The unitary dual of p-adic $SO(5)$

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Abstract

Let F be a p-adic field of characteristic zero. We investigate the composition series of the parabolically induced representations of SO(5,F) and determine the non-cuspidal part of the unitary dual of $SO(5, F)$.

1 Introduction

Let F be a p-adic field of characteristic zero. We investigate the composition series of the parabolically induced representations of the split connected group $SO(5, F)$ and determine the set of equivalence classes of irreducible unitarizable representations of $SO(5, F)$, i.e., we determine the unitary dual of $SO(5, F)$ modulo cuspidal representations. The problem of determining the unitary dual of a reductive group is one of the most important problems in representation theory, with numerous applications in harmonic analysis and the theory of automorphic forms. Similar examples of unitary duals of some other low - rank groups can be found in [11] (for the groups Sp(4) and GSp(4)) or [8] for the simply connected split group of the type G_2.

Here is an outline of the paper. First we introduce some notation related to classical groups which is used through this paper and recall the basic results of the representation theory of these groups. In the third section we determine the unitary dual supported in the minimal parabolic subgroup. The method that we use is that of Jacquet modules ([4], [11], [15]) enhanced by intertwining operator methods ([8], [10], [12]). In the last section, which is included for the sake of completeness, we have an analysis of representations with cuspidal support in the other two parabolic subgroups. It follows completely from the results of Shahidi [12].

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2 Preliminaries

Let F be a p-adic field of characteristic zero and G the F-points of a reductive group defined over F. We denote by $R(G)$ the Grothendieck group of the category of all admissible representations of finite length of G. If σ is an admissible representation of finite length of G, then we write $s.s.(\sigma)$ for its semi-simplification in $R(G)$, but in computations we just write σ instead of $s.s.(\sigma)$.

Let $GL(n, F)$ be the general linear group of type $n \times n$ with entries in F, and I_n the identity matrix in $GL(n, F)$. For some $g \in GL(n, F)$, the transposed matrix of g is denoted by ${}^t g$ and the transposed matrix with respect to the second diagonal is denoted by ${}^\tau g$. The group $SO(2n+1, F)$ is the group of all $(2n+1) \times (2n+1)$ matrices over F with determinant equal 1, and which satisfy ${}^\tau g g = I_{2n+1}$. Let $R(S) = \bigoplus_{n \geq 0} R(SO(2n+1, F))$.

The modulus of F is denoted by $| \ |_F$. We denote by ν the positive valued character $g \mapsto |det(g)|_F$ of $GL(n, F)$. Define R as $R = \bigoplus_{n \geq 0} R(GL(n, F))$. If π is a representation of $GL(n, F)$ and $0 \leq k \leq n$, the normalized Jacquet module of π with respect to the standard parabolic subgroup whose Levi factor is $GL(k, F) \times GL(n-k, F)$ is denoted by $r_{(k)}(\pi)$. If π_1 is an admissible representation of $GL(k, F)$ and π_2 an admissible representation of $GL(n-k, F)$, we write $\pi_1 \times \pi_2$ for the representation of $GL(n, F)$ that is parabolically induced from $\pi_1 \otimes \pi_2$.

We fix a minimal parabolic subgroup P_{min} of $SO(2n+1, F)$ consisting of all upper triangular matrices in the group. A standard parabolic subgroup P of $SO(2n+1, F)$ is a parabolic subgroup containing P_{min}. Recall that an ordered partition of $m \in \mathbb{N}$ is a tuple of the form $\alpha = (n_1, n_2, \ldots, n_k)$ with $n_1, n_2, \ldots, n_k \in \mathbb{N}$ and $n_1 + n_2 + \cdots + n_k = m$. The set of all proper standard parabolic subgroups of $SO(2n+1, F)$ is in one-to-one correspondence with the set of ordered partitions of all m with $0 < m \leq n$. We describe this correspondence. If $\alpha = (n_1, n_2, \ldots, n_k)$ is an ordered partition of such an m, then $(n_1, n_2, \ldots, n_k, 2(n-m)+1, n_k, \ldots, n_2, n_1)$ is an ordered partition of $2n+1$. Define $m_1 = n_1, m_2 =$

$n_2, m_k = n_k, m_{k+1} = 2(n-m)+1, m_{k+2} = n_k, m_{k+3} = n_{k-1}, \ldots, m_{2k+1} = n_1$. Now $(m_1, m_2, \ldots, m_{2k+1})$ is an ordered partition of n. Let P_α be a parabolic subgroup of all block - upper triangular matrices p in group $SO(2n+1,F)$ such that $p = (p_{ij})_{1\leq i,j\leq 2k+1}$, p_{ij} is an $m_i \times m_j$ matrix and p_{ij} is a zero matrix if $i > j$. P_α admits a Levi decomposition $P_\alpha = M_\alpha N_\alpha$, where $M_\alpha = \{\mathrm{diag}(g_1, \ldots, g_k, h, {}^\tau g_k^{-1}, \ldots, {}^\tau g_1^{-1}) : g_i \in GL(m_i, F), h \in SO(2(n-m)+1, F)\}$ and $N_\alpha = \{p \in P_\alpha : p_{ii} = I_{m_i}\}$.

