Damping optimization in mechanical systems with external force

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Abstract

We consider a mechanical system excited by external force. Model of such a system is described by the system of ordinary differential equations: $M\ddot{x}(t) + D\dot{x}(t) + Kx(t) = \widehat{f}(t)$, where matrices M, K (mass and stiffness) are positive definite and the vector $\widehat{f}$ corresponds to an external force. The damping matrix D is assumed to be positive semidefinite and has a small rank. We introduce two criteria that allow damping optimization of mechanical system excited by an external force. Since in general a damping optimization is a very demanding problem, we provide a new formulas which have been used for efficient damping optimization. The efficiency of new formulas is illustrated with a numerical experiment.

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1. Introduction

The motivation for this paper has been posted in [1, Section 17] and it is related to the harmonic response of the mechanical system under the influence of the harmonic force. Our aim is to present clear and more general

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results to this problem, that is our main concern will be the optimization of a linear damped oscillating systems using theirs harmonic responses.

The theory of linear damped oscillations has been studied for more than hundred years and is still of vital interest to researchers in control theory, optimization, and computational aspects.

We have restricted ourselves to the model described with the following second order differential equation

$$M\ddot{x}(t) + D\dot{x}(t) + Kx(t) = \widehat{f}(t)\,, \tag{1}$$

where matrices $M, K \in \mathbb{R}^{n\times n}$ (mass and stiffness) are positive definite, vector $\widehat{f} \in \mathbb{R}^{n\times 1}$ presents external force and $D \in \mathbb{R}^{n\times n}$ presents a (viscous) damping which is positive semidefinite and depends on an optimization parameter $v \in (0, \infty)$ called viscosity. Damping matrix is a small rank matrix and it can be written in a form $D = D_r D_r^*$ where D_r describes the geometry of dampers positions.

We like to emphasize, that our model leaves aside a several important cases such as gyroscopic models free rigid body motion or non-viscous damping. Although these important cases are not considered in the present paper, we believe that the main part of our approach, with appropriate transformations, can be applied to the mentioned cases too.

There is a vast literature in this field of research. Particulary the engineering literature is very rich. For a brief insight we give some selection of some older references: [2, 3, 4, 5], as well as some less old: [6, 7, 8]. In references [9, 10] authors studies topology optimization for the problem of damping optimization. Among further we emphasize two books by G. Chen and J. Zhen [11] and [12], where a thorough presentation of techniques and results in this area are given. Some more recent references are [13, 14, 15] and already mentioned lecture notes [1], where one can find a nice overview of the results connected with viscously damped mechanical systems.

As described above, the main problem considered in the paper reads: for the given mass matrix M, the stiffness matrix K, and the external force $\widehat{f}(t)$ find an optimal (low rank) damping matrix D in order to avoid unwanted oscillations. For such an optimization problem there are lots of different optimization criteria. In general, we can divide them into two categories.

The first category contains criteria which consider the system (1) without external force, that is when $\widehat{f}(t) = 0$. In this category one can consider the spectrum of the matrix of the corresponding linearized problem. The most

popular criteria are: a spectral abscissa criterion which requires the (penalty) function $\max_i \mathrm{Re}(\lambda_i)$ to be minimized, where λ_i are eigenvalues of the matrix of the corresponding linearized problem (see e.g. [16, 17, 18]), a criterion which requires the (penalty) function $\max_i \frac{\mathrm{Re}(\lambda_i)}{|\lambda_i|}$ to be minimized, where λ_i are eigenvalues of corresponding quadratic eigenvalue problem. Sometimes, the above mentioned criteria are combined with an additional constraint on λ_i, that is, we require that $|\max_i \mathrm{Im}(\lambda_i)| \leq a$, for some positive number a (see e.g. [19, 20, 18]).

Next approach in this category (without external force, $\widehat{f}(t) = 0$) considers the solution of the corresponding Lyapunov equation (see e.g. [21, 22, 23, 16, 18, 24]) which is constructed using appropriate linearization of system (1). Within this category, one can consider the following three criteria: minimization of the trace the solution of the corresponding Lyapunov equation, minimization of the 2-norm or the Frobenius norm of the solution of the corresponding Lyapunov equation.

The second category contains criteria which consider the system excited by the external force, that is when $\widehat{f}(t) \neq 0$. For this category we will use following two optimization criteria [1, Chapt. 17]:

i) As the first criteria we will optimize the size of the harmonic answer by the square of its norm, averaged over the period τ which is called *average displacement amplitude*. In this case our aim will be to find the damping matrix D which minimize the *average displacement amplitude*.

ii) Second criteria uses the norm of solution of equivalent system and is called *average energy amplitude*. In this case our aim is to find the damping matrix D which minimize the *average energy amplitude*.

The free oscillations (those with $f = 0$) are also called transient oscillations whereas steady state oscillations, which we will consider through the paper, with harmonic or similar forces which produce responses not largely varying in time. As it can be expected, there are fundamental differences between these two categories, for example in the case of steady state oscillations one has to be aware of possible resonance.

Our approach use the fact that the matrix pair (M, K) is positive definite, which means that the both matrices can be simultaneously diagonalized. This allow us to derive a formula for both above mentioned criteria needed for damping optimization, so that the optimization process can be very efficient.

The paper is organized as follows. In Section 2 we write energy criteria which takes into account periodic function in a special form. In Section 3 we derive an explicit formulae for stated criteria $i)$ and $ii)$ that can be implemented effectively and significantly accelerate optimization procedure. Section 4 includes numerical experiments which illustrate effectiveness of derived formulae.

