# Strongly positive subquotients in a class of induced representations of classical $p$-adic groups

Ivan Matić

*Department of Mathematics, University of Osijek, Trg Ljudevita Gaja 6, Osijek, Croatia*

**Abstract**

We determine under which conditions the induced representation of the form $\delta_1 \times \delta_2 \rtimes \sigma$, where $\delta_1, \delta_2$ are irreducible essentially square integrable representations of a general linear group and $\sigma$ is a discrete series representation of classical $p$-adic group, contains an irreducible strongly positive subquotient.

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## 1. Introduction

According to the Mœglin-Tadić classification, which now holds unconditionally due to recent work of Arthur [1] and Mœglin [14], discrete series of classical groups over $p$-adic fields arise in the natural and inductive way from strongly positive representations. Such representations, which serve as basic building blocks in construction of square-integrable and tempered representations, have been introduced in [13], as a class of discrete series corresponding to the so-called admissible triples of alternated type. An algebraic classification of strongly positive discrete series of metaplectic groups, which also holds in the classical group case, is given in [6]. Some further properties of strongly positive discrete series have been studied in [7] and [9]. We note that the class of strongly positive discrete series contains some prominent parts of the unitary dual, such as the generalized Steinberg representations, regular discrete series representations and discrete series representations obtained by theta-correspondence from the supercuspidal ones (details can be

*Email address:* imatic@mathos.hr (Ivan Matić)

seen in [17] for the symplectic - even orthogonal dual pair and in [7, 8] for the metaplectic - odd orthogonal dual pair).

In this paper we are interested in determining when the induced representation of the form $\delta_1 \times \delta_2 \rtimes \sigma$ contains an irreducible strongly positive subquotient, where $\delta_1, \delta_2$ denote irreducible essentially square integrable representations of the general linear group and $\sigma$ stands for a discrete series representation of the classical $p$-adic group. This problem presents an initial step towards the determination of composition series of the induced representation $\delta_1 \times \delta_2 \rtimes \sigma$, for $\delta_1$, $\delta_2$ and $\sigma$ as above, which should have important applications in the classification of the unitary dual of classical $p$-adic group. Also, it can be viewed as an extension of the description of strongly positive subquotients of generalized principal series, which is implicitly given in [16].

If $\delta_1 \times \delta_2 \rtimes \sigma$ contains a strongly positive subquotient, then $\sigma$ is also strongly positive and there is some irreducible subquotient $\pi$ of $\delta_2 \rtimes \sigma$ (which is either strongly positive or non-tempered) such that $\delta_1 \rtimes \pi$ contains a strongly positive representation. For strongly positive $\pi$, complete composition series of generalized principal series $\delta \rtimes \pi$ have been described by Muić ([16]), so our aim is to obtain necessary and sufficient conditions under which the induced representation $\delta_1 \rtimes L(\delta_2 \rtimes \sigma)$ contains an irreducible strongly positive subquotient, where $L(\delta_2 \rtimes \sigma)$ stands for the Langlands quotient of $\delta_2 \rtimes \sigma$.

To obtain the necessary conditions, we use the Jacquet modules method and description of Jacquet modules of strongly positive representations from [9] (which is also given in Section 7 of [12], using a different approach based on results of [5]). Also, we use results from [16], which enable us to obtain deeper knowledge on the structure of Jacquet modules of the non-tempered representation $L(\delta_2 \rtimes \sigma)$.

It is worth pointing out that non-supercuspidal discrete series naturally appear as subquotients of generalized principal series. In majority of cases, this enables one to realize a discrete series as an irreducible subquotient of the induced representation of the form $\delta_1 \rtimes (\delta_2 \rtimes \sigma)$, where $\delta_2 \rtimes \sigma$ is irreducible. Our results also produce interesting examples of strongly positive discrete series which appear as subquotients of the induced representations of the form $\delta_1 \rtimes L(\delta_2 \rtimes \sigma)$, where $\delta_2 \rtimes \sigma$ reduces. To prove that an induced representation of this type contains the strongly positive subquotient $\sigma_{sp}$, we find an irreducible tempered representation $\tau$ whose Jacquet module with respect to the appropriate standard parabolic subgroup contains an irreducible subquotient of the form $\delta \otimes \sigma_{sp}$. Then we write $\tau$ as a subquotient of the larger representation and, by calculating its Jacquet modules, obtain the induced representation of the form $\delta_1 \rtimes (\delta_2 \rtimes \sigma)$ containing $\sigma_{sp}$. Using results of [16], together with the detailed analysis of Jacquet modules of $\sigma_{sp}$ and $\delta_2 \rtimes \sigma$, we determine the irreducible subquotient $\pi$ of $\delta_2 \rtimes \sigma$ such that $\sigma_{sp}$ is contained in $\delta_1 \rtimes \pi$.

We will now describe the contents of the paper in more details.

In the following section we set up the notation and terminology. In the third section we prove some technical results which reduce our investigation to induced representations of the form $\delta([\nu^a\rho, \nu^b\rho]) \rtimes L(\delta([\nu^c\rho, \nu^d\rho]) \rtimes \sigma)$, with both $a$ and $c$ positive and $a \neq c$. In the same section, we also state our main results. Section 4 provides a detailed analysis of the case $a > \frac{1}{2}$ and $c > \frac{1}{2}$. Section 5 discusses the case $a = \frac{1}{2}$, while Section 6 deals with the remaining case $c = \frac{1}{2}$.

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## 2. Preliminaries

Let $F$ denote a nonarchimedean local field of characteristic different than 2. The groups we are considering are of the following form: we have a tower of symplectic or (full) orthogonal groups $G_n = G(V_n)$, which are the groups of isometries of $F$-vector spaces $V_n$ endowed with the non-degenerate form which is skew-symmetric if the tower is symplectic and symmetric otherwise. Here $n$ stands for the split rank of the group $G_n$, $n \geq 0$.

The set of standard parabolic subgroups will be fixed in a usual way, i.e., we fix a minimal $F$-parabolic subgroup in $G_n$ consisting of upper-triangular matrices in the usual matrix realization of the classical group. Then the Levi factors of standard parabolic subgroups have the form $M = GL(n_1, F) \times \cdots \times GL(n_k, F) \times G_{n'}$, where $GL(m, F)$ denotes a general linear group of rank $m$ over $F$. If $\delta_i, i = 1, 2, \ldots, k$, is a representation of $GL(n_i, F)$ and $\tau$ a representation of $G_m$, then we denote by $\delta_1 \times \cdots \times \delta_k \rtimes \tau$ the normalized parabolically induced representation of the group $G_n$, induced from the representation $\delta_1 \otimes \cdots \otimes \delta_k \otimes \tau$ of the standard parabolic subgroup with the Levi subgroup equal to $GL(n_1, F) \times \cdots \times GL(n_k, F) \times G_m$. Here $n$ equals $n_1 + n_2 + \cdots + n_k + m$.

Let $R_n$ denote the Grothendieck group of admissible representations of finite length of $GL(n, F)$ and define $R = \oplus_{n \geq 0} R_n$. Similarly, let $S_n$ stand for the Grothendieck group of admissible representations of finite length of $G_n$ and define $S = \oplus_{n \geq 0} S_n$.

We will denote by $\nu$ a composition of the determinant mapping with the normalized absolute value on $F$. Let $\rho$ denote an irreducible cuspidal unitary representation of $GL(k, F)$. By a segment of cuspidal representations, which will be denoted by $[\rho, \nu^m \rho]$, we mean the set $\{\rho, \nu\rho, \ldots, \nu^m \rho\}$. To each such segment we attach an irreducible essentially square-integrable representation $\delta([\rho, \nu^m \rho])$ of $GL(m \cdot k, F)$, which is the unique irreducible subrepresentation of $\nu^m \rho \times \cdots \times \nu\rho \times \rho$ (here we use a well known notation for the normalized parabolic induction for the general linear groups with the usual choice of the standard parabolic subgroups). For integers $x, y$, $x \leq y$, we set $[x, y] = \{z \in \mathbb{Z} : x \leq z \leq y\}$.

Let $\delta_i, i = 1, 2, \ldots, k$ be square-integrable representations of $GL(n_i, F)$, $\tau$ an irreducible tempered representation of the group $G_m$, and $s_1, s_2, \ldots, s_k$ real numbers such that $s_1 \geq s_2 \geq \cdots \geq s_k > 0$. Then the induced representation $\nu^{s_1}\delta_1 \times \cdots \times \nu^{s_k}\delta_k \rtimes \tau$ has a unique irreducible quotient, which is called the Langlands quotient and will be denoted by $L(\nu^{s_1}\delta_1 \times \cdots \times \nu^{s_k}\delta_k \rtimes \tau)$.

For representation $\sigma$ of $G_n$ and $1 \leq k \leq n$ we denote by $r_{(k)}(\sigma)$ the normalized Jacquet module of $\sigma$ with respect to the parabolic subgroup $P_{(k)}$ having Levi subgroup equal to $GL(k, F) \times G_{n-k}$. We identify $r_{(k)}(\sigma)$ with its semisimplification in $R_k \otimes S_{n-k}$ and consider

$$\mu^*(\sigma) = 1 \otimes \sigma + \sum_{k=1}^{n} r_{(k)}(\sigma) \in R \otimes S.$$

We take a moment to state the crucial structural formula for our calculations of Jacquet modules ([18]), which is a version of the Geometrical lemma by Bernstein and Zelevinsky ([2]).

**Lemma 2.1.** *Let $\rho$ be an irreducible cuspidal representation of $GL(m, F)$ and $k, l \in \mathbb{R}$ be such that $k + l \in \mathbb{Z}_{\geq 0}$. Let $\sigma$ be an admissible representation of finite length of $G_n$. Write $\mu^*(\sigma) = \sum_{\tau, \sigma'} \tau \otimes \sigma'$. Then the following holds:*

$$\mu^*(\delta([\nu^{-k}\rho, \nu^l \rho]) \rtimes \sigma) = \sum_{i=-k-1}^{l} \sum_{j=i}^{l} \sum_{\tau, \sigma'} \delta([\nu^{-i}\widetilde{\rho}, \nu^k \widetilde{\rho}]) \times \delta([\nu^{j+1}\rho, \nu^l \rho]) \times \tau \otimes$$
$$\otimes\, \delta([\nu^{i+1}\rho, \nu^j \rho]) \rtimes \sigma'.$$

*We omit* $\delta([\nu^x\rho, \nu^y\rho])$ *if* $x > y$.

An irreducible representation $\sigma \in S$ is called strongly positive if for every embedding

$$\sigma \hookrightarrow \nu^{s_1}\rho_1 \times \nu^{s_2}\rho_2 \times \cdots \times \nu^{s_k}\rho_k \rtimes \sigma_{cusp},$$

where $\rho_i \in R$, $i = 1, 2, \ldots, k$, are irreducible cuspidal unitary representations and $\sigma_{cusp} \in S$ is an irreducible cuspidal representation, we have $s_i > 0$ for $i = 1, 2, \ldots, k$.

Obviously, every strongly positive representation is square–integrable. Irreducible strongly positive representations are called strongly positive discrete series.

In this paper we will be concerned only with the non-supercuspidal strongly positive representations. Let us briefly recall an inductive description of non-supercuspidal strongly positive discrete series, which has been obtained in [6].

**Proposition 2.2.** *Suppose that* $\sigma \in S_n$ *is an irreducible strongly positive representation and let* $\rho \in R_m$ *denote an irreducible cuspidal unitary representation such that some twist of* $\rho$ *appears in the cuspidal support of* $\sigma$. *We denote by* $\sigma_{cusp}$ *the partial cuspidal support of* $\sigma$. *Then there exist unique* $a, b \in \mathbb{R}$ *such that* $a > 0, b > 0$, $b - a \in \mathbb{Z}_{\geq 0}$, *and the unique irreducible strongly positive representation* $\sigma'$ *without* $\nu^a\rho$ *in the cuspidal support, with the property that* $\sigma$ *is the unique irreducible subrepresentation of* $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma'$. *Furthermore, there is a non-negative integer* $l$ *such that* $a + l = s$, *for* $s > 0$ *such that* $\nu^s\rho \rtimes \sigma_{cusp}$ *reduces. If* $l = 0$, *there are no twists of* $\rho$ *appearing in the cuspidal support of* $\sigma'$ *and if* $l > 0$ *there exist unique* $b' > b$ *and the unique strongly positive discrete series* $\sigma''$, *which contains neither* $\nu^a\rho$ *nor* $\nu^{a+1}\rho$ *in its cuspidal support, such that* $\sigma'$ *can be written as the unique irreducible subrepresentation of* $\delta([\nu^{a+1}\rho, \nu^{b'}\rho]) \rtimes \sigma''$.