Let π_i be a representation of $GL(n_i, F)$, $1 \leq i \leq k$, and σ a representation of $SO(2(n-m)+1, F)$. Then we consider $\pi_1 \otimes \cdots \otimes \pi_k \otimes \sigma$ as a representation of M_α: $\pi_1 \otimes \cdots \otimes \pi_k \otimes \sigma(diag(g_1, \ldots, g_k, h, {}^\tau g_k^{-1}, \ldots, {}^\tau g_1^{-1})) = \pi_1(g_1) \otimes \cdots \otimes \pi_k(g_k) \otimes \sigma(h)$ and extend it trivially across N_α to the representation of P_α which we denote by the same letter. Normalized induction is written as $\pi_1 \times \cdots \times \pi_k \rtimes \sigma = Ind_{P_\alpha}^{GL(n,F)}(\pi_1 \otimes \cdots \otimes \pi_k \otimes \sigma)$. In this way we get a group homomorphism $R(M_\alpha) \to R(SO(2n+1, F))$.

If σ is a representation of $SO(2n+1, F)$, the normalized Jacquet module of σ with respect to P_α is denoted by $s_\alpha(\sigma)$. In this way we get a group homomorphism $R(SO(2n+1, F)) \to R(M_\alpha)$.

It is worth pointing out that the representations $\pi \rtimes \sigma$ and $\widetilde{\pi} \rtimes \sigma$ have the same composition factors and $\widetilde{\pi \rtimes \sigma} \simeq \widetilde{\pi} \rtimes \widetilde{\sigma}$ (where $\widetilde{\ }$ denotes contragredient).

For each irreducible essentially square integrable representation δ of $GL(n, F)$ there is an $e(\delta) \in \mathbb{R}$ such that $\delta = \nu^{e(\delta)}\delta^u$, where δ^u is unitarizable. We use the letter D to denote the set of equivalence classes of all irreducible essentially square integrable representations of $GL(n, F)$, $n \geq 1$. Let $D_+ = \{\delta \in D : e(\delta) > 0\}$. Further, let $\delta_1, \ldots, \delta_k \in D_+$ such that $e(\delta_1) \geq e(\delta_2) \geq \cdots \geq e(\delta_k)$ and σ an irreducible tempered representation of $SO(2n+1, F)$, $n \in \mathbb{N}$. Then the representation $\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \sigma$ has a unique irreducible quotient, which we denote by $L(\delta_1, \delta_2, \ldots, \delta_k, \sigma)$. Every irreducible representation π of $SO(2n+1, F)$, $n \in \mathbb{N}$, is isomorphic to some $L(\delta_1, \delta_2, \ldots, \delta_k, \sigma)$.

Let π be an admissible irreducible representation of $SO(2n+1, F)$ and let P_α be any standard parabolic subgroup minimal with respect to the property that $s_\alpha(\pi) \neq 0$. Write $\alpha = (n_1, \ldots, n_k)$, where α is partition of $m \leq n$. Let σ be any irreducible subquotient of $s_\alpha(\pi)$. Then we write $\sigma = \rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_k \otimes \rho$.

If all of the following inequalities

$$\begin{aligned} n_1 e(\rho_1) &> 0, \\ n_1 e(\rho_1) + n_2 e(\rho_2) &> 0, \\ &\vdots \\ n_1 e(\rho_1) + n_2 e(\rho_2) + \cdots + n_k e(\rho_k) &> 0 \end{aligned}$$

hold for every α and σ as above, then π is a square integrable representation. Also, if π is a square integrable representation, then all of given inequalities hold for any α and σ as above. The criterion for tempered representations is given by replacing every inequality above with $\geq$.

Since $SO(2n+1, F)$ is a connected reductive group, let $f : Spin(2n+1, \overline{F}) \twoheadrightarrow SO(2n+1, \overline{F})$ be the central isogeny, where $Spin(2n+1, F)$ is the simply - connected double covering of $SO(2n+1, F)$ as algebraic groups (for details see [13]). In the exact sequence

$$1 \to \{\pm 1\} \hookrightarrow Spin(2n+1, F) \xrightarrow{f} SO(2n+1, F) \xrightarrow{\delta} F^\times/(F^\times)^2$$

the homomorphism δ is called spinor norm. Let T be the maximal torus of $SO(2n+1, F)$, $T \simeq \{(x_1, \ldots, x_n) : x_1, \ldots, x_n \in F^\times\}$ and $\delta|_T(x_1, \ldots, x_n) = (x_1 \cdots x_n)(F^\times)^2$. This implies that $\delta|_T$ is an epimorphism and gives its description on T. For $n = 1$, f gives the isomorphism $SO(3, F) \simeq PGL(2, F)$. The spinor norm δ implies that every character of $F^\times/(F^\times)^2$ (i.e., every quadratic character) can also be viewed as a character of $SO(2n+1, F)$. So, for instance, if ζ is a quadratic character and $\alpha_1, \alpha_2 \in \mathbb{R}$, $\nu^{\alpha_1}\zeta \times \nu^{\alpha_2}\zeta \rtimes 1 \simeq \zeta(\nu^{\alpha_1} \times \nu^{\alpha_2} \rtimes 1)$.

Here and subsequently, St_G and 1_G denote the Steinberg and the trivial representation of some reductive group G.

Proposition 2.1 *([5], Proposition 3.1) Let χ, χ_1, χ_2 and ζ be characters of $F^\times$, where $\zeta^2 = 1_{F^\times}$ (i.e., where ζ is a quadratic character).*

The representation $\chi_1 \times \chi_2$ of $GL(2, F)$ reduces if and only if $\chi_1 = \nu^{\pm 1}\chi_2$. We have: $\nu^{\frac{1}{2}}\chi \times \nu^{-\frac{1}{2}}\chi = \chi St_{GL(2)} + \chi 1_{GL(2)}$.

The representation $\chi \rtimes 1$ of $SO(3, F)$ reduces if and only if $\chi^2 = \nu^{\pm 1}$. We have: $\nu^{\frac{1}{2}}\zeta \rtimes 1 = \zeta St_{SO(3)} + \zeta 1_{SO(3)}$.