2. Criteria for damping optimization

As it has been mentioned in the introduction, we proceed with a thoroughly description of the optimization criteria for the proposed problem. For that sake we first assume that the function $\widehat{f}$ in the equation (1) has a special form. If $f\colon [\,0,\,T\,] \to \mathbb{R}$, then

$$\widehat{f}(t) = \sum_{j=1}^{p} f_j^a \cos(\widehat{\omega}_j t) + f_j^b \sin(\widehat{\omega}_j t)$$

with $\widehat{\omega}_j = \frac{2\pi j}{T}$ for $j = 1, \ldots, p$, where index $p \in \mathbb{N}$ corresponds to number of simple oscillating functions in the right-hand side. Excitation function is described by a simple trigonometric functions with different periods, thus T corresponds to a least common multiple of all periods. Motivation for that assumption is related with a Fourier series which decomposes periodic functions or periodic signals into the sum of a set of simple oscillating functions. In continuation we will illustrate how this can be applied for a general periodic function.

Thus, we can write equation (1) as

$$M\ddot{x}(t) + D(v)\dot{x}(t) + Kx(t) = \sum_{j=1}^{p} f_j^a \cos(\widehat{\omega}_j t) + f_j^b \sin(\widehat{\omega}_j t)\,, \tag{2}$$

and in order to simplify notation we assume that both initial conditions are zero, that is

$$x(0) = 0 \quad \text{and} \quad \dot{x}(0) = 0.$$

Using the standard results from the theory of ordinary differential equations it is easy to see that for the solution of (2) we need to calculate solutions x_j of equations

$$M\ddot{x}_j(t) + D(v)\dot{x}_j(t) + Kx_j(t) = f_j^a \cos(\widehat{\omega}_j t) + f_j^b \sin(\widehat{\omega}_j t)\,, \tag{3}$$

$$x_j(0) = 0 \quad \text{and} \quad \dot{x}_j(0) = 0. \tag{4}$$

for $j = 1, \ldots, p$. Then the solution of (2) is given as

$$x(t) = \sum_{j=1}^{p} x_j(t). \tag{5}$$

As it has been shown in [1], with substitution $x_j(t) = x_j^a \cos(\widehat{\omega}_j t) + x_j^b \sin(\widehat{\omega}_j t)$ and by introducing complex quantities

$$x_j^0 = x_j^a - i x_j^b, \quad f_j^0 = f_j^a - i f_j^b \quad \text{for} \quad j = 1, \ldots, p \tag{6}$$

the system (3) is equivalent to

$$(i\widehat{\omega}_j I - A) y_j^0 = F_j^0, \quad \text{where} \quad y_j^0 = \begin{bmatrix} L_1^* x_j^0 \\ i\widehat{\omega}_j L_2^* x_j^0 \end{bmatrix}, \quad F_j^0 = \begin{bmatrix} 0 \\ L_2^{-1} f_j^0 \end{bmatrix}, \tag{7}$$

$$A = \begin{bmatrix} 0 & L_1 L_2^{-*} \\ -L_2^{-1} L_1 & -L_2^{-1} D L_2^{-*} \end{bmatrix}, \quad K = L_1 L_1^*, \quad M = L_2 L_2^*$$

and $y_j(t) = y_j^0 e^{i\widehat{\omega}_j t}$ is a solution of

$$\dot{y}_j(t) = A y_j(t) + F_j^0 e^{i\widehat{\omega}_j t} \quad \text{for} \quad j = 1, \ldots, p. \tag{8}$$

From the equation (8) follows that $y(t) = \sum_{j=1}^{p} y_j(t)$ is solution of equation

$$\dot{y}(t) = A y(t) + \sum_{j=1}^{p} F_j^0 e^{i\widehat{\omega}_j t}.$$

Note that in our setting the solution $x(t)$ depends on damping parameter v, thus in definition of optimization criteria we will emphasize this fact, i.e. we will write $x(t, v)$.

For damping optimization criteria we will use two measures defined in [1, Chapt. 17]:

i) Within the first criterion we will minimize the size of the harmonic response, by the square of norm of $x(t, v)$ and averaged over the period τ, which is given as

$$\mathfrak{F}_1(v) = \frac{1}{\tau} \int_0^\tau \|x(t, v)\|^2 dt\,. \tag{9}$$

This damping criterion is called *average displacement amplitude*. This means that aim will be to minimize $\mathfrak{F}_0(v)$, that is

$$\min_{v \in (0,\infty)} \frac{1}{\tau} \int_0^\tau \|x(t, v)\|^2 dt\,.$$

ii) Within the second criterion we will minimize the square of the norm of $y(t,v)$ averaged over the period τ, which is given as

$$\mathfrak{F}_2(v) = \frac{1}{\tau}\int_0^\tau \|y(t,v)\|^2 dt\,. \tag{10}$$

This damping criterion is called *average energy amplitude*. This means that in the case of the second criterion our aim will be to find a such a parameter v such that

$$\min_{v\in(0,\infty)} \frac{1}{\tau}\int_0^\tau \|y(t,v)\|^2 dt\,.$$

Note, as we have mentioned before, here we use symbol $\mathfrak{F}_s(v)$, $s = 1, 2$ to emphasize that optimization depends on damping parameter v.

We would like to emphasize that in damping optimization both criteria can lead to similar optimal damping matrix, but in general one could expect different optimal values, as it is illustrated in Example 1. Therefore, we cannot say a priori which criterion is better since it strongly depends on applications.