If $\nu^x\rho$ appears in the cuspidal support of $\sigma$, then the results of ([14, 15]) imply $2x \in \mathbb{Z}$ and $\rho \cong \widetilde{\rho}$, where $\widetilde{\rho}$ stands for the contragredient of $\rho$.

By the Mœglin-Tadić classification of discrete series representations for classical groups ([13, 15]), strongly positive discrete series $\sigma$ corresponds to the admissible triple of alternated type. Admissible triple corresponding to discrete series $\sigma$ is an ordered triple of the form $(\mathrm{Jord}, \sigma_{cusp}, \epsilon)$, where $\sigma_{cusp}$ is the partial cuspidal support of $\sigma$, Jord (the set Jordan blocks) is the finite set of pairs $(c, \rho)$, where $c$ is an integer of the appropriate parity, and $\rho \in R$ is an irreducible cuspidal selfcontragredient representation, while

$\epsilon$ is the function defined on the subset of $\mathrm{Jord} \times \mathrm{Jord} \cup \mathrm{Jord}$ into $\{1, -1\}$. For irreducible cuspidal selfcontragredient representation $\rho \in R$, we define $\mathrm{Jord}_\rho = \{c : (c, \rho) \in \mathrm{Jord}\}$ and for $c \in \mathrm{Jord}_\rho$ we write $c_-$ for maximum of the set $\{c' \in \mathrm{Jord}_\rho : c' < c\}$, if this set is non-empty. An admissible triple $(\mathrm{Jord}, \sigma_{cusp}, \epsilon)$ is called a triple of alternated type if for every $(c, \rho) \in \mathrm{Jord}$ such that $c_-$ is defined, we have $\epsilon((c_-, \rho), (c, \rho)) = -1$. By the definition of such triples, a strongly positive discrete series is completely determined by its partial cuspidal support and the set of Jordan blocks. Since all strongly positive discrete series which we study in this paper share a common partial cuspidal support, it suffices to define only the set of Jordan blocks when introducing these strongly positive discrete series. Similar procedure has also been summarized in Proposition 1.2 of [16]. For more details regarding the $\epsilon$-function we refer the reader to [15] and [19].

Throughout the paper we denote by $\mathrm{Jord}(\sigma)$ the set of Jordan blocks corresponding to discrete series $\sigma$. Also, we define $\mathrm{Jord}_\rho(\sigma) = \{c : (c, \rho) \in \mathrm{Jord}(\sigma)\}$.

Previously given description shows that a non-supercuspidal strongly positive representation can always be obtained as an irreducible subrepresentation of the representation induced from the maximal parabolic subgroup having strongly positive discrete series on the classical group part. One can see directly from Proposition 2.2 that the induced representation having tempered but non-strongly positive representation on the classical group part does not contain a strongly positive subquotient. It remains to see what can be said about strongly positive subquotients of induced representations having non-tempered representation on the classical group part.

Let $\delta$ denote an irreducible essentially square integrable representation of $GL(m, F)$ and let $\sigma$ denote a discrete series representation of $G_n$. An induced representation of the form $\delta \rtimes \sigma$ is then called the generalized principal series.

In this paper we discuss when an induced representation of the form $\delta_1 \times \delta_2 \rtimes \sigma$, where $\delta_1, \delta_2$ are irreducible essentially square integrable representations of a general linear group and $\sigma$ a discrete series representation of the classical group, contains a strongly positive subquotient. By [20], $\delta_1$ and $\delta_2$ correspond to segments, and we write $\delta_1 = \delta([\nu^a \rho_1, \nu^b \rho_1])$ and $\delta_2 = \delta([\nu^c \rho_2, \nu^d \rho_2])$. Also, we denote the partial cuspidal support of $\sigma$ by $\sigma_{cusp}$.

Also, it is a direct consequence of the Mœglin-Tadić classification that if the induced representation of the form $\delta([\nu^a \rho_1, \nu^b \rho_1]) \times \delta([\nu^c \rho_2, \nu^d \rho_2]) \rtimes \sigma$ contains a discrete series subquotient, then $2a + 1 - x \in \mathbb{Z}$ and $2c + 1 - y \in \mathbb{Z}$ for $x \in \mathrm{Jord}_{\rho_1}(\sigma)$ and $y \in \mathrm{Jord}_{\rho_2}(\sigma)$. In the sequel we assume that these conditions are satisfied.

Set of Jordan blocks of discrete series subquotient of $\delta_1 \times \delta_2 \rtimes \sigma$ can be directly reconstructed from the cuspidal support of this induced representation (see, for instance, Subsection 4.2 of [3]).

Furthermore, since in the appropriate Grothendieck group holds $\pi \rtimes \sigma = \widetilde{\pi} \rtimes \sigma$ we may assume $a + b \geq 0$ and $c + d \geq 0$.

## 3. Main results

Suppose that the induced representation $\delta_1 \times \delta_2 \rtimes \sigma$ contains a strongly positive subquotient $\sigma_{sp}$. Using property of cuspidal support of strongly positive discrete series given in Lemma 3.3 of [7], and an embedding of the non-strongly positive discrete series given in Theorem 3.5 of the same paper, we conclude that the representation $\sigma$ must also be strongly positive. From $a + b \geq 0$ and $c + d \geq 0$, using Lemma 3.3 of [7] one more time, we deduce $a > 0$ and $c > 0$.

Furthermore, there is some irreducible subquotient $\pi$ of $\delta_2 \rtimes \sigma$ such that $\sigma_{sp}$ is an irreducible subquotient of $\delta_1 \rtimes \pi$. Obviously, $\pi$ is either a strongly positive or a non-tempered representation. The case of strongly positive $\pi$ follows directly from [16], and is summarized in the following proposition.

**Proposition 3.1.** *Induced representation $\delta([\nu^a \rho_1, \nu^b \rho_1]) \times \delta([\nu^c \rho_2, \nu^d \rho_2]) \rtimes \sigma$ contains a strongly positive irreducible subquotient, which is a subquotient of $\delta_1 \rtimes \pi$ for strongly positive subquotient $\pi$ of $\delta_2 \rtimes \sigma$, if and only if $\sigma$ is strongly positive and one of the following holds:*

*(i)* $a > \frac{1}{2}$, $c > \frac{1}{2}$, $2c - 1 \in Jord_{\rho_2}(\sigma)$, $[2c+1, 2d+1] \cap Jord_{\rho_2}(\sigma) = \emptyset$ *and* $2a - 1 \in Jord_{\rho_1}(\pi)$, $[2a+1, 2b+1] \cap Jord_{\rho_1}(\pi) = \emptyset$, *for the unique irreducible strongly positive subquotient* $\pi$ *of* $\delta([\nu^c \rho_2, \nu^d \rho_2]) \rtimes \sigma$.

*(ii)* $a = \frac{1}{2}$, $c > \frac{1}{2}$, $2c - 1 \in Jord_{\rho_2}(\sigma)$, $[2c+1, 2d+1] \cap Jord_{\rho_2}(\sigma) = \emptyset$, $\nu^{\frac{1}{2}} \rho_1$ *does not appear in the cuspidal support of* $\delta([\nu^c \rho_2, \nu^d \rho_2]) \rtimes \sigma$ *and either* $b < x$ *for* $x$ *such that* $2x + 1 = \min(Jord_{\rho_2}(\pi))$ *or* $Jord_{\rho_2}(\pi) = \emptyset$, *for the unique irreducible strongly positive subquotient* $\pi$ *of* $\delta([\nu^c \rho_2, \nu^d \rho_2]) \rtimes \sigma$.

*(iii)* $a > \frac{1}{2}$, $c = \frac{1}{2}$, $\nu^{\frac{1}{2}} \rho_2$ *does not appear in the cuspidal support of* $\sigma$, $2a - 1 \in Jord_{\rho_1}(\pi)$, $[2a+1, 2b+1] \cap Jord_{\rho_1}(\pi) = \emptyset$ *and either* $d < x$ *for* $x$ *such that* $2x + 1 = \min(Jord_{\rho_2}(\sigma))$ *or* $Jord_{\rho_2}(\sigma) = \emptyset$, *for the unique irreducible strongly positive subquotient* $\pi$ *of* $\delta([\nu^c \rho_2, \nu^d \rho_2]) \rtimes \sigma$.

*(iv)* $a = \frac{1}{2}$, $c = \frac{1}{2}$, $\rho_1 \not\cong \rho_2$, $\nu^{\frac{1}{2}}\rho_2$ *does not appear in the cuspidal support of* $\sigma$, $\nu^{\frac{1}{2}}\rho_1$ *does not appear in the cuspidal support of* $\delta([\nu^c\rho_2, \nu^d\rho_2]) \rtimes \sigma$, *either* $d < x$ *for* $x$ *such that* $2x + 1 = \min(Jord_{\rho_2}(\sigma))$ *or* $Jord_{\rho_2}(\sigma) = \emptyset$ *and either* $b < y$ *for* $y$ *such that* $2y+1 = \min(Jord_{\rho_1}(\pi))$ *or* $Jord_{\rho_1}(\pi)$, *for the unique irreducible strongly positive subquotient* $\pi$ *of* $\delta([\nu^c\rho_2, \nu^d\rho_2]) \rtimes \sigma$.

If the reducibility of $\nu^\alpha\rho_i \rtimes \sigma_{cusp}$ occurs at $\alpha = \pm\frac{1}{2}$, for some $i \in \{1, 2\}$, then the fact that $\nu^{\frac{1}{2}}\rho_i$ does not appear in the cuspidal support of $\sigma$ is equivalent to $\mathrm{Jord}_{\rho_i}(\sigma) = \emptyset$.

If $\mathrm{Jord}_{\rho_i}(\sigma) \neq \emptyset$, then the fact that $\nu^{\frac{1}{2}}\rho_i$, $i = 1, 2$, does not appear in the cuspidal support of $\sigma$ is equivalent to $\epsilon_\sigma(\min(\mathrm{Jord}_{\rho_i}(\sigma)), \rho_i) = -1$, where $\epsilon_\sigma$ denotes the $\epsilon$-function corresponding to strongly positive discrete series $\sigma$.

It remains to determine when the induced representation of the form $\delta_1 \rtimes \pi$, where $\delta_1$ is an irreducible essentially square integrable representation of a general linear group and $\pi$ the Langlands quotient of generalized principal series, contains a strongly positive discrete series. Abusing the notation, we again write $\delta_1 = \delta([\nu^a\rho_1, \nu^b\rho_1])$ and $\pi = L(\delta([\nu^c\rho_2, \nu^d\rho_2]) \rtimes \sigma)$, where $\sigma$ is a strongly positive discrete series.

The following result can be deduced from Proposition 7.4 and Definition 7.6 of [4], enhanced by Theorem 9.3 (7) of the same paper. For the sake of completeness, we provide another proof.

**Lemma 3.2.** *If the induced representation of the form* $\delta([\nu^a\rho_1, \nu^b\rho_1]) \rtimes L(\delta([\nu^c\rho_2, \nu^d\rho_2]) \rtimes \sigma)$ *contains an irreducible strongly positive subquotient, then* $\rho_1 \cong \rho_2$.

