \\ \hat{z}^2 - 37d\hat{y}^2 &= 1. \end{aligned}$$

Thus for $i_0, j_0 \in \mathbb{N}$ and $m, n \geq 0$, all solutions of (3.23) and (3.24) are given by

$$\hat{z}+\hat{x}\sqrt{d} = (\hat{z_0}^{(i)} + \hat{x_0}^{(i)}\sqrt{d})(2d-1+2\hat{s}\sqrt{d})^m, \tag{3.25}$$

$$\hat{z}\sqrt{37}+\hat{y}\sqrt{d} = (\hat{z_1}^{(j)}\sqrt{37} + \hat{y_1}^{(j)}\sqrt{d})(74d-1+2\hat{t}\sqrt{37d})^n, \tag{3.26}$$

for $i = 1,\ldots,i_0, j = 1,\ldots,j_0$, respectively. From (3.25) and (3.26), we conclude that $\hat{z} = \hat{v_m}^{(i)} = \hat{w_n}^{(j)}$, for some indices $i, j$ and positive integers $m, n$, where

$$\hat{v_0}^{(i)} = \hat{z_0}^{(i)}, \hat{v_1}^{(i)} = (2d-1)\hat{z_0}^{(i)} + 2d\hat{s}\hat{x_0}^{(i)}, \hat{v}_{m+2}^{(i)} = (4d-2)\hat{v}_{m+1}^{(i)} - \hat{v_m}^{(i)}, \tag{3.27}$$

$$\hat{w_0}^{(j)} = \hat{z_1}^{(j)}, \hat{w_1}^{(j)} = (74d-1)\hat{z_1}^{(j)} + 2d\hat{t}\hat{y_1}^{(j)}, \hat{w}_{n+2}^{(j)} = (148d-2)\hat{w}_{n+1}^{(j)} - \hat{w_n}^{(j)}. \tag{3.28}$$

Now we choose $\hat{z_0}^{(i)} + \hat{x_0}^{(i)}\sqrt{d}$ and $\hat{z_1}^{(j)}\sqrt{37} + \hat{y_1}^{(j)}\sqrt{d}$ with minimal $|\hat{z_0}^{(i)}|$ and $|\hat{z_1}^{(j)}|$, and using [23, Theorem 108] we obtain

$$\begin{aligned} |\hat{z_0}^{(i)}| &< d, \\ |\hat{z_1}^{(j)}| &< d. \end{aligned}$$

From now on, we will also omit the superscripts $(i)$ and $(j)$. Similarly, from (3.27) and (3.28) it follows by induction that

$$\begin{aligned} \hat{v_m} &\equiv (-1)^{m+1}\hat{z_0} \pmod{2d}, \\ \hat{w_n} &\equiv (-1)^{n+1}\hat{z_1} \pmod{2d}. \end{aligned}$$

So, if $\hat{v_m} = \hat{w_n}$ has a solution, we must have $|\hat{z_0}| = |\hat{z_1}|$.

Let $c_0 = (3\hat{z_0}^2 - 1)/d$. Then

$$\begin{aligned} -c_0 - 1 &= \frac{1-d-3\hat{z_0}^2}{c} = -3\hat{x_0}^2, \\ -37c_0 - 1 &= \frac{37-d-111\hat{z_0}^2}{d} = -3\hat{y_1}^2, \\ -c_0 d - 1 &= -3\hat{z_0}^2, \end{aligned}$$

so $\{1, 37, d, -c_0\}$ is a $D(-1)$-quadruple with $0 < c_0 < d$.

We have the following conclusion: by assumption that $D(-1)$-triple $\{1, 37, d_0\}$, $d_0 > 1$ can be extended to $D(-1)$-quadruple $\{1, 37, d_0, -c\}$, we conclude that there exists positive integer $c_0 < d_0 < c$ such that $\{1, 37, d_0, -c_0\}$ is a $D(-1)$-quadruple. But, this is a contradiction with the minimality of $c$.