First, we will rewrite our criterion (10) in a more appropriate form. Since $y(t) = \sum_{j=1}^p y_j(t)$ we have

$$\frac{1}{\tau}\int\limits_0^\tau \|y(t)\|^2 dt = \frac{1}{\tau}\int\limits_0^\tau \sum_{j=1}^p \|y_j(t)\|^2 + \sum_{j=1}^p \sum_{\substack{k=1\\k\neq j}}^p y_j(t)^* y_k(t)\, dt. \tag{11}$$

Using that

$$y_j(t)^* y_k(t) = (y_j^0)^* y_k^0\, e^{i(\widehat{\omega}_k - \widehat{\omega}_j)t} = (y_j^0)^* y_k^0\, (\cos(\widehat{\omega}_k - \widehat{\omega}_j)t + i\sin(\widehat{\omega}_k - \widehat{\omega}_j)t)$$

for $j \neq k$ we obtain that

$$\begin{aligned}
&\frac{1}{\tau}\int\limits_0^\tau \sum_{j=1}^p \sum_{\substack{k=1\\k\neq j}}^p y_j(t)^* y_k(t)\, dt = \\
&= \frac{2}{\tau}\int\limits_0^\tau \sum_{j=1}^p \sum_{\substack{k=1\\k<j}}^p \mathcal{R}e((y_j^0)^* y_k^0)\cos(\widehat{\omega}_k - \widehat{\omega}_j)t - \mathcal{I}m((y_j^0)^* y_k^0)\sin(\widehat{\omega}_k - \widehat{\omega}_j)t\, dt \\
&= 2\sum_{j=1}^p \sum_{\substack{k=1\\k<j}}^p \frac{\mathcal{R}e((y_j^0)^* y_k^0)}{\tau(\widehat{\omega}_k - \widehat{\omega}_j)}\sin(\widehat{\omega}_k - \widehat{\omega}_j)\, t|_0^\tau + \frac{\mathcal{I}m((y_j^0)^* y_k^0)}{\tau(\widehat{\omega}_k - \widehat{\omega}_j)}\cos(\widehat{\omega}_k - \widehat{\omega}_j)\, t|_0^\tau.
\end{aligned}$$

Since parameters $\widehat{\omega}_j = \frac{2\pi j}{T}$, we have that $(\widehat{\omega}_k - \widehat{\omega}_j) = \frac{2\pi(k-j)}{T}$.

Furthermore, since we have trigonometric functions with different periods we take that the period τ is a least common multiple (LCM) of all periods. That is $\tau = LCM(\{\frac{T}{j} : j = 1, \ldots, p\}) = T$ and together with

$$\sin(\widehat{\omega}_k - \widehat{\omega}_j)\, t|_0^\tau = \sin(2\pi(k-j)) = 0,$$
$$\cos(\widehat{\omega}_k - \widehat{\omega}_j)\, t|_0^\tau = \cos(2\pi(k-j)) - 1 = 0,$$

we have that

$$\frac{1}{\tau}\int_0^\tau \sum_{j=1}^{p}\sum_{\substack{k=1\\k\neq j}}^{p} y_j(t)^* y_k(t) = 0.$$

Using (7), equation (11) can be written as

$$\begin{aligned}\frac{1}{\tau}\int_0^\tau \|y(t)\|^2 dt &= \sum_{j=1}^{p}(y_j^0)^* y_j^0 = \sum_{j=1}^{p}(x_j^0)^*(\widehat{\omega}_j^2 M + K)x_j^0 \\ &= \sum_{j=1}^{p}(f_j^0)^*\, \left(-\widehat{\omega}_j^2 M + i\,\widehat{\omega}_j D(v) + K\right)^{-*} \times \\ &\quad \times \left(K + \widehat{\omega}_j^2 M\right)\, \left(-\widehat{\omega}_j^2 M + i\,\widehat{\omega}_j D(v) + K\right)^{-1} f_j^0\,.\end{aligned} \tag{12}$$

Similarly, for the criterion introduced in (9) it can be shown that

$$\begin{aligned}&\frac{1}{\tau}\int_0^\tau \|x(t)\|^2 dt = \\ &= \sum_{j=1}^{p}(f_j^0)^*\, \left(-\widehat{\omega}_j^2 M + i\,\widehat{\omega}_j D(v) + K\right)^{-*}\, \left(-\widehat{\omega}_j^2 M + i\,\widehat{\omega}_j D(v) + K\right)^{-1} f_j^0\,.\end{aligned} \tag{13}$$

Furthermore, using the fact that the matrix pair (M, K) is positive definite there exists a non-singular matrix Φ which simultaneously diagonalizes matrices M and K, that is

$$\Phi^* M \Phi = I, \qquad \Phi^* K \Phi = \Omega^2\,.$$

Then the system (1) can be written equivalently in so-called modal coordinates as:

$$\ddot{z}(t) + v\, CC^* \dot{z}(t) + \Omega^2 z(t) = f(t)\,, \tag{14}$$

where $z = \Phi^{-1}x$, $C = \Phi^* D_r$ and $f(t) = \sum_{j=1}^{p} \Phi^* f_j^a \cos(\widehat{\omega}_j t) + \Phi^* f_j^b \sin(\widehat{\omega}_j t)$.

Since we would like to derive a formulas for the both criteria we will introduce matrices

$$\Delta_j^1 = I \quad \text{and} \quad \Delta_j^2 = \left(\Omega^2 + \widehat{\omega}_j^2 I\right)$$

for $j = 1, \ldots, p$ such that we can write the both criteria using one formula. If we additionally use modal coordinates from (9) and (10) we obtain

$$\mathfrak{F}_s(v) = \sum_{j=1}^{p} f_j^* \left(-\widehat{\omega}_j^2 I + iv\widehat{\omega}_j CC^* + \Omega^2\right)^{-*} \Delta_j^s \left(-\widehat{\omega}_j^2 I + i\widehat{\omega}_j vCC^* + \Omega^2\right)^{-1} f_j \,, \tag{15}$$

where $f_j = \Phi^* f_j^a - i\Phi^* f_j^b$ is as in (6) and for $s = 1$ we have criterion (9), while for $s = 2$ we have criterion (10).