*Proof.* Suppose that there is some strongly positive irreducible subquotient $\sigma_{sp}$ of $\delta([\nu^a\rho_1, \nu^b\rho_1]) \rtimes L(\delta([\nu^c\rho_2, \nu^d\rho_2]) \rtimes \sigma)$ with $\rho_1 \not\cong \rho_2$. By Theorem 4.6 from [9], there is an irreducible constituent $\delta \otimes \sigma'_{sp}$ of $\mu^*(\sigma_{sp})$ such that $\delta$ is an irreducible representation whose cuspidal support contains only twists of $\rho_1$, while $\sigma'_{sp}$ is a strongly positive discrete series having no twists of $\rho_1$ in the cuspidal support. It can be easily seen that there has to be some irreducible constituent $\delta' \otimes \sigma'_{sp}$ of $\mu^*(L(\delta([\nu^c\rho_2, \nu^d\rho_2]) \rtimes \sigma))$, such that $\delta' \times \delta([\nu^a\rho_1, \nu^b\rho_1])$ contains $\delta$. Again, only twists of $\rho_1$ appear in the cuspidal support of $\delta'$.

Since $\rho_1$ and $\rho_2$ are non-isomorphic, from the structural formula for $\mu^*(\delta([\nu^c\rho_2,\ \nu^d\rho_2]) \rtimes \sigma)$ we deduce that there is an irreducible constituent $\delta' \otimes \sigma''_{sp}$ of $\mu^*(\sigma)$, with $\sigma''_{sp}$ strongly positive, such that $\delta([\nu^c\rho_2, \nu^d\rho_2]) \rtimes \sigma''_{sp}$ contains $\sigma'_{sp}$. It follows from Proposition 3.1 (i) and Theorem 5.1 (i) of [16] that then also the induced representation $\delta([\nu^c\rho_2, \nu^d\rho_2]) \rtimes \sigma$ contains a strongly positive subquotient, which we denote by $\pi_{sp}$.

Since Jacquet modules of strongly positive representations are of multiplicity one (by Theorem 4.6 from [9]) and strongly positive representations appear in generalized principal series with multiplicity one (by Proposition 3.1 (i) and Theorem 5.1 (i) of [16]), it can be deduced that $\delta' \otimes \sigma'_{sp}$ appears with multiplicity one in both $\mu^*(\delta([\nu^c\rho_2, \nu^d\rho_2]) \rtimes \sigma)$ and $\mu^*(\pi_{sp})$. Thus, $\mu^*(L(\delta([\nu^c\rho_2, \nu^d\rho_2]) \rtimes \sigma))$ does not contain $\delta' \otimes \sigma'_{sp}$, a contradiction. □

To simplify the notation, in the rest of the paper we write $\rho$ instead of $\rho_1$ and $\rho_2$. The proof of the following lemma follows directly from the description of strongly positive discrete series given in Proposition 2.2.

**Lemma 3.3.** *If the induced representation $\delta([\nu^a\rho, \nu^b\rho]) \rtimes L(\delta([\nu^c\rho, \nu^d\rho]) \rtimes \sigma)$ contains an irreducible strongly positive subquotient then $a \neq c$.*

In the following theorem we state the main results of this paper.

**Theorem 3.4.** *Assume that $a \neq c$ and that $\sigma$ is a strongly positive discrete series.*

*(i) If $a > \frac{1}{2}$ and $c > \frac{1}{2}$, then the induced representation $\delta([\nu^a\rho, \nu^b\rho]) \rtimes L(\delta([\nu^c\rho, \nu^d\rho]) \rtimes \sigma)$ contains an irreducible strongly positive subquotient if and only if one of the following holds:*

  *(a) $a < c$, $c = b+1$, $2a-1 \in Jord_\rho(\sigma)$, $2d+1 \notin Jord_\rho(\sigma)$ and if there is an $x$ such that in $Jord_\rho(\sigma)$ holds $(2x+1)_- = 2a-1$ then $d < x$. The irreducible strongly positive subquotient $\sigma_{sp}$ is characterized by $Jord(\sigma_{sp}) = Jord(\sigma) \setminus \{(2a-1, \rho)\} \cup \{(2d+1, \rho)\}$.*

  *(b) $c < a$, $a = d+1$, $2c-1 \in Jord_\rho(\sigma)$, $2b+1 \notin Jord_\rho(\sigma)$, there is an $x$ such that in $Jord_\rho(\sigma)$ holds $(2x+1)_- = 2c-1$ and $[2a-1, 2b+1] \cap Jord_\rho(\sigma) = \{2x+1\}$. The irreducible strongly positive subquotient $\sigma_{sp}$ is characterized by $Jord(\sigma_{sp}) = Jord(\sigma) \setminus \{(2c-1, \rho)\} \cup \{(2b+1, \rho)\}$.*

*(ii) If $a = \frac{1}{2}$, then the induced representation $\delta([\nu^a\rho, \nu^b\rho]) \rtimes L(\delta([\nu^c\rho, \nu^d\rho]) \rtimes \sigma)$ contains an irreducible strongly positive subquotient if and only if one of the following holds:*

(a) *$Jord_\rho(\sigma) \neq \emptyset$, $\nu^{\frac{1}{2}}\rho$ does not appear in the cuspidal support of $\delta([\nu^c\rho, \nu^d\rho]) \rtimes \sigma$, $c = b+1$ and for $x$ such that $2x+1 = \min(Jord_\rho(\sigma))$ we have $d < x$.*

(b) *$Jord_\rho(\sigma) = \emptyset$ and $c = b + 1$.*

*In both cases, the irreducible strongly positive subquotient $\sigma_{sp}$ is characterized by $Jord(\sigma_{sp}) = Jord(\sigma) \cup \{(2d + 1, \rho)\}$.*

(iii) *If $c = \frac{1}{2}$, then the induced representation $\delta([\nu^a\rho, \nu^b\rho]) \rtimes L(\delta([\nu^c\rho, \nu^d\rho]) \rtimes \sigma)$ contains an irreducible strongly positive subquotient if and only if $\nu^{\frac{1}{2}}\rho$ does not appear in the cuspidal support of $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma$, $Jord_\rho(\sigma) \neq \emptyset$ and one of the following holds:*

(a) *$2a-1 \notin Jord_\rho(\sigma), 2b+1 \notin Jord_\rho(\sigma)$, $d = a-1$ and for $x$ such that $2x+1 = \min(Jord_\rho(\sigma))$ we have $[2a+1, 2b+1] \cap Jord_\rho(\sigma) = \{2x+1\}$.*

(b) *$2a - 1 = \min(Jord_\rho(\sigma)), 2b + 1 \notin Jord_\rho(\sigma)$, $d = a - 1$ and if there is $2x + 1 \in Jord_\rho(\sigma)$ such that $(2x + 1)_- = 2a - 1$ then $b < x$.*

*In both cases, the irreducible strongly positive subquotient $\sigma_{sp}$ is characterized by $Jord(\sigma_{sp}) = Jord(\sigma) \cup \{(2b + 1, \rho)\}$.*

Several possibilities which appear in the previous theorem will be discussed in separate sections.

## 4. Case $a > \frac{1}{2}$ and $c > \frac{1}{2}$

In this section we assume $a > \frac{1}{2}$ and $c > \frac{1}{2}$.

We have divided the proof of Theorem 3.4 (i) in a sequence of lemmas. We start with two technical lemmas which reduce our analysis to the case when the union of segments $[\nu^a\rho, \nu^b\rho]$ and $[\nu^c\rho, \nu^d\rho]$ is again a segment.

**Lemma 4.1.** *If $a < c$ and $c \neq b+1$, the induced representation $\delta([\nu^a\rho, \nu^b\rho]) \rtimes L(\delta([\nu^c\rho, \nu^d\rho]) \rtimes \sigma)$ does not contain an irreducible strongly positive subquotient.*

*Proof.* Suppose, on the contrary, $a < c$, $c \neq b + 1$ and that there is an irreducible strongly positive subquotient $\sigma_{sp}$ of $\delta_1 \rtimes L(\delta_2 \rtimes \sigma)$. Since $a < c$ and $c \neq b + 1$, using a cuspidal support argument analogous to the one from

Subsection 4.2 of [3], we obtain $2a-1, 2c-1 \in \mathrm{Jord}_\rho(\sigma)$, $2b+1, 2d+1 \notin \mathrm{Jord}_\rho(\sigma)$ and $b \neq d$. Furthermore, we have

$$\mathrm{Jord}(\sigma_{sp}) = \mathrm{Jord}(\sigma) \setminus \{(2a-1,\rho),(2c-1,\rho)\} \cup \{(2b+1,\rho),(2d+1,\rho)\}.$$

Let us define $x$ by $2x+1 = \min\{2y+1 \in \mathrm{Jord}_\rho(\sigma_{sp}) : y \geq a\}$. Obviously, $x \leq b$ and (by Theorem 4.6 of [9]) there is a strongly positive discrete series $\sigma'_{sp}$ such that $\mu^*(\sigma_{sp}) \geq \delta([\nu^a\rho,\nu^x\rho]) \otimes \sigma'_{sp}$. This implies $\mu^*(\delta_1 \rtimes L(\delta_2 \rtimes \sigma)) \geq \delta([\nu^a\rho,\nu^x\rho]) \otimes \sigma'_{sp}$ and $\mu^*(\delta_1 \times \delta_2 \rtimes \sigma) \geq \delta([\nu^a\rho,\nu^x\rho]) \otimes \sigma'_{sp}$. We will analyze the last inequality using Lemma 2.1.

There are $a-1 \leq i_1 \leq j_1 \leq b$, $c-1 \leq i_2 \leq j_2 \leq d$ and an irreducible constituent $\delta \otimes \pi$ of $\mu^*(\sigma)$ such that

$$\begin{aligned}\delta([\nu^a\rho,\nu^x\rho]) \leq &\delta([\nu^{-i_1}\rho,\nu^{-a}\rho]) \times \delta([\nu^{j_1+1}\rho,\nu^b\rho]) \times \\ &\delta([\nu^{-i_2}\rho,\nu^{-c}\rho]) \times \delta([\nu^{j_2+1}\rho,\nu^d\rho]) \times \delta\end{aligned}$$

and

$$\sigma'_{sp} \leq \delta([\nu^{i_1+1}\rho,\nu^{j_1}\rho]) \times \delta([\nu^{i_2+1}\rho,\nu^{j_2}\rho]) \rtimes \pi.$$

We see at once $i_1 = a-1$ and $i_2 = c-1$. Two possibilities will be considered separately:

- $x = b$. Since $b \neq d$ and $2b+1 \notin \mathrm{Jord}_\rho(\sigma)$, using Proposition 2.1 from [15] we deduce that $j_1 < b$. Furthermore, $a < c$ and $2a-1 \in \mathrm{Jord}_\rho(\sigma)$, together with Theorem 4.6 from [9], imply $j_1 = a-1$. Now it can be easily seen that if $\mu^*(\delta_1 \rtimes L(\delta_2 \rtimes \sigma)) \geq \delta([\nu^a\rho,\nu^b\rho]) \otimes \sigma'_{sp}$ then $L(\delta_2 \rtimes \sigma) \cong \sigma'_{sp}$, which is impossible.

- $x < b$. This implies $j_1 = b$. Since $a < c$, it follows that $\delta$ is of the form $\delta([\nu^a\rho,\nu^y\rho])$ for some $y \geq a$, contradicting $2a-1 \in \mathrm{Jord}_\rho(\sigma)$ by Theorem 4.6 from [9].

This proves the lemma. □

**Lemma 4.2.** *If $c < a$ and $a \neq d+1$, the induced representation $\delta([\nu^a\rho,\nu^b\rho]) \rtimes L(\delta([\nu^c\rho,\nu^d\rho]) \rtimes \sigma)$ does not contain an irreducible strongly positive subquotient.*

*Proof.* We again suppose, on the contrary, $c < a$, $a \neq d+1$ and that there is an irreducible strongly positive subquotient $\sigma_{sp}$ of $\delta_1 \rtimes L(\delta_2 \rtimes \sigma)$. In the same way as in the proof of the previous lemma we obtain $2a-1, 2c-1 \in \mathrm{Jord}_\rho(\sigma)$, $2b+1, 2d+1 \notin \mathrm{Jord}_\rho(\sigma)$, $b \neq d$ and

$$\mathrm{Jord}(\sigma_{sp}) = \mathrm{Jord}(\sigma) \setminus \{(2a-1,\rho),(2c-1,\rho)\} \cup \{(2b+1,\rho),(2d+1,\rho)\}.$$

We define $x$ by $2x+1 = \min\{2y+1 \in \mathrm{Jord}_\rho(\sigma_{sp}) : y \geq c\}$. Obviously, $x \leq d$ and $x \leq b$.