Therefore, we proved that $d_0 = 1$. This implies $z_0 = z_1 = \pm\tilde{s}$, $x_0 = 0$, $y_1 = 6$ and from (3.13) and (3.14) we conclude that we must consider

$$v_m = \frac{\tilde{s}}{2}\Big((2c+1+2\tilde{s}\sqrt{3c})^m + (2c+1-2\tilde{s}\sqrt{3c})^m\Big), \tag{3.29}$$

$$w_n = \frac{(\tilde{s}\sqrt{111}\pm 6\sqrt{c})(74c+1+2\tilde{t}\sqrt{111c})^n + (\tilde{s}\sqrt{111}\mp 6\sqrt{c})(74c+1-2\tilde{t}\sqrt{111c})^n}{2\sqrt{111}}. \tag{3.30}$$

The following lemma can be proved easily by induction.

**Lemma 3.7**

$$v_m \equiv z_0 + 2cm^2 z_0 + 2cm\tilde{s}x_0 \pmod{8c^2},$$
$$w_n \equiv z_1 + 74cn^2 z_1 + 2cn\tilde{t}y_1 \pmod{8c^2}.$$

**Lemma 3.8** *Let $n \neq 0, v_m = w_n$ and $c = c_k$ is defined by* (3.9).

(*i*) *If $c = 2$, then $n < m < 3.735n$.*

(*ii*) *If $c > 2$, then $n < m < 2.22n$.*

**Proof:** Since $y_1 = 6$, we see that $v_l < w_l$ for $l > 0$, and $v_m = w_n, n \neq 0$ implies that $m > n$. Now we will estimate $v_m$ and $w_n$. From (3.29) and (3.30) we have

$$v_m > \frac{\tilde{s}}{2}(2c+1+2\tilde{s}\sqrt{3c})^m \geq \frac{1}{2}(2c+1+2\tilde{s}\sqrt{3c})^m,$$
$$w_n < \frac{\tilde{s}\sqrt{111}+6\sqrt{c}}{\sqrt{111}}(74c+1+2\tilde{t}\sqrt{111c})^n < \frac{1}{2}(74c+1+2\tilde{t}\sqrt{111c})^{n+\frac{1}{2}}.$$

Thus $v_m = w_n$ implies

$$\frac{2m}{2n+1} < \frac{\ln(74c+1+2\tilde{t}\sqrt{111c})}{\ln(2c+1+2\tilde{s}\sqrt{3c})}. \tag{3.31}$$

(i) If $c = 2$, then (3.31) implies $m < 3.735n$.
(ii) If $c > 2$, then $c \geq 506$, so (3.31) implies $m < 2.22n$.

□

**Lemma 3.9** *If $v_m = w_n$, $n \neq 0$ and $c = c_k > 2$ is defined by* (3.9), *then $m > n > \sqrt[4]{c}/6$.*

**Proof:** Since $v_m = w_n$, $z_0 = z_1 = \pm\tilde{s}$, $x_0 = 0$ and $y_1 = 6$, Lemma 3.7 implies

$$\begin{aligned} m^2\tilde{s} &\equiv 37n^2\tilde{s} \pm 6n\tilde{t} \pmod{4c}, \\ \tilde{s}(m^2-37n^2) &\equiv \pm 6n\tilde{t} \pmod{4c}, \\ 3\tilde{s}^2(m^2-37n^2)^2 &\equiv 108n^2\tilde{t}^2 \pmod{4c}. \end{aligned} \tag{3.32}$$

Since $c+1 = 3\tilde{s}^2, 37c+1 = 3\tilde{t}^2$ we have

$$(c+1)(m^2-37n^2)^2 \equiv 36(37c+1)n^2 \pmod{4c}.$$

which implies

$$(m^2-37n^2)^2 \equiv 36n^2 \pmod{c}. \tag{3.33}$$

Assume that $n \le \sqrt[4]{c}/6$. Since $n < m$ by Lemma 3.8, we have

$$|\tilde{s}(m^2 - 37n^2)| < \sqrt{\frac{c+1}{3}} \cdot 36n^2 \le \sqrt{\frac{c+1}{3}} \cdot \sqrt{c} < c,$$

and

$$(m^2 - 37n^2)^2 < 36^2 n^4 \le c.$$

On the other hand,

$$6\tilde{t}n \le 6 \cdot \sqrt{\frac{37c+1}{3}} \cdot \frac{\sqrt[4]{c}}{6} < c, \qquad 36n^2 \le \sqrt{c} < c.$$

It follows from (3.32) and (3.33) that

$$\tilde{s}(m^2 - 37n^2) = -6\tilde{t}n, \qquad (m^2 - 37n^2)^2 = 36n^2.$$

Hence we have

$$\tilde{s}^2(m^2 - 37n^2)^2 = 36\tilde{t}^2 n^2 = \tilde{t}^2(m^2 - 37n^2)^2,$$

which together with $n \neq 0$ implies $\tilde{s}^2 = \tilde{t}^2$, which is a contradiction.