In the following we will split our presentation in two parts. In the first part we will assume that the external force has the following form

$$\widehat{f}(t) = f_j^a \cos(\widehat{\omega}_j t) + f_j^b \sin(\widehat{\omega}_j t) \,. \tag{16}$$

While in the second part we will assume more general form of the force, that is

$$\widehat{f}(t) = \sum_{j=1}^{p} f_j^a \cos(\widehat{\omega}_j t) + f_j^b \sin(\widehat{\omega}_j t) \,. \tag{17}$$

Thus, the main aim of the next section is to present explicit formulae for the average displacement amplitude and the average energy amplitude for the case when external force is given by (17).

3. Efficient formula for damping criteria with external force

Through this section we are going to derive an efficient formula for the functions $\mathfrak{F}_s(v)$ defined as in (15). As we have mentioned in the previous section, first we will assume that the external force is defined as in (16) and that $E_j^s(v)$ for $s = 1, 2$ denotes the energy that corresponds to this particular external force.

Thus, from (15) we obtain expression for $E_j^s(v)$ as:

$$E_j^s(v) = (f_j)^* \left(-\widehat{\omega}_j^2 I + i\, v\, \widehat{\omega}_j\, CC^* + \Omega^2\right)^{-*} \Delta_j^s \left(-\widehat{\omega}_j^2 I + i\, v\, \widehat{\omega}_j\, CC^* + \Omega^2\right)^{-1} f_j \,, \tag{18}$$

where $f_j = \Phi^* f_j^a - i\Phi^* f_j^b$.

Since, our final scope is to obtain the optimal v which minimize the energy function $E_j^s(v)$, we are going to derive a formula for $E_j^s(v)$ which will be explicit in v (as we will see $E_j^s(v)$ is a rational function in v).

For the calculation of the inverse $\left(-\widehat{\omega}_j^2 I + i\,v\,\widehat{\omega}_j\,CC^* + \Omega^2\right)^{-1}$ we will use the *Sherman-Morrison-Woodbury formula* ([25, (2.1.4),p.51.]). Thus, this inverse can be written as:

$$\left(-\widehat{\omega}_j^2 I + i\,v\,\widehat{\omega}_j\,CC^* + \Omega^2\right)^{-1} = \left(\Omega^2 - \widehat{\omega}_j^2 I\right)^{-1} \tag{19}$$
$$- i\,v\,\widehat{\omega}_j\,\left(\Omega^2 - \widehat{\omega}_j^2 I\right)^{-1} C\,\left(I + i\,v\,\widehat{\omega}_j\,C^*\left(\Omega^2 - \widehat{\omega}_j^2 I\right)^{-1} C\right)^{-1} C^*\left(\Omega^2 - \widehat{\omega}_j^2 I\right)^{-1}$$

For the purpose of easier following, we will introduce some auxiliary notations, thus let

$$T_j = \left(\Omega^2 - \widehat{\omega}_j^2 I\right)\,. \tag{20}$$

Further, note that the matrix $C^* T_j^{-1} C$ is symmetric, thus there exists the following eigenvalue decomposition

$$C^* T_j^{-1} C = U_j \Lambda_j U_j^*\,, \tag{21}$$

where U_j is unitary $r \times r$ matrix and $\Lambda_j = \mathrm{diag}((\lambda_j)_1, \ldots, (\lambda_j)_r)$. If we denote

$$\widehat{C} = C\,U_j$$

(19) can be written as:

$$\left(T_j + i\,v\,\widehat{\omega}_j\,CC^*\right)^{-1} = T_j^{-1} - i\,v\,\widehat{\omega}_j\,T_j^{-1}\widehat{C}\,\left(I + i\,v\,\widehat{\omega}_j\,\Lambda_j\right)^{-1}\widehat{C}^* T_j^{-1} \tag{22}$$

Now using (22) the energy function from (18) can be written as:

$$E_j^s(v) = f_j^*\,\left(T_j^{-1} + i\,v\,\widehat{\omega}_j\,T_j^{-1}\widehat{C}\,\left(I - i\,v\,\widehat{\omega}_j\,\Lambda_j\right)^{-1}\widehat{C}^* T_j^{-1}\right)\,\Delta_j^s\cdot$$
$$\cdot\left(T_j^{-1} - i\,v\,\widehat{\omega}_j\,T_j^{-1}\widehat{C}\,\left(I + i\,v\,\widehat{\omega}_j\,\Lambda_j\right)^{-1}\widehat{C}^* T_j^{-1}\right) f_j\,. \tag{23}$$

If we introduce the following two $r \times r$ diagonal matrices:

$$\Gamma_- = \mathrm{diag}\left(\frac{-i}{1 - i\,v\,\widehat{\omega}_j\,(\lambda_j)_1}, \ldots, \frac{-i}{1 - i\,v\,\widehat{\omega}_j\,(\lambda_j)_r}\right)\,, \tag{24}$$

$$\Gamma_+ = \mathrm{diag}\left(\frac{i}{1 + i\,v\,\widehat{\omega}_j\,(\lambda_j)_1}, \ldots, \frac{i}{1 + i\,v\,\widehat{\omega}_j\,(\lambda_j)_r}\right)\,, \tag{25}$$

then (23) can be written as

$$E_j^s(v) = f_j^* \left(T_j^{-1} - v\,\widehat{\omega}_j\, T_j^{-1}\widehat{C}\,\Gamma_-\,\widehat{C}^* T_j^{-1} \right) \Delta_j^s \cdot \\ \cdot (T_j^{-1} - v\,\widehat{\omega}_j\, T_j^{-1}\widehat{C}\,\Gamma_+\,\widehat{C}^* T_j^{-1}) f_j\,. \quad (26)$$