Remark *From now on, $(\widehat{F^\times})$ stands for the set of unitary characters, while $(\widetilde{F^\times})$ stands for the set of not necessarily unitary characters.*

3 REPRESENTATIONS WITH SUPPORT IN MINIMAL PARABOLIC SUBGROUP

3.1 Non - unitary dual

First we have to see when representations induced from minimal parabolic subgroup reduce. It is well known that unitary principal series for $SO(2n+1,F)$ are irreducible [6], so we investigate non-unitary principal series. The following proposition can be proved in the same way as in section 7 of [15].

Proposition 3.1 *Let $\chi_1, \chi_2 \in (\widetilde{F^\times})$. The non-unitary principal series $\chi_1 \times \chi_2 \rtimes 1$ is reducible if and only if at least one of the following holds:*
(1) $\chi_1^{\pm 1} = \nu^{\pm 1}\chi_2$,
(2) $\chi_i = \nu^{\pm\frac{1}{2}}\zeta$, *for some i, where $\zeta^2 = 1_{F^\times}$.*

In the next two propositions, we determine the admissible dual supported in the minimal parabolic subgroup.

Proposition 3.2 *Let $\chi \in (\widehat{F^\times})$, $s \in \mathbb{R}$, $s \geq 0$, $\zeta \in (\widehat{F^\times})$ such that $\zeta^2 = 1_{F^\times}$. The representations $\nu^s\chi St_{GL(2)} \rtimes 1$ and $\nu^s\chi 1_{GL(2)} \rtimes 1$ are irreducible unless $(s,\chi) = (1,\zeta_1)$ or $(s,\chi) = (0,\zeta_1)$, where $\zeta_1^2 = 1_{F^\times}$. In $R(S)$ we have $\nu^{s+\frac{1}{2}}\chi \times \nu^{s-\frac{1}{2}}\chi \rtimes 1 = \nu^s\chi St_{GL(2)} \rtimes 1 + \nu^s\chi 1_{GL(2)} \rtimes 1$. Also, if $(s,\chi) \neq (1,\zeta_1)$ and $(s,\chi) \neq (0,\zeta_1)$, then $\nu^s\chi St_{GL(2)} \rtimes 1 = L(\nu^s\chi St_{GL(2)}, 1)$ and $\nu^s\chi 1_{GL(2)} \rtimes 1$ is the Langlands quotient of $\nu^{s+\frac{1}{2}}\chi \times \nu^{s-\frac{1}{2}}\chi \rtimes 1$, i.e.,*

$$\nu^s\chi 1_{GL(2)} \rtimes 1 = \begin{cases} L(\nu^{s+\frac{1}{2}}\chi, \nu^{\frac{1}{2}-s}\chi^{-1}, 1) & \text{if } s < \frac{1}{2}, \\ L(\nu\chi, \chi \rtimes 1) & \text{if } s = \frac{1}{2}, \\ L(\nu^{s+\frac{1}{2}}\chi, \nu^{s-\frac{1}{2}}\chi, 1) & \text{if } s > \frac{1}{2}. \end{cases}$$

The representations $\nu^s\chi \rtimes \zeta St_{SO(3)}$ and $\nu^s\chi \rtimes \zeta 1_{SO(3)}$ are irreducible unless $(s,\chi) = (\frac{3}{2},\zeta)$ or $(s,\chi) = (\frac{1}{2},\zeta_2)$, where $\zeta_2^2 = 1_{F^\times}$. In $R(S)$ we have $\nu^s\chi \times \nu^{\frac{1}{2}}\zeta \rtimes 1 = \nu^s\chi \rtimes \zeta St_{SO(3)} + \nu^s\chi \rtimes \zeta 1_{SO(3)}$. Also, if $(s,\chi) \neq (\frac{3}{2},\zeta)$ and $(s,\chi) \neq (\frac{1}{2},\zeta_2)$, then $\nu^s\chi \rtimes \zeta St_{SO(3)} = L(\nu^s\chi, \zeta St_{SO(3)})$ and $\nu^s\chi \rtimes \zeta 1_{SO(3)}$ is the Langlands quotient of $\nu^s\chi \times \nu^{\frac{1}{2}}\zeta \rtimes 1$, i.e.,

$$\nu^s\chi \rtimes \zeta 1_{SO(3)} = \begin{cases} L(\nu^{\frac{1}{2}}\zeta, \chi \rtimes 1) & \text{if } s = 0, \\ L(\nu^{\frac{1}{2}}\zeta, \nu^s\chi, 1) & \text{if } 0 < s < \frac{1}{2}, \\ L(\nu^s\chi, \nu^{\frac{1}{2}}\zeta, 1) & \text{if } s \geq \frac{1}{2}. \end{cases}$$

Proposition 3.3 *Let $\zeta, \zeta_1, \zeta_2 \in (\widehat{F^\times})$ such that $\zeta^2 = \zeta_1^2 = \zeta_2^2 = 1_{F^\times}$ ($\zeta_1 \neq \zeta_2$). The representations $\zeta 1_{GL(2)} \rtimes 1$, $\zeta St_{GL(2)} \rtimes 1$, $\nu^{\frac{1}{2}}\zeta \rtimes \zeta 1_{SO(3)}$ and $\nu^{\frac{1}{2}}\zeta \rtimes \zeta St_{SO(3)}$ are reducible and $\nu^{\frac{1}{2}}\zeta \times \nu^{\frac{1}{2}}\zeta \rtimes 1$ is a representation of length 4. The representations $\zeta St_{GL(2)} \rtimes 1$ and $\nu^{\frac{1}{2}}\zeta \rtimes \zeta 1_{SO(3)}$ (respectively, $\nu^{\frac{1}{2}}\zeta \rtimes \zeta St_{SO(3)}$) have exactly one irreducible subquotient in common. That subquotient is tempered, and is denoted by τ_1 (respectively, τ_2). In $R(S)$ we have:*