We must have $\mu^*(\sigma_{sp}) \geq \delta([\nu^c\rho, \nu^x\rho]) \otimes \sigma'_{sp}$ for some strongly positive discrete series $\sigma'_{sp}$. Using the structural formula for $\mu^*(\delta_1 \rtimes L(\delta_2 \rtimes \sigma))$ we deduce that there are $a-1 \leq i \leq j \leq b$ and an irreducible constituent $\delta \otimes \pi$ of $\mu^*(L(\delta_2 \rtimes \sigma))$ such that

$$\delta([\nu^c\rho, \nu^x\rho]) \leq \delta([\nu^{-i}\rho, \nu^{-a}\rho]) \times \delta([\nu^{j+1}\rho, \nu^b\rho]) \times \delta$$

and

$$\sigma'_{sp} \leq \delta([\nu^{i+1}\rho, \nu^j\rho]) \rtimes \pi.$$

Since $0 < c$ and $c < a$, it follows that $\delta$ is of the form $\delta([\nu^c\rho, \nu^y\rho])$, where $y$ equals either $x$ or $j$ (note that $x \leq b$).

Consequently, if $\delta_1 \rtimes L(\delta_2 \rtimes \sigma)$ contains $\sigma_{sp}$, then $\mu^*(L(\delta_2 \rtimes \sigma))$ contains an irreducible constituent of the form $\delta([\nu^c\rho, \nu^y\rho]) \otimes \sigma'$, where $y \leq d$. Such irreducible constituent also appears in $\mu^*(\delta_2 \rtimes \sigma)$. Thus, there are $c-1 \leq i' \leq j' \leq d$ and an irreducible constituent $\delta' \otimes \pi'$ of $\mu^*(\sigma)$ such that

$$\delta([\nu^c\rho, \nu^y\rho]) \leq \delta([\nu^{-i'}\rho, \nu^{-c}\rho]) \times \delta([\nu^{j'+1}\rho, \nu^d\rho]) \times \delta'$$

and

$$\sigma' \leq \delta([\nu^{i'+1}\rho, \nu^{j'}\rho]) \rtimes \pi'.$$

Condition $2c-1 \in \mathrm{Jord}_\rho(\sigma)$ implies $j' = c-1$ and from $y \leq d$ we obtain that $y = x = d$. Also, it follows that $\sigma'$ is isomorphic to $\sigma$.

Thus, $\delta_2 \otimes \sigma$ is the only irreducible constituent of the form $\delta([\nu^c\rho, \nu^y\rho]) \otimes \sigma'$, with $y \leq d$, appearing in $\mu^*(\delta_2 \rtimes \sigma)$ and appears there with multiplicity one. Furthermore, it follows that $\delta_2 \rtimes \sigma$ has a unique irreducible subrepresentation, which we denote by $\pi$. Obviously, $\mu^*(\pi) \geq \delta_2 \otimes \sigma$. We have

$2c-1 \in \text{Jord}_\rho(\sigma)$ and $2d+1 \notin \text{Jord}_\rho(\sigma)$, so Proposition 3.1 (i) of [16] shows that $\delta_2 \rtimes \sigma$ is representation of length two and in the appropriate Grothendieck group holds $\delta_2 \rtimes \sigma = L(\delta_2 \rtimes \sigma) + \pi$, for $\pi \not\cong L(\delta_2 \rtimes \sigma)$.

Therefore, $\mu^*(L(\delta_2 \rtimes \sigma))$ does not contain $\delta_2 \otimes \sigma$. So, $\delta_1 \rtimes L(\delta_2 \rtimes \sigma)$ does not contain an irreducible strongly positive subquotient. □

The next lemma finishes the proof of the first part of Theorem 3.4 (i).

**Lemma 4.3.** *If $a < c$ and $c = b+1$, the induced representation $\delta([\nu^a\rho, \nu^b\rho]) \rtimes L(\delta([\nu^c\rho, \nu^d\rho]) \rtimes \sigma)$ contains an irreducible strongly positive subquotient if and only if $2a-1 \in \text{Jord}_\rho(\sigma)$, $2d+1 \notin \text{Jord}_\rho(\sigma)$ and if there is an $x$ such that in $\text{Jord}_\rho(\sigma)$ holds $(2x+1)_- = 2a-1$ then $d < x$.*

*Proof.* If there is an strongly positive discrete series subquotient of $\delta_1 \rtimes L(\delta_2 \rtimes \sigma)$, then from the cuspidal support of this representation we obtain $2a-1 \in \text{Jord}_\rho(\sigma)$, $2d+1 \notin \text{Jord}_\rho(\sigma)$. Suppose that there is an $x$ such that in $\text{Jord}_\rho(\sigma)$ holds $(2x+1)_- = 2a-1$. Condition $2d+1 \notin \text{Jord}_\rho(\sigma)$ implies $d \neq x$. To obtain a contradiction, suppose $d > x$ and that there is a strongly positive discrete series subquotient $\sigma_{sp}$ of $\delta_1 \rtimes L(\delta_2 \rtimes \sigma)$. Using the description of Jordan blocks in terms of the cuspidal support, we obtain

$$\text{Jord}(\sigma_{sp}) = \text{Jord}(\sigma) \setminus \{(2a-1, \rho)\} \cup \{(2d+1, \rho)\}.$$

Theorem 4.6 of [9] shows $\mu^*(\sigma_{sp}) \geq \delta([\nu^a\rho, \nu^x\rho]) \otimes \sigma'_{sp}$, for strongly positive discrete series $\sigma'_{sp}$ such that $\text{Jord}(\sigma'_{sp}) = \text{Jord}(\sigma_{sp}) \setminus \{(2x+1, \rho)\} \cup \{(2a-1, \rho)\}$. It follows that $\mu^*(\delta_1 \rtimes L(\delta_2 \rtimes \sigma)) \geq \delta([\nu^a\rho, \nu^x\rho]) \otimes \sigma'_{sp}$ and $\mu^*(\delta_1 \times \delta_2 \rtimes \sigma) \geq \delta([\nu^a\rho, \nu^x\rho]) \otimes \sigma'_{sp}$. Using the structural formula for $\mu^*$ we obtain that there are $a-1 \leq i_1 \leq j_1 \leq b$, $c-1 \leq i_2 \leq j_2 \leq d$ and an irreducible constituent $\delta \otimes \sigma''_{sp}$ of $\mu^*(\sigma)$ such that

$$\begin{aligned} \delta([\nu^a\rho, \nu^x\rho]) \leq & \delta([\nu^{-i_1}\rho, \nu^{-a}\rho]) \times \delta([\nu^{j_1+1}\rho, \nu^b\rho]) \times \\ & \delta([\nu^{-i_2}\rho, \nu^{-c}\rho]) \times \delta([\nu^{j_2+1}\rho, \nu^d\rho]) \times \delta \end{aligned}$$

and

$$\sigma'_{sp} \leq \delta([\nu^{i_1+1}\rho, \nu^{j_1}\rho]) \times \delta([\nu^{i_2+1}\rho, \nu^{j_2}\rho]) \rtimes \sigma''_{sp}.$$

Since $a > 0$, $x < d$ and $2a-1 \in \text{Jord}_\rho(\sigma)$, we get $i_1 = a-1$, $i_2 = c-1$, $j_1 = a-1$ and $j_2 = d$. It follows that $\delta \cong \delta([\nu^{b+1}\rho, \nu^x\rho])$ and, in the same way as in the proof of Lemma 4.1, we deduce that $x > b$ since otherwise we would have $L(\delta_2 \rtimes \sigma) \cong \sigma'_{sp}$. Furthermore, we have $\mathrm{Jord}(\sigma''_{sp}) = \mathrm{Jord}(\sigma) \setminus \{(2x+1, \rho)\} \cup \{(2b+1, \rho)\}$ and

$$\sigma'_{sp} \leq \delta([\nu^c\rho, \nu^d\rho]) \rtimes \sigma''_{sp}.$$

Proposition 3.1 (i) of [16] implies $[2c+1, 2d+1] \cap \mathrm{Jord}_\rho(\sigma''_{sp}) = \emptyset$, i.e., $[2c+1, 2d+1] \cap \mathrm{Jord}_\rho(\sigma) = \{2x+1\}$.

Previous analysis also implies that if $\delta_1 \rtimes L(\delta_2 \rtimes \sigma)$ contains $\sigma_{sp}$ then $\mu^*(L(\delta_2 \rtimes \sigma)) \geq \delta([\nu^{b+1}\rho, \nu^x\rho]) \otimes \sigma'_{sp}$. It can be easily seen that such irreducible constituent appears in $\mu^*(\delta_2 \rtimes \sigma)$ with multiplicity one.

Applying Proposition 3.1 (i) of [16] one more time we deduce that in the appropriate Grothendieck group we have

$$\delta_2 \rtimes \sigma = L(\delta_2 \rtimes \sigma) + L(\delta([\nu^c\rho, \nu^x\rho]) \rtimes \sigma'_{sp}).$$

As $[2c-1, 2x+1] \cap \mathrm{Jord}_\rho(\sigma'_{sp}) = \emptyset$, the induced representation $\delta([\nu^c\rho, \nu^x\rho]) \rtimes \sigma'_{sp}$ is irreducible and isomorphic to its Langlands quotient. Frobenius reciprocity now gives $\mu^*(L(\delta([\nu^c\rho, \nu^x\rho]) \rtimes \sigma'_{sp})) \geq \delta([\nu^{b+1}\rho, \nu^x\rho]) \otimes \sigma'_{sp}$. Thus, $\mu^*(L(\delta_2 \rtimes \sigma))$ does not contain $\delta([\nu^{b+1}\rho, \nu^x\rho]) \otimes \sigma'_{sp}$ and there are no strongly positive discrete series subquotients of $\delta_1 \rtimes L(\delta_2 \rtimes \sigma)$, a contradiction.

Conversely, let us assume $2a-1 \in \mathrm{Jord}_\rho(\sigma)$, $2d+1 \notin \mathrm{Jord}_\rho(\sigma)$ and if there is an $x$ such that in $\mathrm{Jord}_\rho(\sigma)$ holds $(2x+1)_- = 2a-1$ then $d < x$. By Theorem 3.4 of [6], induced representation $\delta([\nu^a\rho, \nu^d\rho]) \rtimes \sigma$ has a unique irreducible subrepresentation, which is strongly positive (by Theorem 4.6 of [6]) and will be denoted by $\sigma_{sp}$. We have the following embeddings and intertwining operator:

$$\begin{aligned}\sigma_{sp} \hookrightarrow & \delta([\nu^a\rho, \nu^d\rho]) \rtimes \sigma \hookrightarrow \delta([\nu^c\rho, \nu^d\rho]) \times \delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma \\ \rightarrow & \delta([\nu^a\rho, \nu^b\rho]) \times \delta([\nu^c\rho, \nu^d\rho]) \rtimes \sigma.\end{aligned}$$

Since $[2c-1, 2d+1] \cap \mathrm{Jord}_\rho(\sigma) = \emptyset$ holds, by Proposition 3.1 (i) of [16] $\delta([\nu^c\rho, \nu^d\rho]) \rtimes \sigma$ is irreducible and we have $L(\delta([\nu^c\rho, \nu^d\rho]) \rtimes \sigma) \cong \delta([\nu^c\rho, \nu^d\rho]) \rtimes \sigma$. Therefore, $\sigma_{sp}$ is an irreducible subquotient of $\delta_1 \rtimes L(\delta_2 \rtimes \sigma)$. This completes the proof. □

The following two lemmas complete the proof of Theorem 3.4 (i).