Note that from (24-25) it follows that the matrices Γ_- and Γ_+ are mutually conjugate. Now we will introduce some additional notation:

$$a_j = \widehat{C}^* T_j^{-1} f_j \qquad \text{and} \qquad b_j = \widehat{C}^* T_j^{-1} \Delta_j^s\, T_j^{-1} f_j\,, \quad (27)$$

and

$$F_0 = f_j^*\, T_j^{-1} \Delta_j^s T_j^{-1} f_j \qquad \text{and} \qquad A_j^s = \widehat{C}^* T_j^{-1} \Delta_j^s\, T_j^{-1} \widehat{C}\,. \quad (28)$$

Using (27) and (28) energy function from (26) can be written as

$$E_j^s(v) = F_0 - v\,\widehat{\omega}_j\, a_j^* \Gamma_- b_j - v\,\widehat{\omega}_j\, b_j^* \Gamma_+ a_j + v^2\,\widehat{\omega}_j^2\, a_j^* \Gamma_- A_j^s \Gamma_+ a_j\,. \quad (29)$$

First note that $F_0 \in \mathbb{R}$, further from the fact that Γ_- and Γ_+ are mutually conjugate it follows that

$$v\,\widehat{\omega}_j\, a_j^* \Gamma_- b_j + v\,\widehat{\omega}_j\, b_j^* \Gamma_+ a_j = 2\,v\,\widehat{\omega}_j\, \mathrm{Re}(a_j^* \Gamma_- b_j)\,.$$

If we write vectors a_j and b_j as

$$a_j = \left[(a_j)_1, (a_j)_2, \ldots, (a_j)_r\right]^T\,, \qquad b_j = \left[(b_j)_1, (b_j)_2, \ldots, (b_j)_r\right]^T\,,$$

then

$$\mathrm{Re}(a_j^* \Gamma_- b_j) = \sum_{k=1}^{r} \frac{(a_j)_k\,(b_j)_k\,v\,\widehat{\omega}_j\,(\lambda_j)_k}{1 + v^2\,\widehat{\omega}_j^2\,(\lambda_j)_k^2}\,.$$

All this together give that the sum of the second and the third term in (29) is equal to

$$v\,\widehat{\omega}_j\, a_j^* \Gamma_- b_j + v\,\widehat{\omega}_j\, b_j^* \Gamma_+ a_j = 2\,v^2\,\widehat{\omega}_j^2 \sum_{k=1}^{r} \frac{(a_j)_k\,(b_j)_k\,(\lambda_j)_k}{1 + v^2\,\widehat{\omega}_j^2\,(\lambda_j)_k^2}\,.$$

It is left to derive the last term in (29). For that purpose note that

$$a_j^* \Gamma_- A_j^s \Gamma_+ a_j = \sum_{l=1}^{r} \sum_{k=1}^{r} \frac{(a_j)_l\,(A_j^s)_{(l,k)}\,(a_j)_k}{(1 - i\,v\,\widehat{\omega}_j\,(\lambda_j)_l)\,(1 + i\,v\,\widehat{\omega}_j\,(\lambda_j)_k)}.$$

Since the energy function is a real function we will rewrite the above equality only using the real arithmetic. Indeed it is easy to see that

$$a_j^* \Gamma_- A_j^s \Gamma_+ a_j = \sum_{l=1}^{r} \sum_{k=1}^{r} \frac{(a_j)_l \, (A_j^s)_{(l,k)} \, (a_j)_k \, (1 + v^2 \, \widehat{\omega}_j^2 \, (\lambda_j)_l (\lambda_j)_k)}{(1 + v^2 \, \widehat{\omega}_j^2 \, (\lambda_j)_l^2) \, (1 + v^2 \, \widehat{\omega}_j^2 \, (\lambda_j)_k^2)}. \tag{30}$$

Finally we are able to write the energy function that corresponds to the force from (16):

$$\begin{aligned} E_j^s(v) = F_0 &- 2 \, v^2 \, \widehat{\omega}_j^2 \sum_{k=1}^{r} \frac{(a_j)_k \, (b_j)_k \, (\lambda_j)_k}{1 + v^2 \, \widehat{\omega}_j^2 \, (\lambda_j)_k^2} \\ &+ v^2 \, \widehat{\omega}_j^2 \sum_{l=1}^{r} \sum_{k=1}^{r} \frac{(a_j)_l \, (A_j^s)_{(l,k)} \, (a_j)_k \, (1 + v^2 \, \widehat{\omega}_j^2 \, (\lambda_j)_l (\lambda_j)_k)}{(1 + v^2 \, \widehat{\omega}_j^2 \, (\lambda_j)_l^2) \, (1 + v^2 \, \widehat{\omega}_j^2 \, (\lambda_j)_k^2)} \, . \end{aligned} \tag{31}$$

REMARK **1.** *In the above derivation of explicit formulas for our objective functions we have used the fact that dampers have the same viscosities. Since this approach is based on the usage of Sherman-Morrison-Woodbury formula, we cannot easily extend it to the case of different viscosities. For that case (of different viscosities) one possible approach could be to write a formula using modal coordinates which will allow usage of the structure of the system matrices and efficient numerical calculation of an approximation of the objective functions.*

Furthermore, our formulas hold for the case without internal or any structural damping. The presented formulas can be extended to the case of modally damped systems (for example Rayleigh damping or critical damping [26]). In this case the matrix T_j from (20) will contain the internal damping and remain diagonal matrix, but succeeding formulas should be derived particulary for this case.