□

Applying Lemma 3.9 we can easily prove the following result:

**Proposition 3.10** *Let $c = c_k > 2$ be defined by* (3.9) *and $x, y, z \in \mathbb{Z}$ be positive integer solutions of the system of simultaneous Pellian equations* (3.10) *and* (3.11)*. Then*

$$\ln y > \left( \frac{\sqrt[4]{c}}{6} - 1 \right) \ln(2c+1).$$

**Proof:** Let $z = v_m$. From $x > 0$ we conclude $m \neq 0$. It follows from (3.13) that

$$x = \frac{3\tilde{s}}{2\sqrt{3c}} \left[ (2c+1+2\tilde{s}\sqrt{3c})^m - (2c+1-2\tilde{s}\sqrt{3c})^m \right].$$

Moreover, $y^2 - 37x^2 = 36 > 0$ implies $y^2 > 37x^2$. Thus we have

$$\begin{aligned} y &> \sqrt{37}x = \frac{3\sqrt{37}\tilde{s}}{2\sqrt{3c}} \left[ (2c+1+2\tilde{s}\sqrt{3c})^m - (2c+1-2\tilde{s}\sqrt{3c})^m \right] \\ &> (2c+1+2\tilde{s}\sqrt{3c})^{m-1} > (2c+1)^{m-1}. \end{aligned}$$

Hence the proposition follows from Lemma 3.9.

□

Let $N = 3\tilde{t}^2, \theta_0 = \sqrt{1 - 1/N}$ and $\theta_2 = \sqrt{1 + 36/N}$. So $\theta_0$ and $\theta_2$ are square roots of rationals which are close to 1. Now we show that every positive integer solution of our problem induce good approximations of these numbers.

**Lemma 3.11** *Let $c = c_k > 2$ is defined by* (3.9)*. All positive integer solutions of* (3.10) *and* (3.11) *satisfy*

$$\max \left\{ \left| \theta_0 - \frac{37z}{\tilde{t}y} \right|, \left| \theta_2 - \frac{37\tilde{s}x}{\tilde{t}y} \right| \right\} < \frac{36.04}{y^2}.$$

**Proof:** We have $\theta_0 = \frac{1}{\tilde{t}}\sqrt{\frac{37c}{3}}$ and $\theta_2 = \frac{\tilde{s}}{\tilde{t}}\sqrt{37}$. Hence,

$$\begin{aligned}
\left|\theta_0 - \frac{37z}{\tilde{t}y}\right| &= \frac{37}{\tilde{t}}\left|\frac{c}{3} - \frac{37z^2}{y^2}\right| \cdot \left|\sqrt{\frac{37c}{3}} + \frac{37z}{y}\right|^{-1} = \frac{37(37+c)}{3\tilde{t}y^2} \cdot \left|\sqrt{\frac{37c}{3}} + \frac{37z}{y}\right|^{-1} \\
&< \frac{1}{y^2} \cdot \frac{37(37+c)}{3\tilde{t}} \cdot \sqrt{\frac{3}{37c}} < \frac{1.1}{y^2}, \\
\left|\theta_2 - \frac{17\tilde{s}x}{\tilde{t}y}\right| &= \frac{\tilde{s}\sqrt{37}}{\tilde{t}}\left|1 - \frac{37x^2}{y^2}\right| \cdot \left|1 + \frac{\sqrt{37}x}{y}\right|^{-1} \\
&< \frac{\tilde{s}\sqrt{37}}{\tilde{t}}\frac{36}{y^2} < \frac{36.04}{y^2}.
\end{aligned}$$

□

Now we are ready to calculate an upper bound for $c$ in the $D(-1)$-quadruple $\{1,37,-c,d\}$. For this we apply the following very useful result of Bennett [4] on simultaneous rational approximations to the square roots of rational numbers, on numbers $\theta_0$ and $\theta_2$.