**Lemma 4.4.** *If $c < a$, $a = d+1$ and the induced representation $\delta([\nu^a\rho, \nu^b\rho]) \rtimes L(\delta([\nu^c\rho, \nu^d\rho]) \rtimes \sigma)$ contains an irreducible strongly positive subquotient then $2c-1 \in Jord_\rho(\sigma)$, $2b+1 \notin Jord_\rho(\sigma)$, there is an $x$ such that in $Jord_\rho(\sigma)$ holds $(2x+1)_- = 2c-1$ and $[2a-1, 2b+1] \cap Jord_\rho(\sigma) = \{2x+1\}$.*

*Proof.* If there is a strongly positive discrete series subquotient $\sigma_{sp}$ of $\delta_1 \rtimes L(\delta_2 \rtimes \sigma)$, then in the same way as before we deduce that $2c-1 \in \mathrm{Jord}_\rho(\sigma)$, $2b+1 \notin \mathrm{Jord}_\rho(\sigma)$. Directly from the cuspidal support of this induced representation, we get

$$\mathrm{Jord}(\sigma_{sp}) = \mathrm{Jord}(\sigma) \setminus \{(2c-1,\rho)\} \cup \{(2b+1,\rho)\}.$$

Let us first assume that there is no $x$ such that in $\mathrm{Jord}_\rho(\sigma)$ holds $(2x+1)_- = 2c-1$, i.e., $2c-1$ is the maximum of $\mathrm{Jord}_\rho(\sigma)$. It follows that $\mu^*(\sigma_{sp})$ contains $\delta([\nu^c\rho,\nu^b\rho]) \otimes \sigma$. Thus, $\mu^*(\delta_1 \rtimes L(\delta_2 \rtimes \sigma))$ also contains $\delta([\nu^c\rho,\nu^b\rho]) \otimes \sigma$ and, by Lemma 2.1, there are $a-1 \leq i \leq j \leq b$ and an irreducible constituent $\delta \otimes \sigma'$ of $\mu^*(L(\delta_2 \rtimes \sigma))$ such that

$$\delta([\nu^c\rho,\nu^b\rho]) \leq \delta([\nu^{-i}\rho,\nu^{-a}\rho]) \times \delta([\nu^{j}\rho,\nu^b\rho]) \times \delta$$

and

$$\sigma \leq \delta([\nu^{i+1}\rho,\nu^{j}\rho]) \rtimes \sigma'.$$

Using $c < a$, $a = d+1$, $2c-1 \in \mathrm{Jord}_\rho(\sigma)$ and the fact that $2c-1$ is the maximum of $\mathrm{Jord}_\rho(\sigma)$, we directly get $i = a-1$, $j = a$, $\delta \cong \delta([\nu^c\rho,\nu^d\rho])$ and $\sigma \cong \sigma'$. It follows that $\mu^*(L(\delta_2 \rtimes \sigma))$ contains $\delta([\nu^c\rho,\nu^d\rho]) \otimes \sigma$. It is not hard to see that such irreducible constituent appears with multiplicity one in $\mu^*(\delta_2 \rtimes \sigma)$. By Proposition 3.1 (i) of [16], $\delta_2 \rtimes \sigma$ is a length two representation whose irreducible subrepresentation is a strongly positive discrete series which contains $\delta([\nu^c\rho,\nu^d\rho]) \otimes \sigma$ in its Jacquet module with respect to the appropriate standard parabolic subgroup. Thus, $\mu^*(L(\delta_2 \rtimes \sigma))$ does not contain $\delta([\nu^c\rho,\nu^d\rho]) \otimes \sigma$, a contradiction.

We will now consider the remaining case. Suppose that there is an $x$ such that in $\mathrm{Jord}_\rho(\sigma)$ holds $(2x+1)_- = 2c-1$. Condition $2b+1 \notin \mathrm{Jord}_\rho(\sigma)$ implies $b \neq x$. If $b < x$, then again $\mu^*(\sigma_{sp})$ contains $\delta([\nu^c\rho,\nu^b\rho]) \otimes \sigma$ and we obtain a contradiction in the same way as in the previous case. Thus, we can assume $b > x$. Now both $\mu^*(\sigma_{sp})$ and $\mu^*(\delta_1 \rtimes L(\delta_2 \rtimes \sigma))$ contain $\delta([\nu^c\rho,\nu^x\rho]) \otimes \sigma'_{sp}$, for strongly positive discrete series $\sigma'_{sp}$ such that $\mathrm{Jord}(\sigma'_{sp}) = \mathrm{Jord}(\sigma) \setminus \{(2x+1,\rho)\} \cup \{(2b+1,\rho)\}$. Lemma 2.1 implies that there are $a-1 \leq i_1 \leq j_1 \leq b$, $c-1 \leq i_2 \leq j_2 \leq d$ and an irreducible constituent $\delta \otimes \sigma''_{sp}$ of $\mu^*(\sigma)$ such that

$$\begin{aligned}\delta([\nu^c\rho,\nu^x\rho]) \leq & \delta([\nu^{-i_1}\rho,\nu^{-a}\rho]) \times \delta([\nu^{j_1+1}\rho,\nu^b\rho]) \times \\ & \delta([\nu^{-i_2}\rho,\nu^{-c}\rho]) \times \delta([\nu^{j_2+1}\rho,\nu^d\rho]) \times \delta\end{aligned}$$

and

$$\sigma'_{sp} \leq \delta([\nu^{i_1+1}\rho, \nu^{j_1}\rho]) \times \delta([\nu^{i_2+1}\rho, \nu^{j_2}\rho]) \rtimes \sigma''_{sp}.$$

It follows immediately that $i_1 = a-1$, $j_1 = b$, $i_2 = c-1$, $j_2 = c-1$ and $\delta \cong \delta([\nu^{d+1}\rho, \nu^x\rho])$. Also, this gives $d \leq x$. Furthermore, we see that $x \neq d$ shows that $\sigma''_{sp}$ is a strongly positive discrete series such that $\text{Jord}(\sigma''_{sp}) = \text{Jord}(\sigma) \setminus \{(2x+1, \rho)\} \cup \{(2d+1, \rho)\}$, while $x = d$ implies $\sigma''_{sp} \cong \sigma$.

Since the induced representation $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma''_{sp}$ has to contain the strongly positive subquotient $\sigma'_{sp}$, from Proposition 3.1 (i) of [16] we get $[2a+1, 2b+1] \cap \text{Jord}_\rho(\sigma'_{sp}) = \emptyset$, i.e., $[2a-1, 2b+1] \cap \text{Jord}_\rho(\sigma) = \{2x+1\}$ and the lemma is proved. □

**Lemma 4.5.** *Suppose that $c < a$ and $a = d+1$. If $2c-1 \in Jord_\rho(\sigma)$, $2b+1 \notin Jord_\rho(\sigma)$ and there is an $x$ such that in $Jord_\rho(\sigma)$ holds $(2x+1)\_ = 2c-1$ and $[2a-1, 2b+1] \cap Jord_\rho(\sigma) = \{2x+1\}$, then the induced representation $\delta([\nu^a\rho, \nu^b\rho]) \rtimes L(\delta([\nu^c\rho, \nu^d\rho]) \rtimes \sigma)$ contains an irreducible strongly positive subquotient.*

*Proof.* First we consider the case $x = d$. Let us denote by $\sigma_{sp}$ the strongly positive discrete series such that $\text{Jord}(\sigma_{sp}) = \text{Jord}(\sigma) \setminus \{(2c-1, \rho)\} \cup \{(2b+1, \rho)\}$. We have the following embeddings and intertwining operator:

$$\begin{aligned}\sigma_{sp} &\hookrightarrow \delta([\nu^c\rho, \nu^d\rho]) \rtimes \sigma'_{sp} \\ &\hookrightarrow \delta([\nu^c\rho, \nu^d\rho]) \times \delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma \\ &\to \delta([\nu^a\rho, \nu^b\rho]) \times \delta([\nu^c\rho, \nu^d\rho]) \rtimes \sigma,\end{aligned}$$

where $\sigma'_{sp}$ stands for the strongly positive discrete series with the property $\text{Jord}(\sigma'_{sp}) = \text{Jord}(\sigma_{sp}) \setminus \{(2d+1, \rho)\} \cup \{(2c-1, \rho)\}$. Since $2d+1 \in \text{Jord}_\rho(\sigma)$, it follows from Proposition 3.1 (i) of [16] that $\delta([\nu^c\rho, \nu^d\rho]) \rtimes \sigma$ is irreducible, hence $L(\delta([\nu^c\rho, \nu^d\rho]) \rtimes \sigma) \cong \delta([\nu^c\rho, \nu^d\rho]) \rtimes \sigma$. It then follows that $\sigma_{sp} \leq \delta([\nu^a\rho, \nu^b\rho]) \rtimes L(\delta([\nu^c\rho, \nu^d\rho]) \rtimes \sigma)$, as needed.

We will now consider the case $x > d$, i.e., $2d+1 \notin \text{Jord}_\rho(\sigma)$. Two possibilities will be studied separately:

- $x = a$.

Let us denote by $\sigma'$ a strongly positive discrete series such that $\text{Jord}(\sigma') = \text{Jord}(\sigma) \setminus \{(2x+1, \rho)\} \cup \{(2b+1, \rho)\}$. It follows from [15], Lemma 4.6, that the induced representation

$$\delta([\nu^{-a}\rho, \nu^a\rho]) \rtimes \sigma'$$

contains two irreducible subquotients, which are non-isomorphic tempered subrepresentations. It can be obtained directly from Lemma 2.1 and Proposition 3.1 (i) of [16] that $\mu^*(\delta([\nu^{-a}\rho, \nu^a\rho]) \rtimes \sigma') \geq \delta([\nu^{-c+1}\rho, \nu^a\rho]) \otimes \sigma_{sp}$, where $\sigma_{sp}$ is strongly positive discrete series such that $\mathrm{Jord}(\sigma_{sp}) = \mathrm{Jord}(\sigma') \setminus \{(2c-1, \rho)\} \cup \{(2a+1, \rho)\}$. We denote by $\tau$ an irreducible subrepresentation of $\delta([\nu^{-a}\rho, \nu^a\rho]) \rtimes \sigma'$ such that $\mu^*(\tau)$ contains $\delta([\nu^{-c+1}\rho, \nu^a\rho]) \otimes \sigma_{sp}$. Furthermore, we denote by $\sigma'_{sp}$ a strongly positive discrete series such that $\mathrm{Jord}(\sigma'_{sp}) = \mathrm{Jord}(\sigma) \setminus \{(2x+1, \rho)\} \cup \{(2d+1, \rho)\}$. Then $\delta([\nu^{-a}\rho, \nu^a\rho]) \rtimes \sigma'$ is a subrepresentation of

$$\delta([\nu^{-a}\rho, \nu^a\rho]) \times \delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma'_{sp}.$$

This implies the following embeddings and intertwining operator:

$$\begin{aligned}\tau \hookrightarrow & \delta([\nu^{-a}\rho, \nu^a\rho]) \rtimes \sigma' \hookrightarrow \delta([\nu^{-a}\rho, \nu^a\rho]) \times \delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma'_{sp}. \\ \rightarrow & \delta([\nu^a\rho, \nu^b\rho]) \times \delta([\nu^{-a}\rho, \nu^a\rho]) \rtimes \sigma'_{sp}.\end{aligned}$$

Consequently, $\mu^*(\delta([\nu^a\rho, \nu^b\rho]) \times \delta([\nu^{-a}\rho, \nu^a\rho]) \rtimes \sigma'_{sp})$ contains $\delta([\nu^{-c+1}\rho, \nu^a\rho]) \otimes \sigma_{sp}$. Using the structural formula for $\mu^*$ we obtain that there are $a-1 \leq i_1 \leq j_1 \leq b$, $-a-1 \leq i_2 \leq j_2 \leq a$ and an irreducible constituent $\delta \otimes \sigma''_{sp}$ of $\mu^*(\sigma'_{sp})$ such that

$$\begin{aligned}\delta([\nu^{-c+1}\rho, \nu^a\rho]) \leq & \delta([\nu^{-i_1}\rho, \nu^{-a}\rho]) \times \delta([\nu^{j_1+1}\rho, \nu^b\rho]) \times \\ & \delta([\nu^{-i_2}\rho, \nu^a\rho]) \times \delta([\nu^{j_2+1}\rho, \nu^a\rho]) \times \delta\end{aligned}$$

and

$$\sigma_{sp} \leq \delta([\nu^{i_1+1}\rho, \nu^{j_1}\rho]) \times \delta([\nu^{i_2+1}\rho, \nu^{j_2}\rho]) \rtimes \sigma''_{sp}.$$

Since $-c+1 \leq 0$ and $c < a$ we get $i_1 = a-1$. Furthermore, since $j_1 + 1 > 0$ and, by Theorem 4.6 of [9], in $\delta$ appear no twists of $\rho$ with negative exponents, we deduce that either $-i_2 = -c+1$ or $j_2 + 1 = -c+1$. It follows that $j_1 = b$ and $\sigma''_{sp} \cong \sigma'_{sp}$. This implies

$$\sigma_{sp} \leq \delta([\nu^a\rho, \nu^b\rho]) \times \delta([\nu^c\rho, \nu^a\rho]) \rtimes \sigma'_{sp}$$

(note that in the appropriate Grothendieck group we have $\pi_1 \times \pi_2 \rtimes \sigma_1 = \pi_1 \times \widetilde{\pi_2} \rtimes \sigma_1$).