$\zeta 1_{GL(2)} \rtimes 1 = L(\nu^{\frac{1}{2}}\zeta, \nu^{\frac{1}{2}}\zeta, 1) + L(\nu^{\frac{1}{2}}\zeta, \zeta St_{SO(3)})$,
$\zeta St_{GL(2)} \rtimes 1 = \tau_1 + \tau_2$,
$\nu^{\frac{1}{2}}\zeta \rtimes \zeta 1_{SO(3)} = L(\nu^{\frac{1}{2}}\zeta, \nu^{\frac{1}{2}}\zeta, 1) + \tau_1$,
$\nu^{\frac{1}{2}}\zeta \rtimes \zeta St_{SO(3)} = L(\nu^{\frac{1}{2}}\zeta, \zeta St_{SO(3)}) + \tau_2$.

The representations $\nu^{\frac{3}{2}}\zeta \rtimes \zeta 1_{SO(3)}$, $\nu^{\frac{3}{2}}\zeta \rtimes \zeta St_{SO(3)}$, $\nu\zeta 1_{GL(2)} \rtimes 1$ and $\nu\zeta St_{GL(2)} \rtimes 1$ are reducible and $\nu^{\frac{3}{2}}\zeta \times \nu^{\frac{1}{2}}\zeta \rtimes 1$ is a representation of length 4. In $R(S)$ we have:

$\nu^{\frac{3}{2}}\zeta \rtimes \zeta 1_{SO(3)} = \zeta 1_{SO(5)} + L(\nu\zeta St_{GL(2)}, 1)$,
$\nu^{\frac{3}{2}}\zeta \rtimes \zeta St_{SO(3)} = \zeta St_{SO(5)} + L(\nu^{\frac{3}{2}}\zeta, \zeta St_{SO(3)})$,
$\nu\zeta 1_{GL(2)} \rtimes 1 = \zeta 1_{SO(5)} + L(\nu^{\frac{3}{2}}\zeta, \zeta St_{SO(3)})$,
$\nu\zeta St_{GL(2)} \rtimes 1 = \zeta St_{SO(5)} + L(\nu\zeta St_{GL(2)}, 1)$.

The representations $\nu^{\frac{1}{2}}\zeta_2 \rtimes \zeta_1 1_{SO(3)}$, $\nu^{\frac{1}{2}}\zeta_2 \rtimes \zeta_1 St_{SO(3)}$, $\nu^{\frac{1}{2}}\zeta_1 \rtimes \zeta_2 1_{SO(3)}$ and $\nu^{\frac{1}{2}}\zeta_1 \rtimes \zeta_2 St_{SO(3)}$ are reducible and $\nu^{\frac{1}{2}}\zeta_1 \times \nu^{\frac{1}{2}}\zeta_2 \rtimes 1$ is a representation of length 4. The representations $\nu^{\frac{1}{2}}\zeta_1 \rtimes \zeta_2 St_{SO(3)}$ and $\nu^{\frac{1}{2}}\zeta_2 \rtimes \zeta_1 St_{SO(3)}$ have exactly one irreducible subquotient in common. That subquotient is square-integrable, we denote it by σ. In $R(S)$ we have:

$\nu^{\frac{1}{2}}\zeta_2 \rtimes \zeta_1 1_{SO(3)} = L(\nu^{\frac{1}{2}}\zeta_1, \zeta_2 St_{SO(3)}) + L(\nu^{\frac{1}{2}}\zeta_1, \nu^{\frac{1}{2}}\zeta_2, 1)$,
$\nu^{\frac{1}{2}}\zeta_2 \rtimes \zeta_1 St_{SO(3)} = L(\nu^{\frac{1}{2}}\zeta_2, \zeta_1 St_{SO(3)}) + \sigma$,
$\nu^{\frac{1}{2}}\zeta_1 \rtimes \zeta_2 1_{SO(3)} = L(\nu^{\frac{1}{2}}\zeta_2, \zeta_1 St_{SO(3)}) + L(\nu^{\frac{1}{2}}\zeta_1, \nu^{\frac{1}{2}}\zeta_2, 1)$,
$\nu^{\frac{1}{2}}\zeta_1 \rtimes \zeta_2 St_{SO(3)} = L(\nu^{\frac{1}{2}}\zeta_1, \zeta_2 St_{SO(3)}) + \sigma$.

This proposition is proved by Jacquet module methods, following the approach of Sally - Tadić [11] and using some results of Jantzen [5] and Muić [9]. In particular, the induced representations $\nu^s\chi 1_{GL(2)} \rtimes 1$ and $\nu^s\chi \rtimes 1_{SO(3)}$ are decomposed in [5].

3.2 Unitary dual

We write $\overline{\pi}$ for the complex conjugate representation of a representation π. An irreducible smooth representation π is called Hermitian if $\pi = \overline{\tilde{\pi}}$.

In the same way as in [15], we get:

$L((\delta_1,\dots,\delta_n,\sigma))^{\sim} = L((\delta_1,\dots,\delta_n,\widetilde{\sigma}))$ and

$L((\delta_1,\dots,\delta_n,\sigma))^{-} = L((\overline{\delta_1},\dots,\overline{\delta_n},\overline{\sigma}))$

The proof of the next proposition is straightforward.