Furthermore, if the force is defined by (17), then for the energy functions from (15) hold

$$\mathfrak{F}_s(v) = \sum_{j=1}^{p} E_j^s(v) \, ,$$

which together with (31) give the formula for the energy functions of the

following form:

$$\mathfrak{F}_s(v) = p \cdot F_0 - 2\, v^2 \sum_{j=1}^{p} \widehat{\omega}_j^2 \sum_{k=1}^{r} \frac{(a_j)_k\, (b_j)_k\, (\lambda_j)_k}{1 + v^2\, \widehat{\omega}_j^2\, (\lambda_j)_k^2} + \\ + v^2 \sum_{j=1}^{p} \widehat{\omega}_j^2 \sum_{l=1}^{r} \sum_{k=1}^{r} \frac{(a_j)_l\, (A_j^s)_{(l,k)}\, (a_j)_k\, (1 + v^2\, \widehat{\omega}_j^2\, (\lambda_j)_l (\lambda_j)_k)}{(1 + v^2\, \widehat{\omega}_j^2\, (\lambda_j)_l^2)\, (1 + v^2\, \widehat{\omega}_j^2\, (\lambda_j)_k^2)}\,, \quad (32)$$

with

$$T_j = \left(\Omega^2 - \widehat{\omega}_j^2 I\right)\,, \qquad F_0 = f_j^*\, T_j^{-1} \Delta_j^s T_j^{-1} f_j\,, \quad (33)$$

$$C^* T_j^{-1} C = U_j \Lambda_j U_j^*\,, \qquad A_j^s = U_j^* C^* T_j^{-1} \Delta_j^s\, T_j^{-1} C U_j\,, \quad (34)$$

$$a_j = U_j^* C^* T_j^{-1} f_j\,, \qquad b_j = U_j^* C^* T_j^{-1} \Delta_j^s\, T_j^{-1} f_j\,, \quad (35)$$

$$\Delta_j^1 = I\,, \qquad \Delta_j^2 = \left(\Omega^2 + \widehat{\omega}_j^2 I\right), \quad (36)$$

where $s \in \{1, 2\}$, $j \in \{1, 2, \ldots, p\}$.

REMARK **2.** Note that in viscosity optimization based on the formula (32) we use matrices defined in (33)-(36) large number of times. Further, the dimensions of these matrices are given in terms of the rank r, i.e. the rank of the damping matrix, which is usually much smaller then the full dimension. Thus, for the given dampers positions we can calculate these matrices in advance which will improve the efficiency of the optimization process. All these matrices can be easily stored using vector, matrix or tensor structure.

As it has been emphasized in the Remark 2, the obtained formula (32) for the energy functions $\mathfrak{F}_s(v)$ allow us to perform the optimization of viscosity v very efficiently. More precisely, the optimization process starts with the diagonalization of the matrix pair (M, K), which needs to be done just once in the whole optimization process.

Further for the calculation of the function $\mathfrak{F}_s(v)$ we first calculate quantities in (33)-(36), for which we need $\mathcal{O}(r^3 p) + \mathcal{O}(n r^2 p) + \mathcal{O}(r n^2 p)$. This has to be done just once for one viscosity optimization. Then, each evaluation of function $\mathfrak{F}_s(v)$ can be calculated in additional $\mathcal{O}(r^2 p)$ floating point operations.

Altogether, for one process of viscosity optimization, if n_{iter} is a number of function evaluations in minimizing procedure, we need

$$\mathcal{O}(r^3 p) + \mathcal{O}(n r^2 p) + \mathcal{O}(r n^2 p) + n_{iter} \cdot \mathcal{O}(r^2 p) \quad (37)$$

floating point operations.

On the other hand, one optimization of viscosity using directly formula (15) requires

$$n_{iter} \cdot \mathcal{O}(n^3 p) \tag{38}$$

floating point operations.

In a practise the damping matrix has a small rank r compared to the full dimension n. That is, usually one applies a very few external dampers on a system. This means that without new explicit formula one needs $O(pn^3)$ floating point operations per iteration, as it has been shown in (38), while using the explicate formulae one needs $\mathcal{O}(nr^2p) + \mathcal{O}(rn^2p)$ plus $O(pr^2)$ floating point operations per iteration, for $r \ll n$, as it has been shown in (37). Taking all this into account it yields that the new approach using new formulas significantly accelerates the optimization process. The acceleration factor for a chosen mechanical system will be illustrated in Example 1.

Remark **3.** It is important to note that from the obtained formulae we can easily calculate explicit formulae for derivations of considered functions $\mathfrak{F}_s(v)$ which can be used for the Newton like methods. Since the Newton like methods require appropriate starting point, whereas in our case one can expect many local minima, in our applications we will use more robust optimization methods such as Brent's method [27] or Nelder-Mead method [28].

4. Numerical experiments

In this section we will present a numerical example which illustrates the advantage of the new formulae for a calculation of energies used for a damping optimization. Computations have been carried out on an Intel(R) Core(TM) i5-4430 CPU with 16GB of RAM and 8 MB cache. Results were calculated in MATLAB (Version 7.14.0.739 (R2012a) 64-bit) installed on Windows.

Example 1. As the numerical illustration of the new approach, we will consider an n-mass oscillator or oscillator ladder with two dampers and an external force that affects the first mass as it is shown in Figure 1. We use an academic model of a simple structure which describes the mechanical system of n masses connecting with $n+1$ springs representing a simple model of e.g. a bridge or a truss excited by an external force caused by earthquake.