There is an irreducible subquotient $\sigma_1$ of $\delta([\nu^c\rho, \nu^a\rho]) \rtimes \sigma'_{sp}$ such that $\sigma_{sp} \leq \delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma_1$. Furthermore, by the description of $\mathrm{Jord}(\sigma_{sp})$, there is some irreducible representation $\sigma_2$ such that $\mu^*(\sigma_{sp}) \geq \delta([\nu^c\rho, \nu^a\rho]) \otimes \sigma_2$. Analyzing $\mu^*(\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma_1)$ it can be deduced that $\mu^*(\sigma_1)$ contains $\delta([\nu^c\rho, \nu^a\rho]) \otimes \sigma_3$ for some irreducible representation $\sigma_3$. Since $2c-1$ is an element of $\mathrm{Jord}_\rho(\sigma'_{sp})$, one can see, using the same argument as before, that only irreducible constituent of the form $\delta([\nu^c\rho, \nu^a\rho]) \otimes \sigma_3$ appearing in $\mu^*(\delta([\nu^c\rho, \nu^a\rho]) \rtimes \sigma'_{sp})$ is $\delta([\nu^c\rho, \nu^a\rho]) \otimes \sigma'_{sp}$ which appears there with multiplicity one.

By Proposition 3.1 (i) of [16], since $a = x$, in the appropriate Grothendieck group we have

$$\delta([\nu^c\rho, \nu^a\rho]) \rtimes \sigma'_{sp} = L(\delta([\nu^c\rho, \nu^a\rho]) \rtimes \sigma'_{sp}) + L(\delta([\nu^c\rho, \nu^d\rho]) \rtimes \sigma)$$

and $L(\delta([\nu^c\rho, \nu^d\rho]) \rtimes \sigma)$ is an irreducible subrepresentation of $\delta([\nu^c\rho, \nu^a\rho]) \rtimes \sigma'_{sp}$. Using Frobenius reciprocity we conclude that $L(\delta([\nu^c\rho, \nu^d\rho]) \rtimes \sigma)$ is the only irreducible subquotient of $\delta([\nu^c\rho, \nu^a\rho]) \rtimes \sigma'_{sp}$ having $\delta([\nu^c\rho, \nu^a\rho]) \otimes \sigma'_{sp}$ in its Jacquet module with respect to the appropriate parabolic subgroup. In consequence, $\sigma_{sp}$ is contained in $\delta([\nu^a\rho, \nu^b\rho]) \rtimes L(\delta([\nu^c\rho, \nu^d\rho]) \rtimes \sigma)$.

- $x > a$.

Let us denote by $\sigma'$ a strongly positive discrete series such that $\mathrm{Jord}(\sigma') = \mathrm{Jord}(\sigma) \setminus \{(2x+1, \rho)\} \cup \{(2d+1, \rho)\}$. Furthermore, we denote by $\sigma_{ds}$ a discrete series which is a subrepresentation of both induced representations $\delta([\nu^{-d}\rho, \nu^a\rho]) \rtimes \sigma$ and $\delta([\nu^{-a}\rho, \nu^x\rho]) \rtimes \sigma'$. This uniquely determines $\sigma_{ds}$ ([10, 15]). By Proposition 4.5 of [10], the induced representation $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma_{ds}$ contains two discrete series subquotients, and exactly one of them contains $\delta([\nu^{-c+1}\rho, \nu^a\rho]) \otimes \sigma_{sp}$ in its Jacquet module with respect to the appropriate standard parabolic subgroup, where $\sigma_{sp}$ stands for the strongly positive discrete series such that $\mathrm{Jord}(\sigma_{sp}) = \mathrm{Jord}(\sigma) \setminus \{(2c-1, \rho)\} \cup \{(2b+1, \rho)\}$. We will denote such discrete series subquotient of $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma_{ds}$ by $\sigma'_{ds}$.

Since $\sigma'_{ds}$ is obviously an irreducible subquotient of the induced representation $\delta([\nu^a\rho, \nu^b\rho]) \times \delta([\nu^{-d}\rho, \nu^a\rho]) \rtimes \sigma$, we deduce

$$\delta([\nu^{-c+1}\rho, \nu^a\rho]) \otimes \sigma_{sp} \leq \mu^*(\delta([\nu^a\rho, \nu^b\rho]) \times \delta([\nu^{-d}\rho, \nu^a\rho]) \rtimes \sigma).$$

The structural formula for $\mu^*$ implies that there are $a - 1 \leq i_1 \leq j_1 \leq b$, $-d-1 \leq i_2 \leq j_2 \leq a$ and an irreducible constituent $\delta \otimes \sigma''$ of $\mu^*(\sigma)$ such that

$$\begin{aligned}\delta([\nu^{-c+1}\rho, \nu^a\rho]) \leq & \delta([\nu^{-i_1}\rho, \nu^{-a}\rho]) \times \delta([\nu^{j_1+1}\rho, \nu^b\rho]) \times \\ & \delta([\nu^{-i_2}\rho, \nu^d\rho]) \times \delta([\nu^{j_2+1}\rho, \nu^a\rho]) \times \delta\end{aligned}$$

and

$$\sigma_{sp} \leq \delta([\nu^{i_1+1}\rho, \nu^{j_1}\rho]) \times \delta([\nu^{i_2+1}\rho, \nu^{j_2}\rho]) \rtimes \sigma''.$$

It follows that $i_1 = a-1$, $j_1 = b$, $\sigma'' \cong \sigma$ and $(i_2, j_2)$ equals either $(-d-1, -c)$ or $(c-1, d)$. In any case, $\sigma_{sp}$ is an irreducible subquotient of $\delta([\nu^a\rho, \nu^b\rho]) \times \delta([\nu^c\rho, \nu^d\rho]) \rtimes \sigma$.

There is an irreducible subquotient $\sigma_1$ of $\delta([\nu^c\rho, \nu^d\rho]) \rtimes \sigma$ such that $\sigma_{sp}$ is an irreducible constituent of $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma_1$. Also, since $\mu^*(\sigma_{sp})$ contains an irreducible quotient of the form $\delta([\nu^c\rho, \nu^x\rho]) \otimes \sigma_2$, using $c < a$, $x < b$ and the structural formula we deduce that $\mu^*(\sigma_1)$ also contains some irreducible constituent of the form $\delta([\nu^c\rho, \nu^x\rho]) \otimes \sigma_3$. Proposition 3.1 (i) of [16] implies that in the appropriate Grothendieck group we have

$$\delta([\nu^c\rho, \nu^d\rho]) \rtimes \sigma = L(\delta([\nu^c\rho, \nu^d\rho]) \rtimes \sigma) + \sigma'_{sp},$$

where $\sigma'_{sp}$ is strongly positive discrete series such that $\mathrm{Jord}(\sigma'_{sp}) = \mathrm{Jord}(\sigma) \setminus \{(2c-1, \rho)\} \cup \{(2d+1, \rho)\}$. But $d < x$ and Theorem 4.6 of [9] show that $\mu^*(\sigma'_{sp})$ does not contain irreducible constituent of the form $\delta([\nu^c\rho, \nu^x\rho]) \otimes \sigma_3$. This clearly implies $\sigma_1 \cong L(\delta([\nu^c\rho, \nu^d\rho]) \rtimes \sigma)$ and the induced representation $\delta([\nu^a\rho, \nu^b\rho]) \rtimes L(\delta([\nu^c\rho, \nu^d\rho]) \rtimes \sigma)$ contains the strongly positive discrete series $\sigma_{sp}$, which is the desired conclusion. □

## 5. Case $a = \frac{1}{2}$

In this section we assume $a = \frac{1}{2}$. Description of strongly positive discrete series, given in Proposition 2.2, implies that if $\delta_1 \rtimes L(\delta_2 \rtimes \sigma)$ contains an irreducible strongly positive subquotient then $\nu^{\frac{1}{2}}\rho$ does not appear in the cuspidal support of $\sigma$.

We will first discuss the case $\mathrm{Jord}_\rho(\sigma) \neq \emptyset$. In this case, necessary and sufficient conditions under which the induced representation $\delta_1 \rtimes L(\delta_2 \rtimes \sigma)$ contains a strongly positive irreducible subquotient are given by the following proposition.

**Proposition 5.1.** *Suppose $Jord_\rho(\sigma) \neq \emptyset$ and define $x$ by $2x+1 = \min(Jord_\rho(\sigma))$. Then the induced representation $\delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \rtimes L(\delta([\nu^c\rho, \nu^d\rho]) \rtimes \sigma)$ contains an irreducible strongly positive subquotient if and only if $\nu^{\frac{1}{2}}\rho$ does not appear in the cuspidal support of $\delta([\nu^c\rho, \nu^d\rho]) \rtimes \sigma$, $c = b+1$ and $d < x$.*

*Proof.* Let us first assume that there is some irreducible strongly positive subquotient $\sigma_{sp}$ of $\delta_1 \rtimes L(\delta_2 \rtimes \sigma)$. We have already observed that then $\nu^{\frac{1}{2}}\rho$ does not appear in the cuspidal support of $\sigma$. If $c \neq b+1$, it can be deduced in the same way as in the proof of Lemma 4.1 that $\delta_1 \rtimes L(\delta_2 \rtimes \sigma)$ does not contain an irreducible strongly positive subquotient.

Thus, it remains to consider the case $c = b + 1$. In this case, $2d + 1 \notin \mathrm{Jord}_\rho(\sigma)$ and we have $\mathrm{Jord}(\sigma_{sp}) = \mathrm{Jord}(\sigma) \cup \{(2d + 1, \rho)\}$. If $x < d$ then $\mu^*(\sigma_{sp}) \geq \delta([\nu^{\frac{1}{2}}\rho, \nu^x\rho]) \otimes \sigma'_{sp}$ for strongly positive discrete series $\sigma'_{sp}$ such that $\mathrm{Jord}(\sigma'_{sp}) = \mathrm{Jord}(\sigma_{sp}) \setminus \{(2x + 1, \rho)\}$ and we have $b \leq x$. If $b = x$, using the same argument as in the previous case we deduce that there are no irreducible strongly positive subquotients of $\delta_1 \rtimes L(\delta_2 \rtimes \sigma)$. Let us now assume $b < x$ and $x < d$. Since $\sigma_{sp}$ is also a subquotient of $\delta_1 \times \delta_2 \rtimes \sigma$, it follows that $\mu^*(\delta_1 \times \delta_2 \rtimes \sigma)$ contains $\delta([\nu^{\frac{1}{2}}\rho, \nu^x\rho]) \otimes \sigma'_{sp}$. We shall analyze $\mu^*(\delta_1 \times \delta_2 \rtimes \sigma)$ using the structural formula from Lemma 2.1.