Proposition 3.4 *Let $\chi,\zeta,\zeta_1,\zeta_2 \in (\widehat{F^\times})$ such that $\zeta^2=\zeta_i^2=1_{F^\times}$, $i=1,2$ ($\zeta_1\neq\zeta_2$). Let $\alpha,\alpha_1,\alpha_2>0$. The following families of the representations are Hermitian and they exhaust all irreducible Hermitian representations of $SO(5,F)$ which are supported in the minimal parabolic subgroup.*

(1) *irreducible tempered representations supported in the minimal parabolic subgroup,*

(2) $L(\nu^\alpha\chi,\nu^\alpha\chi^{-1},1)$, $\chi^2\neq 1_{F^\times}$,

(3) $L(\nu^\alpha\zeta,\chi\rtimes 1)$, $\chi^2\neq 1_{F^\times}$,

(4) $L(\nu^{\alpha_1}\zeta_1,\nu^{\alpha_2}\zeta_2,1)$, $L(\nu^{\alpha_1}\zeta,\nu^{\alpha_2}\zeta,1)$,

(5) $L(\nu^\alpha\zeta_1,\zeta_2\rtimes 1)$, $L(\nu^\alpha\zeta,\zeta\rtimes 1)$,

(6) $L(\nu^\alpha\zeta_1,\zeta_2 St_{SO(3)})$, $L(\nu^\alpha\zeta,\zeta St_{SO(3)})$,

(7) $L(\nu^\alpha\zeta St_{GL(2)},1)$.

We take a moment to recall some well-known complementary series. For $GL(2,F)$, the complementary series is $\nu^\alpha\chi\times\nu^{-\alpha}\chi$, $\chi\in(\widehat{F^\times})$, $0<\alpha<\frac{1}{2}$. For $SO(3,F)$, the complementary series is $\nu^\alpha\zeta\rtimes 1$, $\zeta\in(\widehat{F^\times})$ with $\zeta^2=1_{F^\times}$, $0<\alpha<\frac{1}{2}$.

Theorem 3.5 *Let $\chi,\zeta,\zeta_1,\zeta_2\in(\widehat{F^\times})$ such that $\zeta^2=\zeta_i^2=1_{F^\times}$, $i=1,2$ ($\zeta_1\neq\zeta_2$). The following families of representations are unitary and they exhaust all irreducible unitary representations of $SO(5,F)$ which are supported in the minimal parabolic subgroup.*

(1) *irreducible tempered representations supported in the minimal parabolic subgroup,*

(2) $L(\nu^\alpha\chi,\nu^\alpha\chi^{-1},1)$, $0<\alpha\leq\frac{1}{2}$ $\chi^2\neq 1_{F^\times}$,

(3) $L(\nu^\alpha\zeta,\chi\rtimes 1)$, $0<\alpha\leq\frac{1}{2}$,

(4) $L(\nu^{\alpha_1}\zeta_1,\nu^{\alpha_2}\zeta_2,1)$, $L(\nu^{\alpha_1}\zeta,\nu^{\alpha_2}\zeta,1)$, $\alpha_2\leq\alpha_1$, $\alpha_1\leq\frac{1}{2}$,

(5) $L(\nu^\alpha\zeta_1,\zeta_2 St_{SO(3)})$, $L(\nu^\alpha\zeta,\zeta St_{SO(3)})$, $\alpha\leq\frac{1}{2}$,

(6) $L(\nu^{\frac{3}{2}}\zeta,\nu^{\frac{1}{2}}\zeta,1)=\zeta 1_{SO(5)}$.

Proof: The representations in groups (1) and (6) are obviously unitarizable.

The unitarizability of the representations in (2) and (3) follows since the representations in these two families are unitarily induced from either representations in the complementary series or representations in the ends of the complementary series (noting that representations which occur in ends of complementary series are unitarizable by the results of Miličić [7], see also [14]):

Since $\nu^{\alpha}\chi \times \nu^{-\alpha}\chi \rtimes 1$ and $\chi \times \nu^{\alpha}\zeta \rtimes 1$ are irreducible for $0 < \alpha < \frac{1}{2}$, we have $L(\nu^{\alpha}\chi, \nu^{\alpha}\chi^{-1}, 1) \simeq \nu^{\alpha}\chi \times \nu^{-\alpha}\chi \rtimes 1$, $0 < \alpha < \frac{1}{2}$ and $L(\nu^{\alpha}\zeta, \chi \rtimes 1) \simeq \chi \times \nu^{\alpha}\zeta \rtimes 1$, $0 < \alpha < \frac{1}{2}$. For $\alpha > \frac{1}{2}$ the representations $\nu^{\alpha}\chi \times \nu^{\alpha}\chi^{-1} \rtimes 1$ are $\nu^{\alpha}\zeta \times \chi \rtimes 1$ are irreducible and never unitarizable because their matrix coefficients, which can also be estimated from Jacquet modules, are unbounded for $\alpha > \frac{1}{2}$. For $\alpha = \frac{1}{2}$, we get unitarizability of all irreducible subquotients since the corresponding representations are in the ends of complementary series, and those are $L(\nu^{\frac{1}{2}}\chi, \nu^{\frac{1}{2}}\chi^{-1}, 1) = \chi 1_{GL(2)} \rtimes 1$ and $L(\nu^{\frac{1}{2}}\zeta, \chi \rtimes 1) = \chi \rtimes \zeta 1_{SO(3)}$.

Now we turn our attention to cases (4) and (5). We investigate the two pieces separately and consider only the parts where $\alpha_1 \geq \alpha_2$ (if $\alpha_2 > \alpha_1$ then we can switch them).