**Theorem 3.12** ([4, Theorem 3.2]) *If $a_i, p_i, q$ and $N$ are integers for $0 \le i \le 2$, with $a_0 < a_1 < a_2$, $a_j = 0$ for some $0 \le j \le 2$, $q$ nonzero and $N > M^9$, where*

$$M = \max_{0\le i\le 2}\{|a_i|\} \ge 3,$$

*then we have*

$$\max_{0\le i\le 2}\left\{\left|\sqrt{1+\frac{a_i}{N}} - \frac{p_i}{q}\right|\right\} > (130N\gamma)^{-1}q^{-\lambda},$$

*where*

$$\lambda = 1 + \frac{\ln(32.04N\gamma)}{\ln\left(1.68N^2\prod_{0\le i\le j\le 2}(a_i-a_j)^{-2}\right)}$$

*and*

$$\gamma = \begin{cases} \frac{(a_2-a_0)^2(a_2-a_1)^2}{2a_2-a_0-a_1}, & \textit{ako je } a_2 - a_1 \ge a_1 - a_0, \\ \frac{(a_2-a_0)^2(a_1-a_0)^2}{a_1+a_2-2a_0}, & \textit{ako je } a_2 - a_1 < a_1 - a_0. \end{cases}$$

**Proposition 3.13** *If $c = c_k > 2\,744\,863\,693\,741$ is defined by (3.9), then*

$$\ln y < \frac{2\ln c \cdot \ln(37^7c^2)}{\ln\frac{0.1c}{37^6}}.$$

**Proof:** Let $N = 3\tilde{t}^2, q = \tilde{t}y, a_0 = -1, a_1 = 0, a_2 = 36, p_0 = 37z$ and $p_2 = 37\tilde{s}x$. Note that $c > 2\,744\,863\,693\,741$, implies $N = 3\tilde{t}^2 = 37c + 1 > 36^9$, so we may apply Theorem 3.12. In our case

$$\gamma = \frac{37^2 36^2}{73} \text{ and } \lambda = 1 + \frac{\ln(32.04N\gamma)}{\ln\frac{1.68N^2}{37^2 36^2}}.$$

Theorem 3.12 and Lemma 3.11 together imply

$$\begin{aligned} \frac{73}{691947360\tilde{t}^{\lambda+2}} &< \frac{36.04}{y^{2-\lambda}}, \\ y^{2-\lambda} &< 341613463.8\left(\frac{37c+1}{3}\right)^{\frac{\lambda+2}{2}}. \end{aligned}$$

Nothing $\lambda < 2$, we have

$$y^{2-\lambda} < 341613463.8\left(\frac{37c+1}{3}\right)^{2},$$

so

$$\ln y < \frac{1}{2-\lambda}\left(20 + 2\ln\frac{37c+1}{3}\right) < \frac{1}{2-\lambda}\ln(37^7c^2). \tag{3.34}$$

Moreover,

$$\frac{1}{2-\lambda} = \frac{\ln\frac{1.68N^2}{37^2 36^2}}{\ln\frac{1.68N}{37^2 36^2 32.04\gamma}}.$$

Since $N < 38c$, it follows from

$$\begin{aligned} \frac{1.68N^2}{37^2 36^2} &< c^2, \\ \frac{1.68N}{37^2 36^2 32.04\gamma} &> \frac{0.1c}{37^6}, \end{aligned}$$

that

$$\frac{1}{2-\lambda} < \frac{2\ln c}{\ln\frac{0.1c}{37^6}}. \tag{3.35}$$

From (3.34) and (3.35) we obtain the claim of the proposition.

□

By combining Proposition 3.13 with Proposition 3.10 we will obtain an upper bound for $c$.