There are $-\frac{1}{2} \leq i_1 \leq j_1 \leq b$, $c - 1 \leq i_2 \leq j_2 \leq d$ and an irreducible constituent $\delta \otimes \pi$ of $\mu^*(\sigma)$ such that

$$\begin{aligned}\delta([\nu^{\frac{1}{2}}\rho, \nu^x\rho]) \leq &\delta([\nu^{-i_1}\rho, \nu^{-\frac{1}{2}}\rho]) \times \delta([\nu^{j_1+1}\rho, \nu^b\rho]) \times \\ &\delta([\nu^{-i_2}\rho, \nu^{-c}\rho]) \times \delta([\nu^{j_2+1}\rho, \nu^d\rho]) \times \delta\end{aligned}$$

and

$$\sigma'_{sp} \leq \delta([\nu^{i_1+1}\rho, \nu^{j_1}\rho]) \times \delta([\nu^{i_2+1}\rho, \nu^{j_2}\rho]) \rtimes \pi.$$

Since $c > 0$, $x > 0$ and $x < d$, we directly get $i_1 = -\frac{1}{2}$, $i_2 = c - 1$ and $j_2 = d$. Furthermore, since $\nu^{\frac{1}{2}}\rho$ does not appear in the cuspidal support of $\sigma$, we have $j_1 = -\frac{1}{2}$ and $\delta \cong \delta([\nu^{b+1}\rho, \nu^x\rho])$. It follows from Theorem 4.6 of [9] that $\pi$ is the strongly positive discrete series such that $\mathrm{Jord}(\pi) = \mathrm{Jord}(\sigma) \setminus \{(2x+1, \rho)\} \cup \{(2b+1, \rho)\}$. Since the induced representation $\delta([\nu^c\rho, \nu^d\rho]) \rtimes \pi$ contains the strongly positive discrete series $\sigma'_{sp}$, Proposition 3.1 (i) of [16] implies $[2c+1, 2d+1] \cap \mathrm{Jord}_\rho(\pi) = \emptyset$, i.e., $[2c+1, 2d+1] \cap \mathrm{Jord}_\rho(\sigma) = \{2x+1\}$.

Also, we conclude that if $\delta_1 \rtimes L(\delta_2 \rtimes \sigma)$ contains $\sigma_{sp}$, then $\mu^*(L(\delta_2 \rtimes \sigma))$ contains $\delta([\nu^{b+1}\rho, \nu^x\rho]) \otimes \sigma'_{sp}$. Proposition 3.1 (i) of [16] implies that in the appropriate Grothendieck group we have

$$\delta_2 \rtimes \sigma = L(\delta_2 \rtimes \sigma) + L(\delta([\nu^c\rho, \nu^x\rho]) \rtimes \sigma'_{sp}).$$

Using Proposition 3.1 (ii) of [16], we deduce that the induced representation $\delta([\nu^c\rho, \nu^x\rho]) \rtimes \sigma'_{sp}$ is irreducible, thus it is isomorphic to $L(\delta([\nu^c\rho, \nu^x\rho]) \rtimes$

$\sigma'_{sp}$) and Frobenius reciprocity implies $\mu^*(\delta([\nu^c\rho, \nu^x\rho]) \rtimes \sigma'_{sp}) \geq \delta([\nu^c\rho, \nu^x\rho]) \otimes \sigma'_{sp}$. It can be easily seen that $\delta([\nu^c\rho, \nu^x\rho]) \otimes \sigma'_{sp}$ appears with multiplicity one in $\mu^*(\delta_2 \rtimes \sigma)$ so it does not appear in $\mu^*(L(\delta_2 \rtimes \sigma))$. Consequently, if $x < d$ the induced representation $\delta_1 \rtimes L(\delta_2 \rtimes \sigma)$ does not contain an irreducible strongly positive subquotient.

In this way we have proved that if $\delta_1 \rtimes L(\delta_2 \rtimes \sigma)$ contains an irreducible strongly positive subquotient then $\nu^{\frac{1}{2}}\rho$ does not appear in the cuspidal support of $\sigma$, $c = b + 1$ and $d < x$.

Conversely, let us now assume that $\nu^{\frac{1}{2}}\rho$ does not appear in the cuspidal support of $\sigma$, $c = b + 1$ and $d < x$. Again, we denote by $\sigma_{sp}$ a strongly positive discrete series such that $\mathrm{Jord}(\sigma_{sp}) = \mathrm{Jord}(\sigma) \cup \{(2d + 1, \rho)\}$. Using Proposition 3.1 (ii) of [16], in the same way as in the last part of the proof of Lemma 4.3, we obtain that $\delta_1 \rtimes L(\delta_2 \rtimes \sigma)$ contains $\sigma_{sp}$. This finishes the proof. □

Now we assume $\mathrm{Jord}_\rho(\sigma) = \emptyset$. Then $\nu^{\frac{1}{2}}\rho \rtimes \sigma_{cusp}$ reduces and there are no twists of $\rho$ appearing in the cuspidal support of $\sigma$. The following proposition can be proved in the same way as Lemmas 4.2 and 4.3, details being left to the reader.

**Proposition 5.2.** *Suppose $\mathrm{Jord}_\rho(\sigma) = \emptyset$. Then the induced representation $\delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \rtimes L(\delta([\nu^c\rho, \nu^d\rho]) \rtimes \sigma)$ contains an irreducible strongly positive subquotient if and only if $c = b + 1$.*

## 6. Case $c = \frac{1}{2}$

This section is devoted to the analysis of the remaining case $c = \frac{1}{2}$. Again, if $\delta_1 \rtimes L(\delta_2 \rtimes \sigma)$ contains an irreducible strongly positive subquotient then $\nu^{\frac{1}{2}}\rho$ does not appear in the cuspidal support of $\sigma$. We start with the following result:

**Lemma 6.1.** *If $\mathrm{Jord}_\rho(\sigma) = \emptyset$, the induced representation $\delta([\nu^a\rho, \nu^b\rho]) \rtimes L(\delta([\nu^{\frac{1}{2}}\rho, \nu^d\rho]) \rtimes \sigma)$ does not contain an irreducible strongly positive subquotient.*

*Proof.* From $\mathrm{Jord}_\rho(\sigma) = \emptyset$ we conclude that $\nu^{\frac{1}{2}}\rho \rtimes \sigma_{cusp}$ reduces and there are no twists of $\rho$ appearing in the cuspidal support of $\sigma$. Suppose, contrary to our assumption, that there is some strongly positive irreducible subquotient $\sigma_{sp}$ of $\delta_1 \rtimes L(\delta_2 \rtimes \sigma)$. By Theorem 5.1 of [16], induced representation $\delta_2 \rtimes \sigma$

contains a strongly positive discrete series which will be denoted by $\pi_{sp}$. We note that $\pi_{sp}$ is also the unique irreducible subrepresentation of $\delta_2 \rtimes \sigma$.

Using the description of Jacquet modules of strongly positive representations we deduce that $\mu^*(\sigma_{sp})$ contains an irreducible constituent of the form $\delta([\nu^{\frac{1}{2}}\rho, \nu^y\rho]) \otimes \sigma'_{sp}$, for some strongly positive discrete series $\sigma'_{sp}$. Thus, $\mu^*(\delta_1 \rtimes L(\delta_2 \rtimes \sigma))$ also contains $\delta([\nu^{\frac{1}{2}}\rho, \nu^y\rho]) \otimes \sigma'_{sp}$.

Using Lemma 2.1 we obtain that $\mu^*(L(\delta_2 \rtimes \sigma))$ contains $\delta([\nu^{\frac{1}{2}}\rho, \nu^d\rho]) \otimes \sigma'_{sp}$, which is impossible since such irreducible constituent is contained with multiplicity one in both $\mu^*(\delta_2 \rtimes \sigma)$ and $\mu^*(\pi_{sp})$. This completes the proof. □

In the rest of this section we assume $\mathrm{Jord}_\rho(\sigma) \neq \emptyset$. In the next couple of lemmas we will finish the proof of Theorem 3.4 (iii).

**Lemma 6.2.** *Suppose $2a-1 \notin Jord_\rho(\sigma)$. If the induced representation $\delta([\nu^a\rho, \nu^b\rho]) \rtimes L(\delta([\nu^{\frac{1}{2}}\rho, \nu^d\rho]) \rtimes \sigma)$ contains an irreducible strongly positive subquotient then $\nu^{\frac{1}{2}}\rho$ does not appear in the cuspidal support of $\sigma$, $2b+1 \notin Jord_\rho(\sigma)$, $d = a-1$ and for $x$ such that $2x+1 = \min(Jord_\rho(\sigma))$ we have $[2a+1, 2b+1] \cap Jord_\rho(\sigma) = \{2x+1\}$.*

*Proof.* It follows directly from the cuspidal support of $\delta_1 \rtimes L(\delta_2 \rtimes \sigma)$ that if such induced representation contains a discrete series subquotient then $2b+1 \notin \mathrm{Jord}_\rho(\sigma)$ and $d = a-1$. Furthermore, if $\sigma_{sp}$ is an irreducible strongly positive subquotient of $\delta_1 \rtimes L(\delta_2 \rtimes \sigma)$ then $\mathrm{Jord}(\sigma_{sp}) = \mathrm{Jord}(\sigma) \cup \{(2b+1, \rho)\}$. Let us define $y$ by $2y+1 = \min(\mathrm{Jord}_\rho(\sigma_{sp}))$. Obviously, $y \leq b$. If $y = b$, then $2d+1$ is less than $\min(\mathrm{Jord}_\rho(\sigma))$ and, by Theorem 5.1 of [16], the induced representation $\delta_2 \rtimes \sigma$ contains a strongly positive discrete series. Now, in the same way as in the proof of Lemma 6.1, we deduce that $\delta_1 \rtimes L(\delta_2 \rtimes \sigma)$ does not contain an irreducible strongly positive subquotient.

Consequently, $y < b$. This also gives $2y+1 = \min(\mathrm{Jord}_\rho(\sigma))$. Using the description of Jacquet modules of strongly positive representations, we deduce that $\mu^*(\sigma_{sp})$ contains an irreducible constituent of the form $\delta([\nu^{\frac{1}{2}}\rho, \nu^y\rho]) \otimes \sigma'_{sp}$, for some strongly positive discrete series $\sigma'_{sp}$. Using Lemma 2.1 together with Theorem 4.6 of [9], we deduce that $\sigma'_{sp}$ is an irreducible subquotient of the induced representation of the form $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma''_{sp}$, where $\sigma''_{sp}$ is the strongly positive discrete series such that $\mathrm{Jord}(\sigma''_{sp}) = \mathrm{Jord}(\sigma) \setminus \{(2y+1, \rho)\} \cup \{(2a-1, \rho)\}$. Proposition 3.1 (i) of [16] gives $[2a+1, 2b+1] \cap \mathrm{Jord}_\rho(\sigma) = \{2y+1\}$ and the lemma is proved. □

The following lemma completes the proof of the first part of Theorem 3.4 (iii).

**Lemma 6.3.** *Suppose that* $\nu^{\frac{1}{2}}\rho$ *does not appear in the cuspidal support of* $\sigma$, $2a-1, 2b+1 \notin Jord_\rho(\sigma)$, $d = a-1$ *and for* $x$ *such that* $2x+1 = \min(Jord_\rho(\sigma))$ *we have* $[2a+1, 2b+1] \cap Jord_\rho(\sigma) = \{2x+1\}$*. Then the induced representation* $\delta([\nu^a\rho, \nu^b\rho]) \rtimes L(\delta([\nu^{\frac{1}{2}}\rho, \nu^d\rho]) \rtimes \sigma)$ *contains an irreducible strongly positive subquotient.*

*Proof.* Two possibilities will be considered separately:

- $x = a$.