(i) Let $\zeta_1 \neq \zeta_2$.

For $\alpha_1 \neq \frac{1}{2}$ and $\alpha_2 \neq \frac{1}{2}$ the representation $\nu^{\alpha_1}\zeta_1 \times \nu^{\alpha_2}\zeta_2 \rtimes 1$ is irreducible. For $\alpha_1 = \alpha_2 = 0$ we have the unitarizable representation $\zeta_1 \times \zeta_2 \rtimes 1$. Now we see, in a standard way, what happens in a neighborhood of zero.

Let $w \in W$ be the longest element of the Weyl group of $SO(5, F)$. We denote by $A(\alpha_1, \alpha_2, \zeta_1, \zeta_2, w)$ the standard long intertwining operator, $A(\alpha_1, \alpha_2, \zeta_1, \zeta_2, w) : \nu^{\alpha_1}\zeta_1 \times \nu^{\alpha_2}\zeta_2 \rtimes 1 \rightarrow \nu^{-\alpha_1}\zeta_1 \times \nu^{-\alpha_2}\zeta_2 \rtimes 1$. The image of A is isomorphic to $L(\nu^{\alpha_1}\zeta_1, \nu^{\alpha_2}\zeta_2, 1)$ and $L(\nu^{\alpha_1}\zeta_1, \nu^{\alpha_2}\zeta_2, 1)$ is hermitian. Next, we realize the 2-parameter family of hermitian representations $X_{\alpha_1,\alpha_2} = \nu^{\alpha_1}\zeta_1 \times \nu^{\alpha_2}\zeta_2 \rtimes 1$, $\alpha_1 \geq \alpha_2 > 0$ on the same space X, compact image of induced representations. From this, we get a continuous family of hermitian forms:

$$\langle f, g\rangle_{\alpha_1,\alpha_2} = \int_{SO_5(\mathfrak{o})} A(\alpha_1, \alpha_2, \zeta_1, \zeta_2, w) f_{\alpha_1,\alpha_2}(k)\overline{g_{\alpha_1,\alpha_2}(k)}dk; f, g \in X$$

where f_{α_1,α_2} and g_{α_1,α_2} are the sections of f and g while $\mathfrak{o}$ is the ring of the integers in F ($SO_5(\mathfrak{o})$ can be replaced with any good compact subgroup of $SO_5(F)$). All forms $\langle \cdot, \cdot\rangle_{\alpha_1,\alpha_2}$ are X_{α_1,α_2}-invariant and non-degenerate on $L(\nu^{\alpha_1}\zeta_1 \times \nu^{\alpha_2}\zeta_2 \rtimes 1)$. Fix α_1, α_2 and form the 1-parameter family of hermitian representations $X_t = X_{t\alpha_1,t\alpha_2}$, $t \geq 0$; then choose a polynomial

$P(t)$ with real coefficients such that $A(t) = P(t)A(t\alpha_1, t\alpha_2, \zeta_1, \zeta_2, w)$ is holomorphic and non-zero for $t \geq 0$. In this way we get continuous family of hermitian forms

$$\langle f, g\rangle_t = \int_{SO_5(\mathfrak{o})} A(t) f_t(k)\overline{g_t(k)} dk; f, g \in X.$$

We may, and we do, assume that X_0 is positive definite. This implies that the form $\langle \cdot, \cdot\rangle_t$ is positive definite in some neighborhood of $t = 0$, until X_t becomes reducible.

We conclude:

- for $\alpha_1 < \frac{1}{2}$, the representation $\nu^{\alpha_1}\zeta_1 \times \nu^{\alpha_2}\zeta_2 \rtimes 1$ is irreducible and unitarizable, $\nu^{\alpha_1}\zeta_1 \times \nu^{\alpha_2}\zeta_2 \rtimes 1 = L(\nu^{\alpha_1}\zeta_1, \nu^{\alpha_2}\zeta_2, 1)$
- for $\alpha_1 > \frac{1}{2}$ and $\alpha_2 \neq \frac{1}{2}$, the representation $\nu^{\alpha_1}\zeta_1 \times \nu^{\alpha_2}\zeta_2 \rtimes 1$ is irreducible and never unitarizable
- for $\alpha_1 = \frac{1}{2}$, all irreducible subquotients are unitarizable, $\nu^{\alpha_1}\zeta_1 \times \nu^{\alpha_2}\zeta_2 \rtimes 1 = \nu^{\alpha_2}\zeta_2 \rtimes \zeta_1 St_{SO(3)} + \nu^{\alpha_2}\zeta_2 \rtimes \zeta_1 1_{SO(3)}$, i.e., $L(\nu^{\alpha}\zeta_2, \zeta_1 St_{SO(3)})$ and $L(\nu^{\frac{1}{2}}\zeta_1, \nu^{\alpha}\zeta_2, 1)$ are unitarizable for $\alpha \leq \frac{1}{2}$
- for $\alpha_1 > \frac{1}{2}$ and $\alpha_2 = \frac{1}{2}$, the representations $\nu^{\alpha_1}\zeta_1 \rtimes \zeta_2 St_{SO(3)} = L(\nu^{\alpha_1}\zeta_1, \zeta_2 St_{SO(3)})$ and $\nu^{\alpha_1}\zeta_1 \rtimes \zeta_2 1_{SO(3)} = L(\nu^{\alpha_1}\zeta_1, \nu^{\frac{1}{2}}\zeta_2, 1)$ are irreducible and never unitarizable.