**Proposition 3.14** *If $c = c_k > 2$ defined by* (3.9) *is minimal for which the equations* (3.10) *and* (3.11) *have a nontrivial solution, then $c < 1100 \cdot 37^6$.*

**Proof:** Suppose $c \geq 1100 \cdot 37^6$. By Propositions 3.10 and 3.13 we have

$$\left(\frac{\sqrt[4]{c}}{6} - 1\right)\ln(2c+1) < \ln c < \frac{2\ln c \cdot \ln(37^7c^2)}{\ln\frac{0.1c}{37^6}}. \tag{3.36}$$

Because of

$$\begin{aligned} \ln c &< \ln(2c+1), \\ 0.16c^{\frac{1}{4}} &< \frac{\sqrt[4]{c}}{6} - 1, \\ \ln\frac{0.1c}{37^6} &\geq \ln 110, \\ \ln(37^7c^2) &< \ln c^3, \end{aligned}$$

from (3.36) we obtain

$$0.1c^{\frac{1}{4}} \quad < \quad \ln c.$$

Since

$$f(c) := 0.1c^{\frac{1}{4}} - \ln c$$

is positive and increasing function for $c \geq 1100 \cdot 37^6$, we have a contradiction.

□

Next step in solving of our equation $v_m = w_n$ is to find an explicit upper bound for index $m$ or $n$. To do that we use the standard method in dealing with such kind of problems. We use Baker's theory on linear forms in logarithms on algebraic numbers, which yields us an upper bound for $n$.

Now, we are ready to prove:

**Lemma 3.15** *Assume that $c = c_k \geq 2$ is defined by (3.9). If $v_m = w_n$ and $n \neq 0$, then*

$$0 < n\ln(74c+1+2\tilde{t}\sqrt{111c}) - m\ln(2c+1+2\tilde{s}\sqrt{3c}) + \ln\frac{\tilde{s}\sqrt{111}\pm 6\sqrt{c}}{\tilde{s}\sqrt{111}} < 5.33\cdot(148c)^{-n}. \quad (3.37)$$

**Proof:** Putting

$$P = \tilde{s}(2c+1+2\tilde{s}\sqrt{3c})^m, \quad Q = \frac{1}{\sqrt{111}}(\tilde{s}\sqrt{111}\pm 6\sqrt{c})(74c+1+2\tilde{t}\sqrt{111c})^n, \quad (3.38)$$

we have

$$P^{-1} = \frac{1}{\tilde{s}}(2c+1-2\tilde{s}\sqrt{3c})^m, \quad Q^{-1} = \frac{\sqrt{111}}{c+37}(\tilde{s}\sqrt{111}\mp 6\sqrt{c})(74c+1-2\tilde{t}\sqrt{111c})^n.$$

If $v_m = w_n$, then from (3.29) and (3.30) we obtain

$$P + \tilde{s}^2 P^{-1} = Q + \frac{c+37}{111}Q^{-1}. \quad (3.39)$$

It is obvious that $P > 1$. Moreover,

$$Q \geq \frac{1}{\sqrt{111}}(\tilde{s}\sqrt{111}-6\sqrt{c})(74c+1+2\tilde{t}\sqrt{111c}) > 7c(\tilde{s}\sqrt{111}-6\sqrt{c}) > 7c > 1. \quad (3.40)$$

Furthermore,

$$\begin{aligned} P - Q &= \frac{c+37}{111}Q^{-1} - \frac{c+1}{3}P^{-1} \\ &< \frac{c+1}{3}(P-Q)P^{-1}Q^{-1}, \end{aligned} \quad (3.41)$$

$$P - \frac{c+1}{3} = \frac{c+1}{3}\left(\frac{(2c+1+2\tilde{s}\sqrt{3c})^m}{\tilde{s}} - 1\right) > 0.$$

Thus

$$P \quad > \quad \frac{c+1}{3}. \quad (3.42)$$

If $P > Q$, it follows from (3.41) that $PQ < (c+1)/3$. Since $Q > 1$, using (3.42) we get a contradiction. Therefore, $Q > P$.

Using (3.39), we conclude that

$$P > Q - \frac{c+1}{3}P^{-1} > Q - 1. \tag{3.43}$$

Thus from (3.40) and (3.43) we obtain

$$\frac{Q-P}{Q} < Q^{-1}. \tag{3.44}$$

On the other hand,

$$Q^{-1} \leq \frac{\sqrt{111}}{c+37}(\tilde{s}\sqrt{111}+6\sqrt{c})(74c+1+2\tilde{t}\sqrt{111c})^{-n} < 5.27618 \cdot (74c+1+2\tilde{t}\sqrt{111c})^{-n}. \tag{3.45}$$

Now we will bound linear form $\ln \frac{Q}{P}$ in logarithms.