Let us denote by $\sigma_{sp}$ strongly positive discrete series such that $\mathrm{Jord}(\sigma_{sp}) = \mathrm{Jord}(\sigma) \setminus \{(2x+1, \rho)\} \cup \{(2b+1, \rho)\}$. Furthermore, let us denote by $\tau$ a tempered subrepresentation of the induced representation $\delta([\nu^{-a}\rho, \nu^a\rho]) \rtimes \sigma_{sp}$ such that $\mu^*(\tau) \geq \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \otimes \sigma_{sp}$. The induced representation $\delta([\nu^{-a}\rho, \nu^a\rho]) \rtimes \sigma_{sp}$ contains two tempered subrepresentations and such a subrepresentation $\tau$ is unique by Corollary 4.5 of [19]. Transitivity of Jacquet modules shows that the Jacquet module of $\tau$ with respect to the appropriate parabolic subgroup contains $\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \otimes \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \otimes \sigma_{sp}$. Consequently, there is some irreducible constituent $\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \otimes \pi$ of $\mu^*(\tau)$ such that $\mu^*(\pi) \geq \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \otimes \sigma_{sp}$. Using the structural formula for $\mu^*(\delta([\nu^{-a}\rho, \nu^a\rho]) \rtimes \sigma_{sp})$ we get that $\pi$ is subquotient of $\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_{sp}$, since $\nu^{\frac{1}{2}}\rho$ does not appear in the cuspidal support of $\sigma_{sp}$. It now follows directly that $\pi$ has to be the unique irreducible subrepresentation of $\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_{sp}$, which is also strongly positive by Theorem 4.6 of [6].

We have an embedding

$$\sigma_{sp} \hookrightarrow \delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma'_{sp},$$

where $\sigma'_{sp}$ is the unique strongly positive discrete series such that $\mathrm{Jord}(\sigma'_{sp}) = \mathrm{Jord}(\sigma_{sp}) \setminus \{(2b+1, \rho)\} \cup \{(2d+1, \rho)\}$. Now $\tau$ is contained in

$$\delta([\nu^a\rho, \nu^b\rho]) \times \delta([\nu^{-a}\rho, \nu^a\rho]) \rtimes \sigma'_{sp}.$$

This implies $\mu^*(\delta([\nu^a\rho, \nu^b\rho]) \times \delta([\nu^{-a}\rho, \nu^a\rho]) \rtimes \sigma'_{sp}) \geq \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \otimes \pi$ and using Lemma 2.1 two times we obtain $\delta([\nu^a\rho, \nu^b\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma'_{sp} \geq \pi$.

By Theorem 5.1 (i) of [16], in the appropriate Grothendieck group we have

$$\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma'_{sp} = L(\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma'_{sp}) + L(\delta_2 \rtimes \sigma),$$

and, in the same way as in the proof of Lemma 4.5, we obtain that $\delta_1 \rtimes L(\delta_2 \rtimes \sigma)$ contains the strongly positive discrete series $\pi$.

- $x > a$.

Let us denote by $\sigma'$ a strongly positive discrete series such that $\mathrm{Jord}(\sigma') = \mathrm{Jord}(\sigma) \setminus \{(2x+1, \rho)\} \cup \{(2d+1, \rho)\}$. Furthermore, we denote by $\sigma_{ds}$ the unique discrete series representation which is a subrepresentation of both induced representations

$$\delta([\nu^{-d}\rho, \nu^a\rho]) \rtimes \sigma.$$

and

$$\delta([\nu^{-a}\rho, \nu^x\rho]) \rtimes \sigma'.$$

Proposition 4.5 of [10] shows that the generalized principal series $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma_{ds}$ contains two discrete series subquotients. We denote by $\sigma'_{ds}$ a discrete series subquotient of $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma_{ds}$ with the property that there is some irreducible representation $\pi$ such that $\mu^*(\sigma'_{ds}) \geq \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \otimes \pi$. It follows from Proposition 3.4 of [11] that $\pi$ is the unique strongly positive discrete series subrepresentation of the induced representation $\delta([\nu^{\frac{1}{2}}\rho, \nu^x\rho]) \rtimes \sigma_{sp}$, for strongly positive discrete series $\sigma_{sp}$ such that $\mathrm{Jord}(\sigma_{sp}) = \mathrm{Jord}(\sigma) \setminus \{(2x+1, \rho)\} \cup \{(2b+1, \rho)\}$. Since $\sigma'_{ds}$ is an irreducible subquotient of

$$\delta([\nu^a\rho, \nu^b\rho]) \times \delta([\nu^{-d}\rho, \nu^a\rho]) \rtimes \sigma,$$

it follows that $\mu^*(\delta([\nu^a\rho, \nu^b\rho]) \times \delta([\nu^{-d}\rho, \nu^a\rho]) \rtimes \sigma)$ contains $\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \otimes \pi$. Since $a > 0$ and $\mathrm{Jord}_\rho(\sigma) \cap \{2d+1, 2a+1\} = \emptyset$, using Lemma 2.1 two times we get

$$\pi \leq \delta([\nu^a\rho, \nu^b\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^d\rho]) \rtimes \sigma.$$

In the appropriate Grothendieck group we have (by Theorem 5.1 (i) of [16])

$$\delta([\nu^{\frac{1}{2}}\rho, \nu^d\rho]) \rtimes \sigma = L(\delta([\nu^{\frac{1}{2}}\rho, \nu^d\rho]) \rtimes \sigma) + \pi_{sp},$$

for strongly positive discrete series $\pi_{sp}$ such that $\mathrm{Jord}(\pi_{sp}) = \mathrm{Jord}(\sigma) \cup \{(2d+1, \rho)\}$. Following the same lines as in the proof of Lemma 4.5 we deduce that $\delta_1 \rtimes L(\delta_2 \rtimes \sigma)$ contains the strongly positive discrete series $\pi$. This completes the proof. □

To complete the proof of Theorem 3.4 (iii), we prove

**Lemma 6.4.** *Suppose $2a-1 \in Jord_\rho(\sigma)$. Then the induced representation $\delta([\nu^a\rho, \nu^b\rho]) \rtimes L(\delta([\nu^{\frac{1}{2}}\rho, \nu^d\rho]) \rtimes \sigma)$ contains an irreducible strongly positive subquotient if and only if $\nu^{\frac{1}{2}}\rho$ does not appear in the cuspidal support of $\sigma$, $2a-1 = \min(Jord_\rho(\sigma))$, $2b+1 \notin Jord_\rho(\sigma)$, $d = a-1$ and if there is $2x+1 \in Jord_\rho(\sigma)$ such that $(2x+1)_- = 2a-1$ then $b < x$.*

*Proof.* Using the cuspidal support argument as before, we deduce that if $2b+1 \in \mathrm{Jord}_\rho(\sigma)$ then there are no discrete series subquotients of $\delta_1 \rtimes L(\delta_2 \rtimes \sigma)$. Thus, we may assume $2b+1 \notin \mathrm{Jord}_\rho(\sigma)$. If we again denote by $\sigma_{sp}$ irreducible strongly positive subquotient of $\delta_1 \rtimes L(\delta_2 \rtimes \sigma)$, it follows directly that $\mathrm{Jord}(\sigma_{sp}) = \mathrm{Jord}(\sigma) \setminus \{(2a-1,\rho)\} \cup \{(2d+1,\rho),(2b+1,\rho)\}$.

Since $\nu^{\frac{1}{2}}\rho$ appears in the cuspidal support of $\sigma_{sp}$, for some strongly positive discrete series $\sigma'_{sp}$ we have $\mu^*(\sigma_{sp}) \geq \delta([\nu^{\frac{1}{2}}\rho, \nu^{x_{\min}}\rho]) \otimes \sigma'_{sp}$, where $x_{\min}$ is defined by $2x_{\min}+1 = \min(\mathrm{Jord}_\rho(\sigma_{sp}))$. Since $\nu^{\frac{1}{2}}\rho$ does not appear in the cuspidal support of $\sigma$, using Lemma 2.1 we get that if $\mu^*(\delta_1 \times \delta_2 \rtimes \sigma)$ contains some irreducible constituent of the form $\delta([\nu^{\frac{1}{2}}\rho, \nu^y\rho]) \otimes \pi'$, then $y \geq d$. Thus, it follows that $x_{\min}$ equals $d$, since we have $2d+1 \in \mathrm{Jord}_\rho(\sigma_{sp})$.

Consequently, $\sigma'_{sp}$ is an irreducible subquotient of $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma$. Since $a > \frac{1}{2}$, by Proposition 3.1 (i) of [16] we have $[2a+1, 2b+1] \cap \mathrm{Jord}_\rho(\sigma) = \emptyset$. Therefore, if there is $2x+1 \in \mathrm{Jord}_\rho(\sigma)$ such that $(2x+1)_- = 2a-1$ then $b < x$.

Also, we have $\mu^*(\delta([\nu^a\rho, \nu^b\rho]) \rtimes L(\delta([\nu^{\frac{1}{2}}\rho, \nu^d\rho]) \rtimes \sigma)) \geq \delta([\nu^{\frac{1}{2}}\rho, \nu^d\rho]) \otimes \sigma'_{sp}$. Again, using $a > \frac{1}{2}$ we deduce that $\mu^*(L(\delta([\nu^{\frac{1}{2}}\rho, \nu^d\rho]) \rtimes \sigma))$ contains an irreducible constituent of the form $\delta([\nu^{\frac{1}{2}}\rho, \nu^d\rho]) \otimes \pi$. It can be easily seen that the only irreducible constituent of such form which appears in $\mu^*(\delta([\nu^{\frac{1}{2}}\rho, \nu^d\rho]) \rtimes \sigma)$ is $\delta([\nu^{\frac{1}{2}}\rho, \nu^d\rho]) \otimes \sigma$, which appears there with multiplicity one. Now Frobenius reciprocity and Theorem 5.1 (i) of [16] imply that $L(\delta([\nu^{\frac{1}{2}}\rho, \nu^d\rho]) \rtimes \sigma)$ has to be the unique irreducible subrepresentation of $\delta([\nu^{\frac{1}{2}}\rho, \nu^d\rho]) \rtimes \sigma$, which is then irreducible and applying the same theorem one more time we deduce that $2d+1 \in \mathrm{Jord}_\rho(\sigma)$.

Since $2d+1 \in \mathrm{Jord}_\rho(\sigma) \cap \mathrm{Jord}_\rho(\sigma_{sp})$, from the description of $\mathrm{Jord}(\sigma_{sp})$ in terms of $\mathrm{Jord}(\sigma)$ and description of the cuspidal support of strongly positive discrete series we obtain that $d = a-1$. Furthermore, $2d+1 = \min(\mathrm{Jord}_\rho(\sigma_{sp}))$ now also implies $2d+1 = \min(\mathrm{Jord}_\rho(\sigma))$, i.e., $2a-1 = \min(\mathrm{Jord}_\rho(\sigma))$.

Conversely, let us assume that $\nu^{\frac{1}{2}}\rho$ does not appear in the cuspidal support of $\sigma$, $2a-1 = \min(\mathrm{Jord}_\rho(\sigma)), 2b+1 \notin \mathrm{Jord}_\rho(\sigma)$, $d = a-1$ and if there is $2x+1 \in \mathrm{Jord}_\rho(\sigma)$ such that $(2x+1)_- = 2a-1$ then $b < x$. It follows from Theorem 3.4 of [6] that the induced representation

$$\delta([\nu^{\frac{1}{2}}\rho, \nu^d\rho]) \times \delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma.$$

contains a unique irreducible subrepresentation, which is strongly positive (by Theorem 4.6 of [6]) and we will denote it by $\sigma_{sp}$. Since, by Theorem 5.1 of [16], the induced representation $\delta([\nu^{\frac{1}{2}}\rho, \nu^d\rho]) \rtimes \sigma$ is irreducible, in the same way as before we deduce that $\sigma_{sp}$ is an irreducible subquotient of $\delta([\nu^a\rho, \nu^b\rho]) \rtimes L(\delta([\nu^{\frac{1}{2}}\rho, \nu^d\rho]) \rtimes \sigma)$, and the lemma is proved. □


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