For such a mechanical system the mathematical model is given by (1), where the mass and stiffness matrices are

$$M = \mathrm{diag}(m_1, m_2, \ldots, m_n),$$

$$K = \begin{pmatrix} k_1 + k_2 & -k_2 & & & & \\ -k_2 & k_2 + k_3 & -k_3 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -k_{n-1} & k_{n-1} + k_n & -k_n & \\ & & & -k_n & k_n + k_{n+1} \end{pmatrix},$$

with the following configuration

$$n = 1200; \quad k_j = 300, \quad \forall j; \qquad m_j = \begin{cases} 801 - j, & j = 1, \ldots, 600, \\ j - 400, & j = 601, \ldots, n. \end{cases}$$

The geometry of external damping placed between k-th and $k+1$-st and j-th and $j+1$-st mass is given by

$$D = v(e_k - e_{k+1})(e_k - e_{k+1})^T + v(e_j - e_{j+1})(e_j - e_{j+1})^T, \quad 1 \le k < j \le n. \quad (39)$$

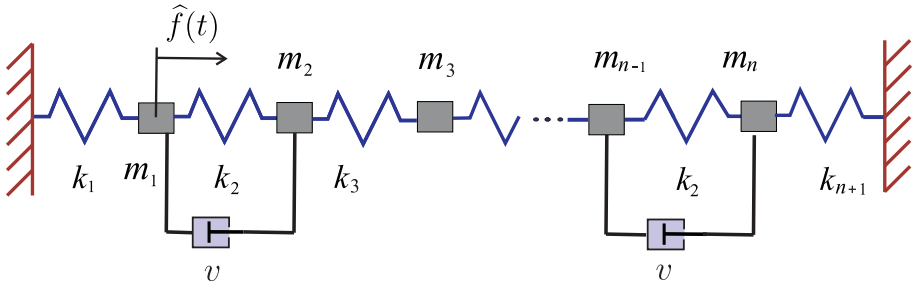

Figure 1: n-mass oscillator with two dampers with external force

For example, for the damping presented in Figure 1, one has $k = 1$ and $j = n - 1$ in formula (39). But we introduce a general indexes k and j, since we would like to examine behavior of the system for a different dampers positions.

For the external force, we use the data from Loma Prieta Earthquake given in Matlab. Data represents earthquake from the October 17, 1989 Loma Prieta in the Santa Cruz Mountains. We took 1000 data points from east-west acceleration which we have shown in Figure 2.

For our purpose, we first interpolated the data by a piecewise polynomial form of the cubic spline interpolant using the MATLAB function `spline`, then, this interpolant we approximated by the first 200 terms of Fourier series. Thereby we get the data of the force acting on the first mass, which we shown in Figure 2 together with the corresponding Fourier approximation.

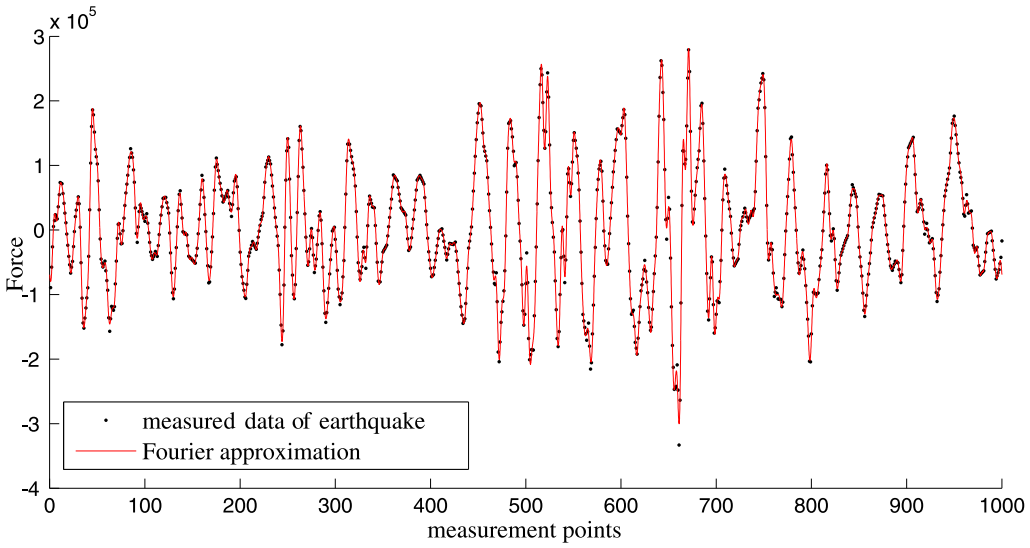

Figure 2: Date of Loma Prieta Earthquake and corresponding Fourier approximation

Without the new formulae the considered optimization process is extremely demanding, since it requires evaluation of formulae (15) for a large number of damping positions and different viscosities. Thus, in the following we will present the advantages of usage of the formula given in (32) which calculates the same value with significant time acceleration that allows an efficient damping optimization for various dampers positions.

Instead of performing optimization over all damping positions (complete optimization), in order to illustrate behavior of optimal viscosities and complexity of its calculation, we will illustrate the efficiency of the new formula on the equidistant mesh of damping positions from (39) where k and j vary as:

$$k = 1 : 10 : n, \quad j = k + 1 : 10 : n - 1, \tag{40}$$

which yields a 7260 different damping configurations.

For the viscosity optimization we use the Nelder-Mead algorithm [28] implemented in MATLAB's function `fminsearch`. The termination tolerance for the optimization variable (viscosity) and the function value is set to 0.0001. Since there may exist a several local minima for a given configuration of positions for each (j,k) we have performed several optimization procedures with different starting points $v_0 \in \{100, 500, 900, 1300, 1700, 2100\}$.

In Table 1 the first and second column shows configuration of positions (needed for formula (39)) that correspond to a minimal value of function $\mathfrak{F}_2(v)$ given in (32). Similarly, the fifth and sixth column determine configuration of positions that correspond to a minimal value of function $\mathfrak{F}_1(v)$. Optimal viscosity that corresponds to the configuration given in the first two columns is presented in third column and optimal viscosity that corresponds to configuration given in the fifth and sixth column is presented in the seventh column. Corresponding function values for $\mathfrak{F}_1(v)$ and $\mathfrak{F}_2(v)$ are given in fourth column and eight column, respectively.