By [24, Lemma B2], it follows from (3.44) and (3.45) that

$$\begin{aligned} 0 &< \ln \frac{Q}{P} = -\ln\left(1 - \frac{Q-P}{Q}\right) < -\ln(1 - Q^{-1}) \\ &< 5.33 \cdot (74c+1+2\tilde{t}\sqrt{111c})^{-n} \\ &< 5.33 \cdot (148c)^{-n}. \end{aligned} \tag{3.46}$$

Since

$$\ln \frac{Q}{P} = n \ln(74c+1+2\tilde{t}\sqrt{111c}) - m \ln(2c+1+2\tilde{s}\sqrt{3c}) + \ln \frac{\tilde{s}\sqrt{111} \pm 6\sqrt{c}}{\tilde{s}\sqrt{111}}, \tag{3.47}$$

combining (3.46) and (3.47) we obtain the claim of the lemma.

□

Let

$$\Lambda = n \ln(74c+1+2\tilde{t}\sqrt{111c}) - m \ln(2c+1+2\tilde{s}\sqrt{3c}) + \ln \frac{\tilde{s}\sqrt{111} \pm 6\sqrt{c}}{\tilde{s}\sqrt{111}}.$$

Now we will apply the following result of Baker and Wüstholz to the form $\Lambda$.

**Theorem 3.16** ([3, Theorem]) *For a linear form $\Lambda \neq 0$ in logarithms of $l$ algebraic numbers $\alpha_1, \dots, \alpha_l$ with rational integer coefficients $b_1, \dots, b_l$,*

$$\ln|\Lambda| \geq -18(l+1)!\,l^{l+1}(32d)^{l+2}\ln(2ld)h'(\alpha_1)\cdots h'(\alpha_l)\ln B,$$

*where $d = [\mathbb{Q}(\alpha_1, \dots, \alpha_l) : \mathbb{Q}], B = \max\{|b_1|, \dots, |b_l|\}, h'(\alpha) = \max\{h(\alpha), \frac{1}{d}|\ln \alpha|, \frac{1}{d}\}$, and $h(\alpha)$ logarithmic Weil height of $\alpha$.*

We have $l = 3, d = 8, B = m$ and

$$\alpha_1 = 74c+1+2\tilde{t}\sqrt{111c}, \quad \alpha_2 = 2c+1+2\tilde{s}\sqrt{3c}, \quad \alpha_3 = \frac{\tilde{s}\sqrt{111} \pm 6\sqrt{c}}{\tilde{s}\sqrt{111}}. \tag{3.48}$$

Minimal polynomials of $\alpha_1, \alpha_2, \alpha_3$ are

$$\begin{aligned} P_{\alpha_1}(x) &= x^2 - (148c+2)x + 1, \quad P_{\alpha_2}(x) = x^2 - (4c+2)x + 1, \\ P_{\alpha_3}(x) &= \frac{37c+37}{3}x^2 - \frac{74c+74}{3}x + \frac{37+c}{3}. \end{aligned}$$

Since $\tilde{s} \le \sqrt{c}$ and $\tilde{t} \le 4\sqrt{c}$ we obtain

$$\begin{aligned} h'(\alpha_1) &= \frac{1}{2}\ln\alpha_1 < \frac{1}{2}\ln(159c), \\ h'(\alpha_2) &= \frac{1}{2}\ln\alpha_2 < \frac{1}{2}\ln(6c), \\ h'(\alpha_3) &= \frac{1}{2}\ln(37\tilde{s}^2 + 2\tilde{s}\sqrt{111c}) < \frac{1}{2}\ln(59c). \end{aligned}$$

Now from Lemma 3.15 we have $\ln\Lambda < \ln(5.33\cdot(148c)^{-n})$ and it follows from Theorem 3.16 that

$$1.862\cdot 10^{16}\cdot\ln(159c)\cdot\ln(6c)\cdot\ln(59c)\cdot\ln m > n\ln(148c) - 1.68. \tag{3.49}$$

Using Lemma 3.8 from (3.49) we conclude the following:

(i) If $c=2$, then

$$1.28\cdot 10^{18}\ln(3.735n) > 5.69n - 1.68. \tag{3.50}$$

(ii) If $c \ge 506$, then

$$1.862\cdot 10^{16}\cdot\ln(159c)\cdot\ln(6c)\cdot\ln(59c)\cdot\ln(2.22n) > n\ln(148c) - 1.68. \tag{3.51}$$

Since $c_6 > 1100\cdot 37^6$, by Proposition 3.14 to complete the proof of Theorem 3.6 we have to check if there is any nontrivial solution of the system of equations (3.10) and (3.11) for $c \in \{c_0,\ldots,c_5\}$. We use the following reduction method of Dujella and Pethő, and for each $0 \le k \le 5$, i.e. $c \in \{c_0,\ldots,c_5\}$ find a much better upper bound for $n$. And at the end, we directly check if we have any nontrivial solution of the system $v_m = w_n$, with small indices $m$ and $n$. Because the procedure here is pretty standard we will not give all details.

From (3.37) dividing by $\ln(2c+1+2\tilde{s}\sqrt{3c})$ we obtain the inequality

$$0 < n\kappa - m + \mu < A\cdot B^{-n}, \tag{3.52}$$

where

$$\kappa = \frac{\ln(74c+1+2\tilde{t}\sqrt{111c})}{\ln(2c+1+2\tilde{s}\sqrt{3c})}, \mu_{\pm} = \frac{\ln\frac{\tilde{s}\sqrt{111}\pm 6\sqrt{c}}{\tilde{s}\sqrt{111}}}{\ln(2c+1+2\tilde{s}\sqrt{3c})}, \quad A = \frac{5.33}{\ln(2c+1+2\tilde{s}\sqrt{3c})}, \quad B = 148c.$$

**Lemma 3.17** ([13, Lemma 5a]) *Suppose that $N$ is a positive integer. Let $p/q$ be the convergent of the continued fraction expansion of $\kappa$ such that $q > 6N$ and let $\varepsilon = ||\mu q|| - N\cdot||\kappa q||$, where $||\cdot||$ denotes the distance from the nearest integer.*

*If $\varepsilon > 0$, then there is no solution of the inequality (3.52) in integers $m$ and $n$ with*

$$\frac{\ln\frac{Aq}{\varepsilon}}{\ln B} \le n \le N.$$

We apply Lemma 3.17 with $N$ the upper bound for $n$ in the each case. If $k=0$, then $c=c_0=2$ and to find an upper bound for $n$ we use the relation (3.50). In other cases we use the relation (3.51). Using the Lemma 3.8 we find the corresponding $m$. We obtain following results:

1) $c = c_5 = 230102245802, \tilde{s} = 276949, \tilde{t} = 1684615, N = 8\cdot 10^{20}$.

   In the first step of reduction we obtain $n=1, m=2$.

2) $c = c_4 = 1131059666, \tilde{s} = 19417, \tilde{t} = 118109, N = 6 \cdot 10^{20}$.

   If $\mu = \mu_+$, then in the first step of reduction follows $n = 1, m = 2$ and $n = 2$, $m = 3, 4$. For $\mu = \mu_-$ we obtain $n = 1, m = 2$.

3) $c = c_3 = 10795826, \tilde{s} = 1897, \tilde{t} = 11539, N = 4 \cdot 10^{20}$.

   In the first step of reduction we obtain $n = 1, m = 2$ and $n = 2$, $m = 3, 4$.

4) $c = c_2 = 53066, \tilde{s} = 133, \tilde{t} = 809, N = 2 \cdot 10^{20}$.

   In the first step of reduction we conclude that $n = 1, m = 2$, $n = 2, m = 3, 4$ and $n = 3, m = 4, 5, 6$.

5) $c = c_1 = 506, \tilde{s} = 13, \tilde{t} = 79$. For the first step of reduction we set $N = 8 \cdot 10^{19}$, and for the second step we take $N = 4$. It follows $n = 1, m = 2$.

6) $c = c_0 = 2, \tilde{s} = 1, \tilde{t} = 5$. In the first step of reduction we have $N = 2 \cdot 10^{19}$. After second step, where $N = 8$, we obtain $n = 1, m = 2, 3$.

For determined small indices $m$ and $n$ it is easy to check that in the each case there are no solutions of the equation $v_m = w_n$.

This completes the proof of Theorem 3.6.

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