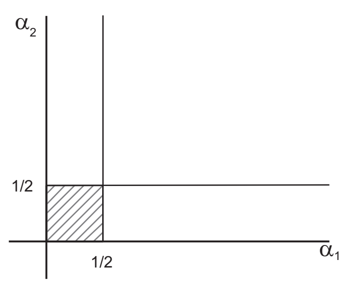

Figure 1

(ii) Let $\zeta_1 = \zeta_2 = \zeta$.

The representation $\nu^{\alpha_1}\zeta \times \nu^{\alpha_2}\zeta \rtimes 1$ reduces for $\alpha_1 = \frac{1}{2}$, $\alpha_2 = \frac{1}{2}$, $\alpha_1 - \alpha_2 = 1$, $\alpha_2 - \alpha_1 = 1$, $\alpha_1 + \alpha_2 = 1$.

We investigate the unitarizability of representations belonging to areas on Figure 2. Unitarizability of representations which belong to I follows in the same way as in (i). The point $T(\frac{3}{2}, \frac{1}{2})$ corresponds to $\nu^{\frac{3}{2}}\zeta \times \nu^{\frac{1}{2}}\zeta \rtimes 1$, which we know is isomorphic to $\zeta(\nu^{\frac{3}{2}} \times \nu^{\frac{1}{2}} \rtimes 1)$.

$\nu^{\frac{3}{2}}\zeta \times \nu^{\frac{1}{2}}\zeta \rtimes 1 = \zeta St_{SO(5)} + L(\nu^{\frac{3}{2}}\zeta, \zeta St_{SO(3)}) + \zeta 1_{SO(5)} + L(\nu\zeta St_{GL(2)}, 1)$,
$\nu\zeta St_{GL(2)} \rtimes 1 = \zeta St_{SO(5)} + L(\nu\zeta St_{GL(2)}, 1)$,
$\nu\zeta 1_{GL(2)} \rtimes 1 = \zeta 1_{SO(5)} + L(\nu^{\frac{3}{2}}\zeta, \zeta St_{SO(3)})$.

The results of Casselman [3] imply that exactly one of the irreducible subquotients of the representation $\nu\zeta St_{GL(2)} \rtimes 1$ (resp., $\nu\zeta 1_{GL(2)} \rtimes 1$) is unitarizable. More precisely: $\zeta St_{SO(5)}$ and $\zeta 1_{SO(5)}$ are unitarizable, while $L(\nu\zeta St_{GL(2)}, 1)$ and $L(\nu^{\frac{3}{2}}\zeta, \zeta St_{SO(5)})$ are non-unitarizable. Because there are some non-unitarizable irreducible subquotients of $\nu^{\frac{3}{2}}\zeta \times \nu^{\frac{1}{2}}\zeta \rtimes 1$, it follows that there are no unitarizable representations in II, III, IV and V (the point T is contained in closures of all of this regions). Applying the same argument, we get that there are no unitarizable representations on the line $\alpha_2 = \frac{1}{2}$, from $\alpha_1 = \frac{1}{2}$ to $\alpha_1 = \frac{3}{2}$.

For the final part, we take a look at representations on the line $\alpha_1 - \alpha_2 = 1$, from the point $(1, 0)$ to the point T. At the point $(1, 0)$ we have the representation $\nu\zeta \times \zeta \rtimes 1$ so Proposition 3.2 implies that we have two continuous families of irreducible representations: $\nu^\alpha\zeta St_{GL(2)} \rtimes 1$ and $\nu^\alpha\zeta 1_{GL(2)} \rtimes 1$, $\frac{1}{2} \leq \alpha < 1$. If there is some $\frac{1}{2} \leq \beta < 1$ such that the representation $\nu^\beta\zeta St_{GL(2)} \rtimes 1$ is unitarizable, then all the representations $\nu^\alpha\zeta St_{GL(2)} \rtimes 1$ are unitarizable, for all $\frac{1}{2} \leq \alpha < 1$. If all these representations were unitarizable, then also all the irreducible subquotients of $\nu\zeta St_{GL(2)} \rtimes 1$ would have to be unitarizable, but that contradicts the non-unitarizability of the representation $L(\nu\zeta St_{GL(2)}, 1)$. This implies that all the representations $\nu^\alpha\zeta St_{GL(2)} \rtimes 1$ are non-unitarizable for $\frac{1}{2} \leq \alpha < 1$. In the same way we also conclude that all the representations $\nu^\alpha\zeta 1_{GL(2)} \rtimes 1$ are non-unitarizable for $\frac{1}{2} \leq \alpha < 1$. Now we directly get that the irreducible subquotients of $\nu\zeta \times \zeta \rtimes 1$ are non - unitarizable and there are no unitarizable representations in VI. (see Figure 2) ■

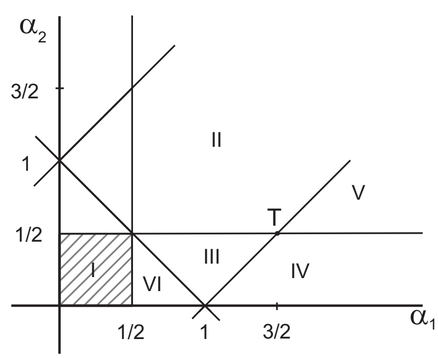

Figure 2

4 REPRESENTATIONS WITH SUPPORT IN OTHER PARABOLIC SUBGROUPS

First we consider the case of the representations which have cuspidal support in $P_{(2)}$. The representations listed in the next proposition exhaust all the irreducible unitary representations of $SO(5, F)$ which are supported in $P_{(2)}$.