k	j	optimal v_2 for $s=2$	$\mathfrak{F}_2(v_2)$ $(\cdot 10^{10})$	k	j	optimal v_1 for $s=1$	$\mathfrak{F}_1(v_1)$ $(\cdot 10^{11})$
21	22	1379.7	3.2422	21	1152	1532.7	2.4842
41	182	1048.7	3.3331	1151	1152	1882.9	2.5637
21	372	1053.9	3.3830	1171	1182	1299.3	2.6315
21	1152	998.02	3.3884	51	1152	1329.3	2.6413
21	182	916.30	3.4229	281	1092	1508.2	2.6440
51	1152	931.95	3.4446	51	1112	1436.4	2.6474
41	282	1046.2	3.4949	1131	1172	1371.9	2.6537
51	952	1012.4	3.5264	281	882	1637.7	2.6575
131	1112	953.65	3.5729	1111	1112	1272.9	2.6629
21	1072	1078.8	3.5771	341	1152	1601.3	2.6648

Table 1: Optimal viscosities and function values for functions $\mathfrak{F}_1(v)$ and $\mathfrak{F}_2(v)$

The Table 2 illustrates that for a given positions (k,j) the function values $\mathfrak{F}_1(v_2)$ is always larger than $\mathfrak{F}_1(v_1)$ and similarly $\mathfrak{F}_2(v_1)$ is always larger than $\mathfrak{F}_2(v_2)$ which verifies the optimizations results. Note that in Table 2 we have chosen positions close to optimum in both criteria.

Figure 3 shows 40 configurations that correspond to smallest function values, where configurations that correspond to function $\mathfrak{F}_2(v)$ are presented on left plot while configurations that correspond to function $\mathfrak{F}_1(v)$ are presented

k	j	optimal v_2 for $s=2$	$\mathfrak{F}_2(v_2)$ $(\cdot 10^{10})$	$\mathfrak{F}_2(v_1)$ $(\cdot 10^{10})$	optimal v_1 for $s=1$	$\mathfrak{F}_1(v_1)$ $(\cdot 10^{11})$	$\mathfrak{F}_1(v_2)$ $(\cdot 10^{11})$
21	1152	998.02	3.3884	3.6250	1532.7	2.4842	2.6601
51	1152	931.95	3.4446	3.6259	1329.3	2.6413	2.7899
51	1142	930.08	3.6323	3.8926	1403.2	2.7119	2.9344
21	1182	757.50	3.6824	4.2307	1208.6	2.6667	2.9676

Table 2: Function values at different viscosities

on the right plot. Recall that $\mathfrak{F}_i(v)$ for $i = 1, 2$ are defined in (32). This figure shows that there exists a several areas appropriate for posting damper and on some of them the both criteria overlap e.g. Table 1 shows that configurations with positions $(21, 1152)$ and $(51, 1152)$ are in the first 10 "optimal" positions using both criteria. On the other hand we can summarize, that in general, criteria introduced in (9) and (10) give different optimal positions and different optimal viscosities.

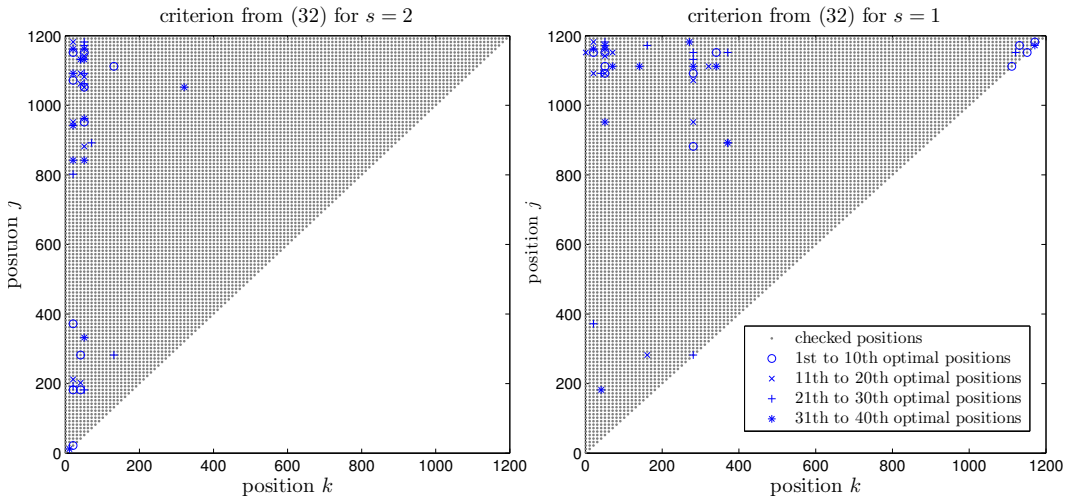

Figure 3: Optimal positions

Furthermore, we would like to emphasize that using a new formulas we have significantly accelerated the optimization process. More precisely, the ratio between the time required for the one viscosity optimization using the new formulas (given in (32)) and without the new formulas (using expression

(15)) is in average around 100. This means, that on the presented example, the new approach requires approximately 100 times less computation time, which leads to a considerably faster viscosity optimization process.

5. Conclusions

In this paper we have provided two criteria, based on the minimization of the energy functions, that allow a damping optimization in mechanical systems with external force. This optimization problem is a very demanding due to the numerous linear systems that have to be solved. For that purpose we have derived the new formulas which allow us to calculate energy functions very efficiently. The number of floating point operations needed for the one viscosity optimization (for one damping) using this new formula is given in (37) and it is equal to $\mathcal{O}(r^3p) + \mathcal{O}(nr^2p) + \mathcal{O}(rn^2p) + n_{iter} \cdot \mathcal{O}(r^2p)$. On the other hand the number of floating point operations for the same optimization without the new formula is equal to $n_{iter} \cdot \mathcal{O}(n^3p)$ (see (38)). Since, the number of dampers r is usually much smaller than dimension of the system n, this shows that new formulas accelerate optimization process considerably. Moreover, the numerical experiment presented in the paper confirms that we have accelerated viscosity optimization by factor of 100.

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