Proposition 4.1 *Let ρ be an irreducible unitarizable supercuspidal representation of $GL(2, F)$. There is at most one $s \geq 0$ such that $\nu^s \rho \rtimes 1$ reduces.*

(i) If $\rho \neq \widetilde{\rho}$ then $\rho \rtimes 1$ is irreducible and unitarizable. Also, the representations $\nu^s \rho \rtimes 1$, $s > 0$ are irreducible and never unitarizable.

(ii) If $\rho = \widetilde{\rho}$ and $\rho \rtimes 1$ reduces (that is the case when $\omega_\rho = 1$, where ω_ρ denotes the central character of ρ), all of the representations $\nu^s \rho \rtimes 1$, $s > 0$ are non - unitarizable.

(iii) If $\rho = \widetilde{\rho}$ and $\rho \rtimes 1$ is irreducible (that is the case when $\omega_\rho \neq 1$), then the unique $s > 0$ such that the representation $\nu^s \rho \rtimes 1$ reduces equals $\frac{1}{2}$. For $0 < s < \frac{1}{2}$ the representation $\nu^s \rho \rtimes 1 \simeq L(\nu^s \rho, 1)$ is unitarizable, while it is never unitarizable for $s > \frac{1}{2}$. All irreducible subquotients of $\nu^{\frac{1}{2}} \rho \rtimes 1$ are unitarizable.

The unique $s > 0$ such that representation $\nu^s \rho \rtimes 1$ reduces is obtained by determining the poles of the Plancherel measure, which in this case coincide with the poles of

$$\frac{L(1 - 2s, \rho, Sym^2 \rho_2) L(1 + 2s, \rho, Sym^2 \rho_2)}{L(2s, \rho, Sym^2 \rho_2) L(-2s, \rho, Sym^2 \rho_2)}$$

(this quotient is equal to the Plancherel measure $\mu(s, \rho)$ up to a monomial in q^s).

Now we consider the case of the representations which have cuspidal support in $P_{(1)}$. The representations listed in the next proposition exhaust all the irreducible unitary representations of $SO(5, F)$ which are supported in $P_{(1)}$.

Proposition 4.2 *Let $\chi \in (\widehat{F^\times})$ and let σ be an irreducible unitarizable supercuspidal representation of $SO(3, F) \simeq PGL(2, F)$ (observe that σ is generic). There is at most one $s \geq 0$ such that $\nu^s \chi \rtimes \sigma$ reduces.*

(i) If $\chi \neq \chi^{-1}$, $\chi \rtimes \sigma$ is irreducible and unitarizable, while $\nu^s \chi \rtimes \sigma$ is irreducible and non - unitarizable for $s > 0$.

(ii) If $\chi = \chi^{-1}$, then $\nu^s \chi \rtimes \sigma$ reduces only for $s = \frac{1}{2}$. For $0 < s < \frac{1}{2}$ the representation $\nu^s \chi \rtimes \sigma \simeq L(\nu^s \chi, \sigma)$ is unitarizable, while it is never unitarizable for $s > \frac{1}{2}$. All irreducible subquotients of $\nu^{\frac{1}{2}} \chi \rtimes \sigma$ are unitarizable.

References

[1] A.-M. Aubert, *Dualité dans le groupe de Grothendieck de la catégorie des représentations lisses de longueur finie d'un groupe réductif p-adique*, Trans. Amer. Math. Soc., 347 (1995), pp. 2179–2189.

[2] ——, *Erratum: "Duality in the Grothendieck group of the category of finite-length smooth representations of a p-adic reductive group" [Trans. Amer. Math. Soc.* ***347*** *(1995), no. 6, 2179–2189; MR1285969 (95i:22025)]*, Trans. Amer. Math. Soc., 348 (1996), pp. 4687–4690.

[3] W. Casselman, *A new nonunitarity argument for p-adic representations*, J. Fac. Sci. Univ. Tokyo Sect. IA Math., 28 (1981), pp. 907–928 (1982).

[4] M. Hanzer and G. Muić, *On an algebraic approach to the Zelevinsky classification for classical p-adic groups*, J. Algebra, 320 (2008), pp. 3206–3231.

[5] C. Jantzen, *Degenerate principal series for orthogonal groups*, J. Reine Angew. Math., 441 (1993), pp. 61–98.

[6] C. D. Keys, *On the decomposition of reducible principal series representations of p-adic Chevalley groups*, Pacific J. Math., 101 (1982), pp. 351–388.

[7] D. Miličić, *On C^*-algebras with bounded trace*, Glasnik Mat. Ser. III, 8(28) (1973), pp. 7–22.

[8] G. Muić, *The unitary dual of p-adic G_2*, Duke Math. J., 90 (1997), pp. 465–493.

[9] ——, *Composition series of generalized principal series; the case of strongly positive discrete series*, Israel J. Math., 140 (2004), pp. 157–202.

[10] ——, *A geometric construction of intertwining operators for reductive p-adic groups*, Manuscripta Math., 125 (2008), pp. 241–272.

[11] P. J. Sally, Jr. and M. Tadić, *Induced representations and classifications for GSp(2,F) and Sp(2,F)*, Mém. Soc. Math. France (N.S.), (1993), pp. 75–133.

[12] F. Shahidi, *A proof of Langlands' conjecture on Plancherel measures; complementary series for p-adic groups*, Ann. of Math. (2), 132 (1990), pp. 273–330.

[13] T. A. Springer, *Reductive groups*, in Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 3–27.

[14] M. Tadić, *An external approach to unitary representations*, Bull. Amer. Math. Soc. (N.S.), 28 (1993), pp. 215–252.

[15] ——, *Representations of p-adic symplectic groups*, Compositio Math., 90 (1994), pp. 123–181.

[16] ——, *On reducibility of parabolic induction*, Israel J. Math., 107 (1998), pp. 29–91.

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